Title	Problem of large deflection of coiled spring, continued report
Sub Title	
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Publisher	慶応義塾大学藤原記念工学部
Publication year	1970
Jtitle	Proceedings of the Fujihara Memorial Faculty of Engineering Keio University Vol.23, No.94/95 (1970.) ,p.141- 159
JaLC DOI	
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Notes	
Genre	Departmental Bulletin Paper
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=KO50001004-00230095-0141

Proc. Fujihara Mem. Fac. Eng. Keio Univ.
Vol. 23, Nos. 3 & 4, 1970, pp. 141—159.

Problem of Large Deflection of Coiled Spring, Continued Report*

(Received March 17, 1971)

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Abstract

The discussion of the deflection of coiled springs is generally based on the theory of finite deformation. It seems, however, that some of its basic assumptions are not properly used for the investigation of deformed springs. Therefore, it is of particular interest for us to treat with them in the light of a theory of large deformation. Now that the theory of large deformation does not seem to have been developed fully enough, it would be the more so.

In this paper, we shall discuss the problem concerning the deflection of coiled springs under oblique loads. Dealing with such a general case, we could probably solve many questions under various boundary conditions.

We do not, however, directly consider the problem about oblique loads. Instead, we are going to discuss the post-buckling deflection of coiled springs first, and then translate its theoretical outcomes into the problem of deflection under oblique loads.

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The experimental results have shown that our theory is applicable for springs with a wide variety of slenderness.

1. Introduction

The theory of finite deformation provides us with powerful means in the investigation of elastic stability of bars. In this theory the squares of deformation angles are assumed to be negligible when compared to unit. But it is the case that if the elements of a spring are regarded to have infinitesimal strains in the elastic range, the spring itself is largely deformed on the whole. For this reason, the deflection curve of the central line of a coiled spring should be treated as a problem of large deflection. However, this theory seems not to have achieved a desirable success. The subject of

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* The previous report is reference (1).

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this paper is to develop the theory further; the examination of its practical problems will be reported later.

Now we shall consider a coiled spring under oblique loads. General analysis on such a case would enable us to solve various problems with given particuler boundary conditions (see 3. 3). We do not, however, directly consider the problem about oblique loads, instead, we'll discuss the post-buckling deflection of coiled springs first, and then translate its theoretical results into the problem of deflection under oblique loads by the translation and rotation of coordinates. The reasons for this are the following two. The first reason is that the process to the goal is simplified and clarified by our method. The second is that the difficulty of experiments to determine whether the theory is good or not is greatly decreased. The detailed discussions of translation and rotation of coordinates will be given in 3. 1. Let us notice that the load R Fig. 1 is larger than the critical load and that it is no longer impossible to elucidate R by the theory of finite deformation. And therefore, we'll have to use the theory of large deflection.

As is mentioned above, we'll discuss in this paper the deflection of the central line of a coiled spring, that is, the central line will be treated as an elastic bar with a certain rigidity. Here we should note that our spring has several characteristics as follows.

(1) As the pitch varies from one point to another along its central line, we must consider the spring as a bar with a gradually varying rigidity.

(2) The total length of a spring is not a constant, but a variable.

(3) The curvature of its central line caused by a varying shear is comparable to the one caused by bending and this curvature cannot be neglected.

As for the above notion of shearing deflection, we must remember two different methods:

(a) to find the shearing force by the slope of inflection curve; and

(b) to find it by the rotational angle of the cross section.

By the definition of shearing strains, the method (b) is more precise than that of (a), but the analysis on method (a) is easy and its results are on the safety-side. These factors are worth noticing. However, method (a) leads to a noticeable error in the case of shearing effect is large. Springs in practical use are largely affected by shearing effects, so we will adopt the method (b) this time. In the previous report, we have analysed the problem by the method (a). Fig. 4 shows both theoretical curves in order to compare with each other.

