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STRESS CORRECTION FACTOR FOR HELICAL SPRINGS

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ABSTRACT

In this paper, the stress correction factor for helical spring having small pitch angles with an epitrochoidal cross section is discussed, applying the result (MIZUNO, 1969) of which one of the authors had reported before. The stress correction factor for helical springs has been calculated by many researchers. But these methods are based on the assumption that the cross section of the spring remains circular after coiling. However, the deformed shape of the section of the circular wire after coiling is reported to be an epitrochoid (MIZUNO, 1969).

The correction factor calculated considering this effect is similar to the Röver's result which is standing under the assumption that the cross section remains circular.

1. The maximum shearing stress of helical springs with circular cross section.

The maximum shearing stress, τ_{max}, which occurs at the inside of helical springs with the circular cross section, has been calculated by many researchers. This τ_{max} is expressed as K times as large as the maximum shearing stress of a twisted bar with the circular cross section, as follows:

$$\tau_{max} = \frac{8PD}{\pi d^3} K, \tag{1}$$

where,

K: stress correction factor,
P: tensile or compressive load for the spring,
D: mean coil diameter,
d: bar diameter,

and, K is the function of the spring index, $C = D/d$.

RÖVER (1913) obtained K of Eq. (1) as,

$$K=\frac{C}{C-1}+\frac{1}{4C}=1+\frac{5}{4C}+\frac{1}{C^2}+\frac{1}{C^3}+\frac{1}{C^4}+\cdots, \tag{2}$$

for the springs with small pitch angles (pitch angle $\alpha\approx0$) considering effects of curvature.

According to WAHL's result (1929),

$$K=\frac{4C-1}{4C-4}+\frac{0.615}{C}\approx1+\frac{5.46}{4C}+\frac{3}{4}\left(\frac{1}{C^2}+\frac{1}{C^3}+\frac{1}{C^4}+\cdots\right). \tag{3}$$

Assuming the shape of coils of helical springs to be a torus, K can be calculated in the following.

Taking cylindrical coodinates, in Fig. 1, the stress function φ, that satisfies the following equations, is obtained:

$$\left.\begin{array}{ll} \text{at the cross section,} & \dfrac{\partial^2\varphi}{\partial\rho^2}+\dfrac{\partial^2\varphi}{\partial z^2}-\dfrac{3}{\rho}\cdot\dfrac{\partial\varphi}{\partial\rho}=\text{const.,} \\ \text{at the boundary,} & \varphi=0. \end{array}\right\} \tag{4}$$

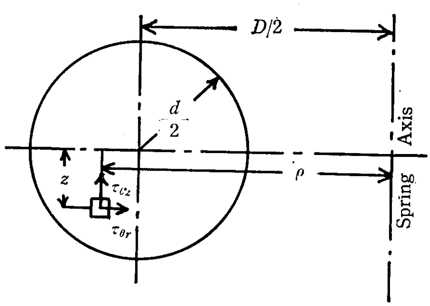

Fig. 1.

From this stress function, $\tau_{\theta z}$ and $\tau_{\theta r}$, the components of the shearing stress, are obtained.

The approximate solutions of the stress function φ, were given by GÖHNER (1930, 1931, 1932), and he showed K in the following,

$$K=\frac{\dfrac{C}{C-1}+\dfrac{1}{4C}+\dfrac{1}{16C^2}}{1+\dfrac{3}{16}\cdot\dfrac{1}{C^2-1}}=1+\frac{5}{4C}+\frac{7}{8C^2}+\frac{55}{64C^3}+\frac{148}{256C^4}+\cdots, \tag{5}$$

$$K=\frac{C}{C-1}+\frac{1}{4C}-\frac{1}{8C^2}=1+\frac{5}{4C}+\frac{7}{8C^2}+\frac{1}{C^3}+\frac{1}{C^4}+\cdots, \qquad (5)'$$

(for $C\geq 3$, $\alpha\leq 16°$),

$$K=1+\frac{5}{4C}+\frac{7}{8C^2}+\frac{1}{C^3}, \qquad \text{(for } C\geq 3,\ \alpha\leq 12°\text{)}. \qquad (5)''$$

The exact solution of the stress function φ that satisfies Eq. (4) was expressed in series form using Legendre function by Freibeger. HENRICI (1955) expanded this exact solution in power series form of $1/C$. That is:

$$K=1+\frac{5}{4C}+\frac{7}{8C^2}+\frac{155}{256C^3}+\frac{11{,}911}{24{,}576C^4}+\cdots. \qquad (6)$$

From the comparison of these results, approximate solutions are larger than that of Eq. (6) and are in the safty-side.

The effects of the pitch angle α are considered only in GÖHNER's Eqs. (5)$'$, and (5)$''$, but it may be said that these are very small, because the Eqs. (5)$'$ and (5)$''$ coincide with Eq. (5) up to the terms of $1/C^2$.

If $C\geq 3$ and $\alpha\leq 16°$, thus springs may be regarded as closed ones.

These results are calculated from the assumption that the section is circular after deformation of the material.

But, spring material is subjected to plastic deformation in coiling, and if the cross section of the material is circular before coiling, the shape of the deformed cross section after coiling is obtained to be an epitrochoid (MIZUNO, 1969). Therefore the influence of this deformation of the cross section on the shearing stress should be considered.

2. Torsion of a bar with epitrochoidal cross section.

The boundary of the cross section is conformably mapped on a unit circle, $\zeta=e^{i\theta}$, in order to solve the problem of a torsional bar with an epitrochoidal cross section. The equations of an epitrochoid after coiling are given in the following (MIZUNO, 1969):

$$\left.\begin{aligned} x&=r_0\left(\cos\theta+\frac{1}{4C}\cos 2\theta\right),\\ y&=r_0\left(\sin\theta+\frac{1}{4C}\sin 2\theta\right), \end{aligned}\right\} \qquad (7)$$

where,

r_0: bar radius,

C: spring index,

therefore,

$$z=x+iy$$

$$=r_0\left[\cos\theta+i\sin\theta+\frac{1}{4C}(\cos 2\theta+i\sin 2\theta)\right]$$

$$=r_0\left[\zeta+\frac{1}{4C}\zeta^2\right].$$

Thus the conformable mapping function $w(\zeta)$ is

$$z=w(\zeta)=r_0\left[\zeta+\frac{1}{4C}\zeta^2\right]. \tag{8}$$

The shearing stresses, τ_{zx}, τ_{zy}, are expressed as follows (SOKOLNIKOFF, 1946):

$$\tau_{zx}-i\tau_{zy}=G\alpha_0\left[\frac{f'(\zeta)}{w'(\zeta)}-i\bar{w}(\zeta)\right], \tag{9}$$

where,

G : shearing modulus of elasticity,
α_0 : twisting angle per unit length,
$\bar{w}(\zeta)$: conjugate function of $w(\zeta)$,
$f(\zeta)$: complex stress function.

If the mapping function is written in the following:

$$z=w(\zeta)=\sum_{n=0}^{\infty}a_n\zeta^n, \tag{10}$$

where,

a_n: const.,

then, the function $f(\zeta)$ is given easily in the following manner:

$$f(\zeta)=\varphi+i\psi=i\sum_{n=0}^{\infty}b_n\zeta^n, \tag{11}$$

where,

φ: torsional function,
ψ: conjugate function of φ,
b_n: $\sum_{p=0}^{\infty}a_{n+p}\bar{a}_p$.

For the epitrochoidal cross section, from Eqs. (8) and (10),

$$a_1=r_0\,, \qquad a_2=\frac{r_0}{4C}\,,$$

$$b_0=r_0^2+\left(\frac{r_0}{4C}\right)^2, \qquad b_1=r_0\left(\frac{r_0}{4C}\right).$$

Substituting these terms into Eq. (11),

$$\left.\begin{aligned} f(\zeta) &= ir_0^2\left[1+\left(\frac{1}{4C}\right)^2+\frac{1}{4C}\zeta\right], \\ f'(\zeta) &= ir_0^2\frac{1}{4C}. \end{aligned}\right\} \tag{12}$$

From Eq. (8),

$$\begin{aligned} \overline{w}(\zeta) &= r_0\left(\bar{\zeta}+\frac{1}{4C}\bar{\zeta}^2\right) \\ &= r_0\left(\cos\theta+\frac{1}{4C}\cos 2\theta\right)-ir_0\left(\sin\theta+\frac{1}{4C}\sin 2\theta\right), \\ w'(\zeta) &= r_0\left(1+\frac{1}{2C}\zeta\right) \\ &= r_0\left[1+\frac{1}{2C}\left(\cos\theta+i\sin\theta\right)\right]. \end{aligned}$$

Putting these equations into Eq. (9), the shearing stresses of a torsional bar with an epitrochoidal cross section at the boundary are obtained as follows:

$$\left.\begin{aligned} \tau_{zx} &= G\alpha_0 r_0\left[\frac{\sin\theta}{8\left(C^2+C\cos\theta+\frac{1}{4}\right)}-\left(\sin\theta+\frac{1}{4C}\sin 2\theta\right)\right], \\ \tau_{zy} &= G\alpha_0 r_0\left[\left(\cos\theta+\frac{1}{4C}\cos 2\theta\right)-\frac{2C+\cos\theta}{8\left(C^2+C\cos\theta+\frac{1}{4}\right)}\right]. \end{aligned}\right\} \tag{13}$$