2. Post-Buckling Deformation of Coiled Springs

2. 1 *Fundamental Equations*

We consider the coiled spring with uniform pitches h_0 and with straight central

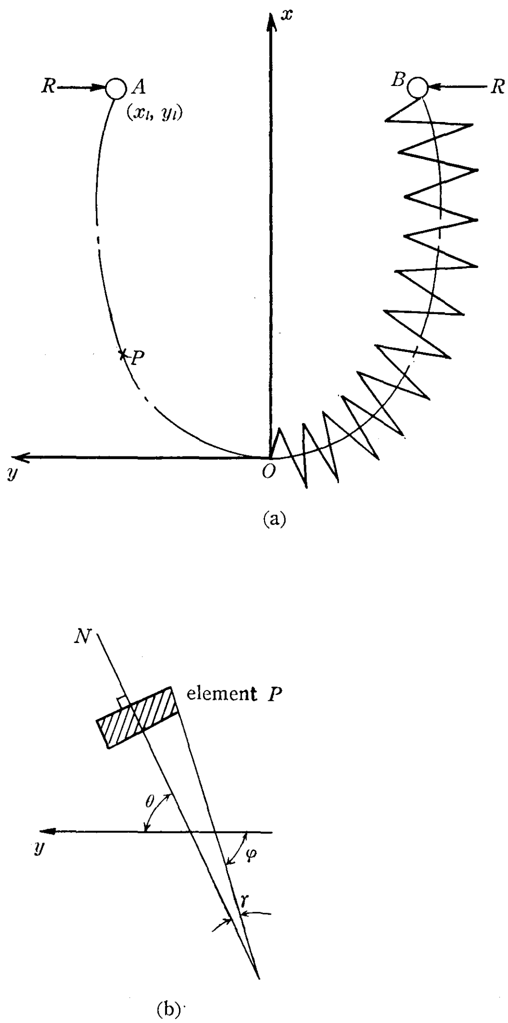

Fig. 1.

line in its undeformed state, buckled under the pressure R acting on both hinged ends (Fig. 1–a). The deformed curve is symmetrically related to a vertical line through its central point O i.e. x-axis. Then let point O be the origin and the tangent at O parallel to the force R be the y-axis. Fig. 1–b shows an element P of the elastic prismatic bar replacing the spring, in a deformed state. If θ denotes the angle through which the original normal section has turned and γ the shearing angle, then the total deformation of the element

$$\varphi=\theta+\gamma$$

Further, let

l_0=the total length of the unloaded spring,

l =the total length of the loaded spring,
h_0=the pitch at P under no load,
h =the pitch at P under load R,
S =the length $0P$ along the deflection curve of the central line of the spring,
S_0=the length in the unloaded state.

Now, let T, Q, M denote the axial force, the shearing force and the bending moment respectively at the element P and let tension be positive. Then we get

$$\left.\begin{aligned} &T=-R\cos\theta, \qquad Q=R\sin\theta, \\ &M=R\int_s^{l/2}\sin\varphi\, ds. \end{aligned}\right\} \qquad \cdots\cdots(2\cdot a)$$

Denoting by A, B, C the rigidities for the axial force, the flexure and the shear of the spring under load R, and by A_0, B_0, C_0 their unloaded values respectively,

$$\frac{T}{A_0}=\frac{ds-ds_0}{ds_0}=\frac{h-h_0}{h_0}$$

or

$$\frac{h}{h_0}=1+\frac{T}{A_0}=1-\frac{R\cos\theta}{A_0}, \qquad \cdots\cdots(2\cdot b)$$

so

$$\left.\begin{aligned} A&=A_0\frac{h}{h_0}=A_0\left(1-\frac{R}{A_0}\cos\theta\right), \\ B&=B_0\frac{h}{h_0}=B_0\left(1-\frac{R}{A_0}\cos\theta\right), \\ C&=C_0\frac{h}{h_0}=C_0\left(1-\frac{R}{A_0}\cos\theta\right). \end{aligned}\right\} \qquad \cdots\cdots(2\cdot c)$$

On the other hand, the shearing strain is expressed in the following equation:

$$\gamma=\frac{Q}{C}=\frac{R\sin\theta}{C_0\left(1-\frac{R}{A_0}\cos\theta\right)},$$

so we get

$$\varphi=\theta+\gamma=\theta+\frac{R\sin\theta}{C_0\left(1-\frac{R}{A_0}\cos\theta\right)}. \qquad \cdots\cdots(2\cdot 1)$$

And we find that the curvature by the bending moment is given by

$$\frac{d\theta}{ds}=\frac{M}{B}=\frac{R\int_s^{l/2}\sin\varphi\, ds}{B_0\left(1-\frac{R}{A_0}\cos\theta\right)}. \qquad \cdots\cdots(2\cdot 2)$$

Eq. (2·1) and Eq. (2·2) are the foundamental equations of the deflection curve. Then we introduce non-dimensional variables to make the calculations convenient hereafter:

$$\left.\begin{aligned} &\sigma=s/l\,, &&\lambda=R/A_0\,, \\ &\mu=l^2R/B_0\,, &&\nu=R/C_0\,. \end{aligned}\right\} \quad \cdots\cdots(2\cdot\mathrm{d})$$

With these variables, we rewrite the fundamental equations as

$$\varphi=\theta+\frac{\nu\sin\theta}{1-\lambda\cos\theta} \quad \cdots\cdots(2\cdot 1)'$$

$$\frac{d\theta}{d\sigma}=\frac{\mu\int_\sigma^{1/2}\sin\varphi\, d\sigma}{1-\lambda\cos\theta} \quad \cdots\cdots(2\cdot 2)'$$

and express Eq. (2·2)′ in a differential form

$$\frac{d}{d\sigma}\left\{(1-\lambda\cos\theta)\frac{d\theta}{d\sigma}\right\}=-\mu\sin\varphi \cdots\cdots(2\cdot 2)''$$