From Eq. (13), the shearing stress is only τ_{zy} for $\theta=0$, and its value is

$$\tau_{\theta=0} = G\alpha_0 r_0\left[1+\frac{1}{8C\left(C+\frac{1}{2}\right)}\right]. \tag{14}$$

This value $\tau_{\theta=0}$ is compared with the shearing stress of circular cross section. The maximum shearing stress τ_0 of a torsional bar of the circular section is

$$\tau_0 = \frac{16M}{\pi d^3} = \frac{M}{I_p}\cdot\frac{d}{2} = G\alpha_0 r_0, \tag{15}$$

where,

M: twisting moment,

d : diameter of a bar,

I_p : polar moment of inertia of the circular cross section $\left(\frac{\pi d^4}{32}\right)$.

Hence, the ratio of $\tau_{\theta=0}$ to τ_0 becomes:

$$\eta_{\theta=0}=\frac{\tau_{\theta=0}}{\tau_0}=1+\frac{1}{8C\left(C+\frac{1}{2}\right)}. \tag{16}$$

The value of $\eta_{\theta=0}$ due to the spring index C is obtained in Table 1.

Table 1.

η \ C	1	2	3	4	5	7	10
$\eta_{\theta=0}$	1.083	1.025	1.012	1.007	1.005	1.002	1.001

3. The maximum shearing stress of the closed coiled helical springs with epitrochoidal cross sections.

When the maximum shearing stress $\tau_{\max}$ of the springs with epitrochoidal cross sections is considered, its value should be refered as follows:

$$\tau_{\max}=\frac{8PD}{\pi d^3}(\eta_{\theta=0})\cdot K. \tag{17}$$

The ratio η becomes the maximal at $\theta=0$ and $\theta=\pi$, and at $\theta=\pi$ it reaches its maximum value. But the maximum shearing stress $\tau_{\max}$ occurs at the inside of the coil, because the effect of curvature is the largest.

The Eq. (16), showing the increase of the shearing stress, may be expanded in series form as follows:

$$\eta_{\theta=0}=1+\frac{1}{8C\left(C+\frac{1}{2}\right)}=1+\frac{1}{8C^2}-\frac{1}{16C^3}+\frac{1}{32C^4}-\cdots. \tag{18}$$

Substituting Eq. (18) into Eq. (17), and using Henrici's series form for the value of K,

$$\tau_{\max}=\frac{8PD}{\pi d^3}\left(1+\frac{5}{4C}+\frac{1}{C^2}+\frac{179}{256C^3}+\frac{13{,}447}{24{,}576C^4}+\cdots\right)=\frac{8PD}{\pi d^3}K', \tag{19}$$

where,

$$K'=1+\frac{5}{4C}+\frac{1}{C^2}+\frac{179}{256C^3}+\frac{13{,}447}{24{,}576C^4}+\cdots. \tag{19$'$}$$

Comparing Eq. (19)′ with Eqs. (2), (3), (5), and (5)″, it is found that RÖVER's result is most close to Eq. (19)′ and agrees well up to the third term.

Its value is greater than Eq. (19)′ and is in the safe side and has a simple form. Therefore, the RÖVER's formula is thought to be useful, considering deformation of the cross section of the springs.

In the above analysis, the effects of bending stress, direct tension component and the secondary shearing stresses are not yet considered. but these are negligiblly small.

APPENDIX

The material shape before coiling of helical springs with the rectangular cross section.

In order to obtain a helical spring with a rectangular or square cross section, the cross section of the material before coiling should be trapezoidal. The shape of the cross section before coiling is found empirically by the graphical method of Fig. 2 and Fig. 3.

Fig. 2 shows how to find the shape of the cross section of the material before coiling in the case of the square cross section after coiling, where, D is the mean coil diameter, A is the length of a side.

Fig. 3 shows the case of the rectangular cross section, where, A is the length of a short side, B is the length of a long side.

In the case of the rectangular cross section, (In the case of square: $A=B$.), the lower side of the cross section in Fig. 3 is (MIZUNO, 1969),

$$y=\frac{A}{2}=y_0+v_0=y_0+\frac{x_0y_0}{2R}, \tag{a}$$

where,

R: mean coil radius ($D/2$),

x_0, y_0: coordinates before deformation,

v_0: displacement of y direction.