Substituting Eq. (2·1)′ into Eq. (2·2)″, we have

$$\frac{d}{d\sigma}\left\{(1-\lambda\cos\theta)\frac{d\theta}{d\sigma}\right\}=-\mu\sin\left(\theta+\frac{\nu\sin\theta}{1-\lambda\cos\theta}\right)$$

or

$$\left\{(1-\lambda\cos\theta)\frac{d\theta}{d\sigma}\right\}\frac{d}{d\sigma}\left\{(1-\lambda\cos\theta)\frac{d\theta}{d\sigma}\right\}d\sigma$$

$$=-\sin\left(\theta+\frac{\nu\sin\theta}{1-\lambda\cos\theta}\right)(1-\lambda\cos\theta)\,d\theta \quad \cdots\cdots(2\cdot 3)$$

Following Taylor's procedure, we work out all the trigonometrical expressions in the right-hand side of Eq. (2·3). Then

$$\text{the right-hand side } =-\mu(p\theta+q\theta^3),$$

where

$$\left.\begin{aligned} &p=1-\lambda+\nu\,, \\ &q=\frac{1}{6}\left\{p-3(\lambda-\nu)+\frac{3\nu^2}{1-\lambda}+\frac{\nu^3}{(1-\lambda)^2}\right\}. \end{aligned}\right\} \quad \cdots\cdots(2\cdot\mathrm{e})$$

Integrating Eq. (2·3), we get

$$(1-\lambda\cos\theta)^2\left(\frac{d\theta}{d\sigma}\right)^2 \doteqdot -2\mu\int(p\theta-q\theta^3)\,d\theta$$

$$=-\mu\left(p\theta^2-\frac{1}{2}q\theta^4\right)+H,$$

Where, H is an integration constant. If α denotes the angle θ at the point A, we have

$$H=\mu\left(p\alpha^2-\frac{1}{2}q\alpha^4\right).$$

So we get

$$(1-\lambda\cos\theta)\left(\frac{d\theta}{d\sigma}\right)\doteqdot\sqrt{\mu\left\{p(\alpha^2-\theta^2)-\frac{1}{2}q(\alpha^4-\theta^4)\right\}}.$$

In the root of the above equation, the terms θ^5 and higher powers are neglected, then we must expand the left-hand side up to this order.

$$\sqrt{\mu}d\sigma=\frac{(1-\lambda)+\dfrac{1}{2}\lambda\theta^2}{\sqrt{p(\alpha^2-\theta^2)-\dfrac{1}{2}q(\alpha^4-\theta^4)}}d\theta \quad \cdots\cdots(2\cdot 4)$$

Actually the above equation contains all terms up to order θ^2, neglecting higher powers of θ. Remembering $d\sigma$ is the non-dimensional value of ds, we find Eq. (2·4) the differential equation expressed by natural coordinate, that is, the fundamental equation for the coiled springs with an arbitrary wire section.

2·2. *Coiled Spring with a Circular Wire Section*

Since the wire section of coiled springs in practical use are circular, we continue to discuss a spring with a circular section. The case of a wire with a cross section other than circular, e.g. rectangular, could be treated by the similar method. The rigidities in the unloaded state are given as follows:[2)]

$$\left.\begin{aligned} A_0&=\frac{GIh_0}{\pi r^3}, \qquad B_0=\frac{2EGIh_0}{\pi r(E+2G)},\\ C_0&=\frac{EIh_0}{\pi r^3},\end{aligned}\right\} \quad \cdots\cdots(2\cdot\mathrm{f})$$

where,

E=Young's modulus of elasticity,

G=the shear modulus,

I=the moment of inertia of the cross sectional area of a wire,

r=mean radious of the coil.

$$\left.\begin{aligned} \mu&=\frac{1+2(G/E)}{2}\lambda\left(\frac{l}{r}\right)^2,\\ \nu&=(G/E)\lambda.\end{aligned}\right\} \quad \cdots\cdots(2\cdot\mathrm{f})'$$

These formulae (2·f)′ express the dependence of μ and ν on λ.

[2·2·1] Relations with Ordinary Formulae

If we transform Eq. (2·4) as follows,

$$\sqrt{2q\mu}d\sigma=\{2(1-\lambda)+\lambda\beta^2\}\frac{d\theta}{\sqrt{(\beta^2-\alpha^2)(\alpha^2-\theta^2)}}$$

$$-\lambda\sqrt{\frac{\beta^2-\theta^2}{\alpha^2-\theta^2}}\,d\theta\,,$$

where,

$$\beta^2=2p/q-\alpha^2 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(2\cdot g)$$

the right-hand side has only the terms expressed in an elliptic function. Then let us integrate this from point O to point A.

$$\sqrt{\mu}=\sqrt{\frac{2}{q}}\left[\frac{2(1-\lambda)}{\beta}F\left(\frac{\alpha}{\beta},\ \frac{\pi}{2}\right)+\lambda\beta\left\{F\left(\frac{\alpha}{\beta},\ \frac{\pi}{2}\right)-E\left(\frac{\alpha}{\beta},\ \frac{\pi}{2}\right)\right\}\right],\cdots\cdots(2\cdot5)$$

where $F(\alpha/\beta,\ \pi/2)$ and $E(\alpha/\beta,\ \pi/2)$ are complete elliptic integrals of the first and the second kind respectively.

Now we consider $\alpha=0$ in Eq. (2·5). In such a state the spring is not buckled yet, so the equation should coincide with the corresponding formula based on the theory of finite deformation. In fact, when $\alpha=0$, $\beta^2=2p/q$ and $F(0,\ \pi/2)=E(0,\ \pi/2)=\pi/2$, then, from Eq. (2·5), we get

$$\sqrt{\mu_{cr.}}=\frac{1-\lambda_{cr.}}{\sqrt{q_{cr.}}}\pi\,. \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(2\cdot6)$$

The quantities with suffices cr. denote the values under buckling load R. Substituting Eq. (2·d) in Eq. (2·6), we have

$$\frac{R_{cr.}l_{cr.}^2}{B_0}=\frac{\left(1-\dfrac{R_{cr.}}{A_0}\right)^2\pi^2}{1-\dfrac{R_{cr.}}{A_0}+\dfrac{R_{cr.}}{C_0}}\,.$$

And, as the central line is straight under $R_{cr.}$

$$l_{cr.}=(1-\lambda_{cr.})l_0\,.$$

Therefore, we get

$$\frac{R_{cr.}}{A_0}=\frac{1-\sqrt{1-\dfrac{4\pi^2B_0}{l_0^2A_0}\left(1-\dfrac{A_0}{C_0}\right)}}{2\left(1-\dfrac{A_0}{C_0}\right)}\,. \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(2\cdot6)'$$

Eq. (2·6)′ is no other than that of the ordinary theory of coiled springs[3] under buckling load. From Eq. (2·b) and Eq. (2·4),

$$\frac{l_0}{2}=\int_0^{l/2}\frac{ds}{1-\lambda\cos\theta}=\frac{l}{\sqrt{\mu}}\int_0^{\alpha}\frac{d\theta}{\sqrt{p(\alpha^2-\theta^2)-\frac{1}{2}q(\alpha^4-\theta^4)}}$$

$$=\frac{l}{\beta}\sqrt{\frac{2}{\mu q}}F\left(\frac{\alpha}{\beta},\ \frac{\pi}{2}\right). \quad \cdots\cdots(2\cdot7)$$

Further, substituting Eq. (2·g) into this, we get

$$\frac{l_0}{r}=\frac{4}{\beta\sqrt{\lambda q\{1+2(G/E)\}}}F\left(\frac{\alpha}{\beta},\ \frac{\pi}{2}\right). \quad \cdots\cdots(2\cdot7)'$$

Let $\alpha=0$ in Eq. (2·7), then

$$\frac{l_0}{l}=\frac{1}{1-\lambda}. \quad \cdots\cdots(2\cdot8)$$

And let

δ_y=the deflection of the spring before buckling,

n=number of turns$=l_0/h_0$,

d=the diameter of the wire,

then, Eq. (2·8) can be written as follows:

$$\delta_y=l_0-l=\lambda l_0=\frac{64nr^3}{d^4G}R \quad \cdots\cdots(2\cdot9)$$

which coincides with the ordinary formula of coiled springs.

[2·2·2] The Coordinates of Point $A\,(x_l,\ y_l)$

Let us multiply Eq. (2·4) by $\sin\varphi$, and integrate the result from point O to point A.

$$\sqrt{\mu}\int_0^{1/2}\sin\varphi\,d\sigma\doteqdot p\int_0^{\alpha}\frac{\theta\,d\theta}{\sqrt{p(\alpha^2-\theta^2)-\frac{1}{2}(\alpha^4-\theta^4)}}.$$

Geometrically considering, we find

$$\text{the left-hand side }\left(\int_0^{1/2}\sin\varphi d\sigma\right)=x_l/l$$

so we can rewrite the above equation as follows;

$$\sqrt{\mu}\,\frac{x_l}{l}=\sqrt{\frac{2}{q}}\,p\ln\frac{\alpha+\sqrt{\alpha^2+2(p/q-\alpha^2)}}{\sqrt{2(p/q-\alpha^2)}}.$$

Substituting Eq. (2·g) in this

$$\frac{x_l}{l_0}=\frac{2p}{\sqrt{\{1+2(G/E)\}\lambda q}}\left(\frac{r}{l_0}\right)\ln\frac{\alpha+\sqrt{\alpha^2+2(p/q-\alpha^2)}}{\sqrt{2(p/q-\alpha^2)}}. \quad \cdots\cdots(2\cdot10)$$