Thus,

$$y_0=\frac{A/2}{1+x_0/2R}.$$

Since, $|x_0|<B/2$, and $1\gg B/4R$,

$$y_0\approx\frac{A}{2}\times\left(1-\frac{x_0}{2R}\right), \tag{b}$$

this shows that the lower side before coiling is a straight line, and from the symmetry the upper side is also a straight line. And, at the points of four corners,

$$x=\pm\frac{B}{2}(=x_0+u_0)=x_0+\frac{x_0^2+y_0^2}{4R}, \tag{c}$$

where,

u_0: displacement of x direction.

Putting Eq. (b) into Eq. (c), and calculating approximately,

$$
\begin{aligned}
&x_0^2+4Rx_0+\frac{A^2}{4}\left(1-\frac{x_0}{2R}\right)^2\mp 2BR\\
&=\left(1+\frac{A^2}{16R^2}\right)x_0^2+\left(4-\frac{A}{4R^2}\right)Rx_0+\frac{A^2}{4}\mp 2BR\\
&\approx x_0^2+4Rx_0\mp 2BR=0,
\end{aligned}
$$

$$x_0=-2R\pm 2R\sqrt{1\pm B/2R}\approx 2R\left(1-1\pm\frac{B}{4R}\right)=\pm\frac{B}{2}.$$

Thus,

$$x_0=\pm\frac{B}{2}. \tag{d}$$

Putting Eq. (d) into Eq. (b),

$$A_{\frac{1}{2}}=2y_0\approx A\left(1\pm\frac{B}{2D}\right). \tag{e}$$

The equations of Fig. 3 are approximately,

$$A_1=\frac{2AD}{2D-B}=\frac{A}{1-B/2D}\approx A\left(1+\frac{B}{2D}\right), \tag{e)$'$}$$

$$A_2=\frac{2A(D-B)}{2D-B}=\frac{A(1-B/D)}{1-B/2D}\approx A\left(1-\frac{B}{D}\right)\left(1+\frac{B}{2D}\right)\approx A\left(1-\frac{B}{2D}\right). \tag{e)$''$}$$

Then it can be seen that Eq. (e) is same as Eqs. (e)′ and (e)″.

That is, the empirical formula gives the same results as that obtained from the method of 'Plastic Deformation of a Wire with Circular Section in Coiling' (MIZUNO, 1969).

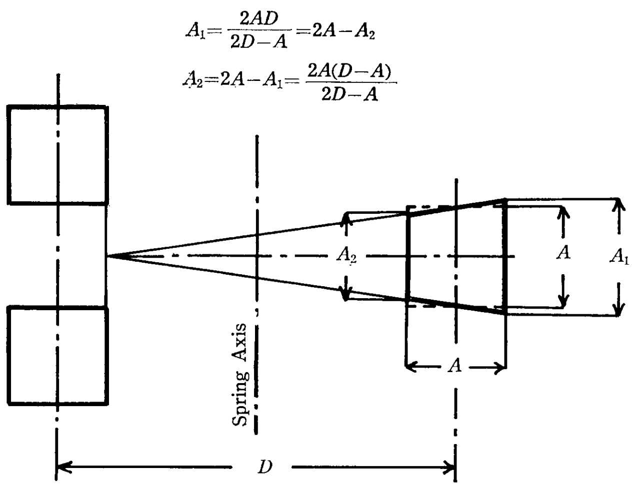

Fig. 2.

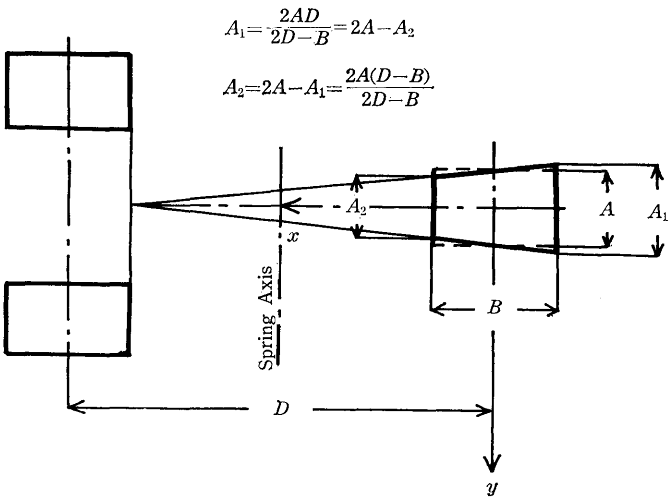

Fig. 3.

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