Thus, we have the x-compornent of the point A expressed in a non-dimensional form. Similarly, let us find y_l, multiplying Eq. (2·4) by $\cos\varphi$,

$$\sqrt{\mu}(\cos\varphi\, d\sigma)=\frac{(1-\lambda\cos\theta)\cos\varphi\, d\theta}{\sqrt{p(\alpha^2-\theta^2)-\frac{1}{2}q(\alpha^4-\theta^4)}}$$

$$\doteqdot\sqrt{\frac{2}{q}}\{(1-\lambda)-t\beta^2\}\frac{d\theta}{\sqrt{(\beta^2-\theta^2)(\alpha^2-\theta^2)}}\quad+\sqrt{\frac{2}{q}}t\sqrt{\frac{\beta^2-\theta^2}{\alpha^2-\theta^2}}d\theta,$$

where,

$$t=\frac{1}{2}\left\{\left(1-\frac{\nu}{1-\lambda}\right)^2-\lambda\right\}, \quad\cdots\cdots(2\cdot\mathrm{h})$$

and integrating this from point O to the point A, we get

$$\frac{y_l}{l_0}=\frac{2}{\sqrt{\{1+2(G/E)\}\lambda q}}\left(\frac{r}{l_0}\right)\left[\left(\frac{1-\lambda}{\beta}-t\beta\right)F\left(\frac{\alpha}{\beta},\ \frac{\pi}{2}\right)+t\beta E\left(\frac{\alpha}{\beta},\ \frac{\pi}{2}\right)\right].\cdots\cdots(2\cdot11)$$

Further, substituting Eq. (2·7)′ into Eq. (2·11), we have the y-compornent.

$$\frac{y_l}{l_0}=\frac{1-\lambda-t\beta^2}{2\beta}+\frac{2t\beta}{\sqrt{\{1+2(G/E)\}\lambda q}}\left(\frac{r}{l_0}\right)E\left(\frac{\alpha}{\beta}.\ \frac{r}{2}\right).\quad\cdots\cdots(2\cdot11)'$$

[2·2·3] Numerical Calculations

Eq. (2·10) and Eq. (2·11)′ are expressed by the material constant of the spring (G/E); its dimension (l_0/r); the non-dimensional value of load (λ); and the angle θ at the point $A(\alpha)$. Other values p, q and t are determined by the above four variables: Eq. (2·e) and Eq. (2·h) determine p, q and t by (λ) and ν and further, Eq. (2·f) determines ν by (G/E) and (λ). Thus we find p, q and t the dependent variables.

Since Eq. (2·7)′ expresses the dependency of (l_0/r) on (λ), (G/E) and (α), we can extinguish one out of them. This has the following simple physical meaning: the deflection (α) of a certain spring $((G/E), (l_0/r))$ must be determined by the load (λ). Therefore, we can take (α) as a dependent variables.

Let us work out complete elliptic integral of the first kind.

$$F\left(\frac{\alpha}{\beta},\ \frac{\pi}{2}\right)\doteqdot\frac{\pi}{2}\left\{1+\frac{1}{4}\left(\frac{\alpha}{\beta}\right)^2\right\}.$$

Substitution of this relation and Eq. (2·g) into Eq. (2·7) gives

$$J\sqrt{2(p/q)-\alpha^2}=\frac{\pi}{2}+\frac{\pi}{8}\ \frac{\alpha^2}{2(p/q)-\alpha^2},$$

where,

$$J=\frac{\sqrt{\{1+2(G/E)\}pq}}{4}\left(\frac{l_0}{r}\right),\quad\cdots\cdots(2\cdot\mathrm{i})$$

or

$$\alpha^2=\frac{(2p/q)\left\{(2p/q)J^2-\dfrac{\pi^2}{4}\right\}}{2J^2(2p/q)-\dfrac{\pi^2}{8}}. \qquad \cdots\cdots(2\cdot12)$$

From Eq. (2·12), the value of α for each (G/E), (l_0/r) and (λ) is determined. Now, we have prepared all relations necessary for the following numerical calculations.

If we assume that $(E/G)=2.6$*, then Eq. (2·1) and Eq. (2·12) give

$$J=0.11\,\lambda q\left(\frac{l_0}{r}\right)^2 \qquad \cdots\cdots(2\cdot\mathrm{i})'$$

$$\alpha^2=\frac{0.44p^2(\lambda/q)(l_0/r)^2-p\pi^2/(2q)}{0.44p\lambda(l_0/r)-r^2/8} \qquad \cdots\cdots(2\cdot12)'$$

and Eq. (2·6)′ gives

$$\lambda_{\mathrm{cr.}}=0.8125\{1-\sqrt{1-27.46(r/l_0)^2}\} \qquad \cdots\cdots(2\cdot6)''$$

Now introducing a new variable

$$K=\frac{\lambda}{\lambda_{\mathrm{cr.}}}\left(=\frac{R}{R_{\mathrm{cr.}}}\right), \qquad \cdots\cdots(2\cdot\mathrm{j})$$

we can express λ with K and (l_0/r). From Eq. (2·9), it is distinct that the longitudinal deflection δ_y is proportional to the load K for $K\leq1$. But the equation

$$\delta_y=l_0-2y_l$$

holds after buckling. So we substitute Eq. (2·11)′ into this equation, and find (δ_y/l_0) instead of (y_l/l_0).

On the other hand, (x_l/l_0) is given by Eq. (2·10). Fig. 2 shows the culculated results of them. In this (x_l/l_0) are represented by the family of curves which intersect the left ordinate at $K=1$ (the central line of the spring for $K\leq1$ is a straight line) and (δ_y/l_0) are represented by the family of curves which intersect the right ordinate at $K=0$.

N. B. the origin of (δ_y/l_0) will be at the right corner as is shown in the bracket under the abscissa.

[2·2·4] A Comparison of the Calculated Results and the Experiments

Fig. 2 shows the relations between (x_l/l_0) and (δ_y/l_0) with parameter K. These relations allow a simplification of the experiments: we do not have to measure the

* this means Poisson's number $m=\frac{10}{3}$.

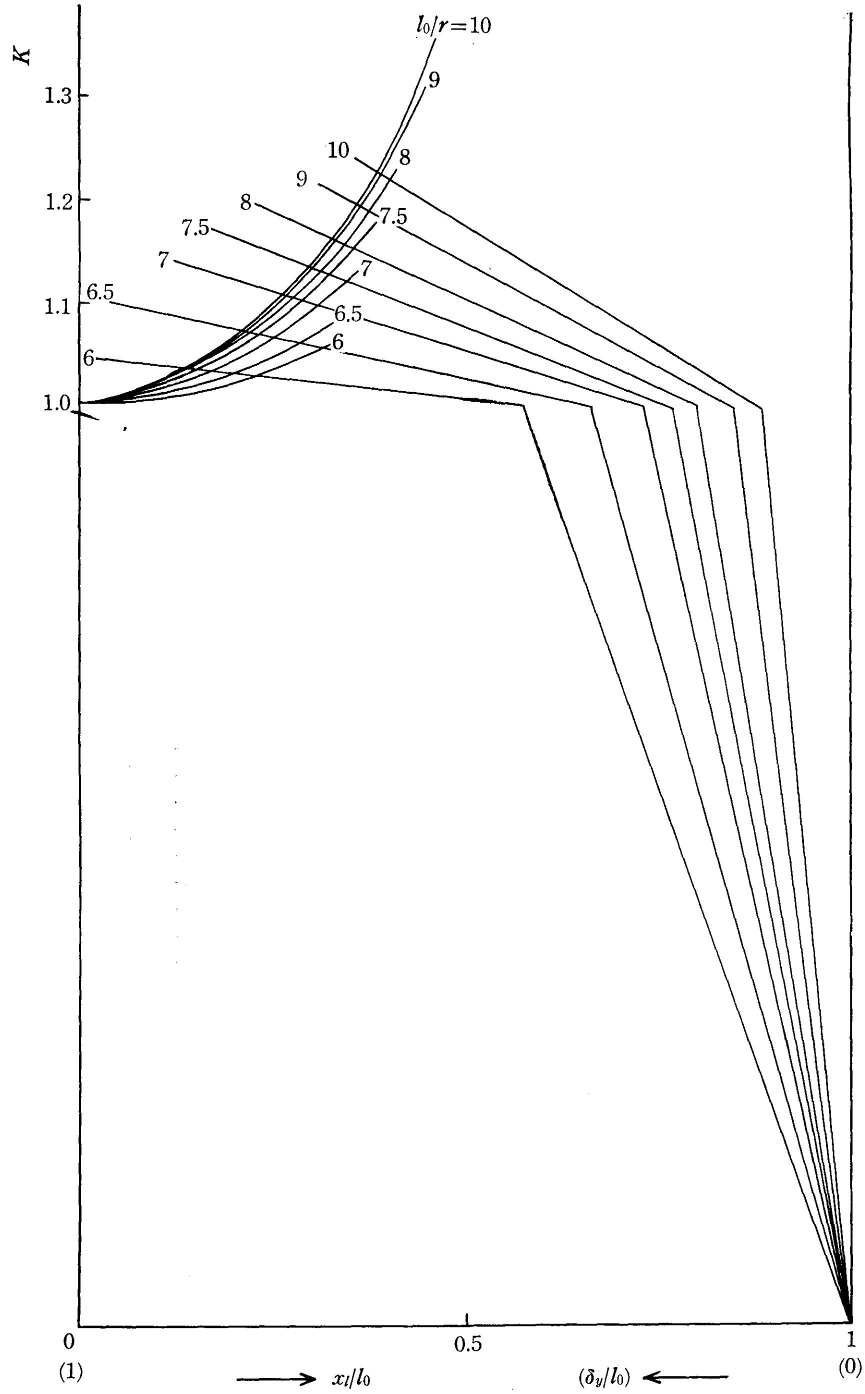
K
1.3
1.2
1.1
1.0
$l_0/r=10$
9
8
7.5
7
6.5
6
10
9
8
7.5
7
6.5
6
0
(1)
0.5
1
(0)
x_l/l_0
(δ_y/l_0)

Fig. 2.

loads, and to worry about suitability of the capacity of the test machine to the spring.

As the properties of materials except (E/G) do not clearly affect the measured results of this experiment, these were not researched in detail. So we have used springs made of ordinary steel wire in order to compare with the theoretical results in which $(E/G)=2.6$. And in the dimensions of the springs, the only quantity we need is (l_0/r). As we have examined the springs with $r=1$ mm, the length l_0 was calculated by the corresponding value of (l_0/r) in theoretical results.

Fig. 3 shows the appearance of the testing device There $C_{1,2}$ are metallic struts fixing the ends of a spring in an exact direction of its central line, and their cylindrical surface has a spiral groove into which springs are screwed. P is a plate on

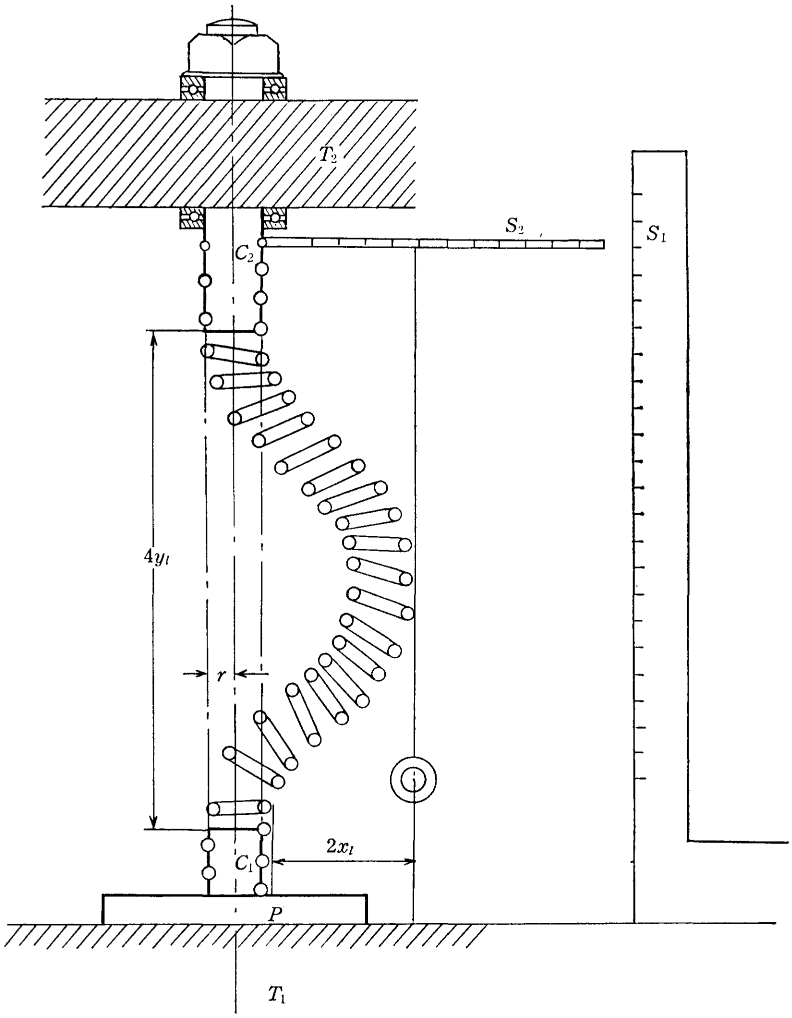

Fig. 3.

which C_1 is fixed, to protect against the movement of C_1 when the spring is buckled. C_2 is fixed to the table T_2 by a nut. And bearings on both sides of T_2 are put in order to set springs free from twisting.

The experiment was performed with the following steps. We have attached springs to $C_{1,2}$ and set the horizontal free length $2l_0$ by adjusting the length the screwing-into $C_{1,2}$. If we shorten the distance between the two tables by driving the testing machine*, the spring will be buckled at a certain position. We stop the

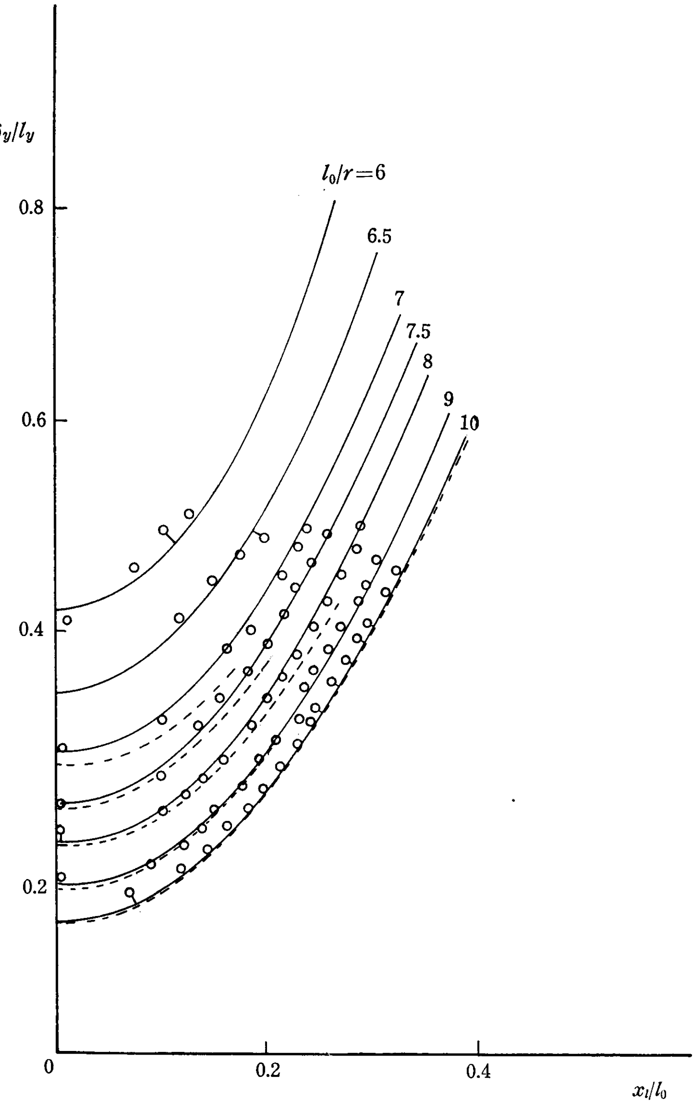

Fig. 4.

* Shimazu's universal testing machine with the capacity of 2 tons.

machine at certain proper positions after the buckling point. As a string hangs down from the scale S_2, we move it the point where it touches the contour of the spring and measure the distance $2x_l$. And with another scale S_1, we measure the distance between T_1 and T_2. We repeat this process several times so far as the coils of the spring do not touch each other. The experimental results obtained as described above are plotted in Fig. 4. The curves in this figure are theoretical results obtained from Fig. 2. The coincidence of the experimental results with the theoretical ones, as is seen from Fig. 4, is said to be satisfactory. The broken lines in Fig. 4 show the theoretical results[1] obtained by the other method (a) concerning shearing force (see Chap. 1). Fig. 4 shows that the less the value (l_0/r), the more the discrepancy between two lines. This indicates that a stumpy spring is largely affected by the shearing effect. From Eq. (2·9), the intersections of the curves and the ordinate in Fig. 4 are the values $(\lambda_{cr.})$.

Fig. 5 shows the relations between $(\lambda_{cr.})$ and (l_0/r). In this figure (a) and (b) are the same as used in Chap. 1. N. B. the curve (b) is Eq. (2·6)'' itself.

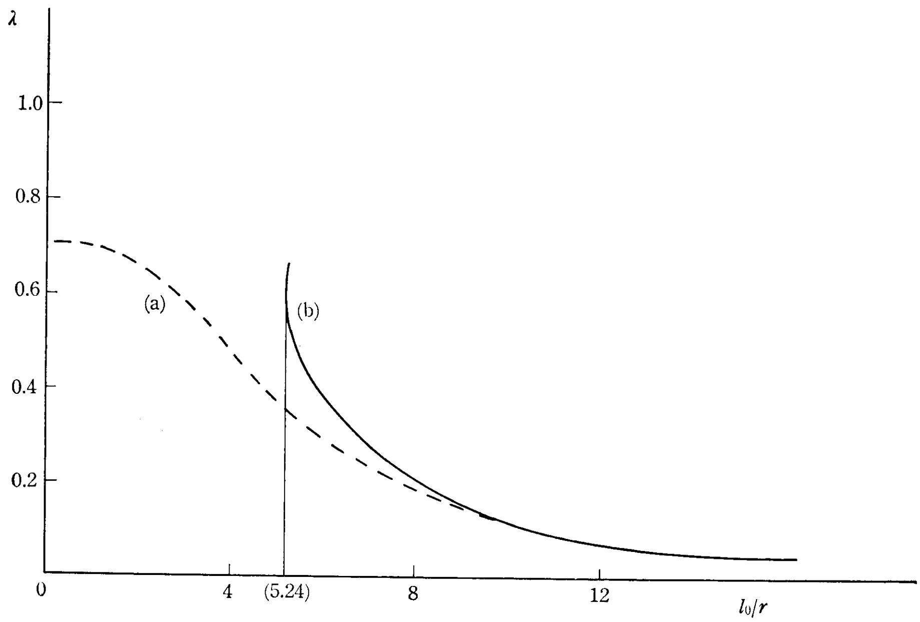

Fig. 5.

3. Deflection under Oblique Loads

3·1 *Translation and Rotation of Coordinates*

We consider a coiled spring with one end fixed at O, deflected under an oblique load R acting on the other end A (Fig. 6·a). Then let the central line of the spring in the unloaded state be y-axis, and the line perpendicular to it be x-axis. Further,

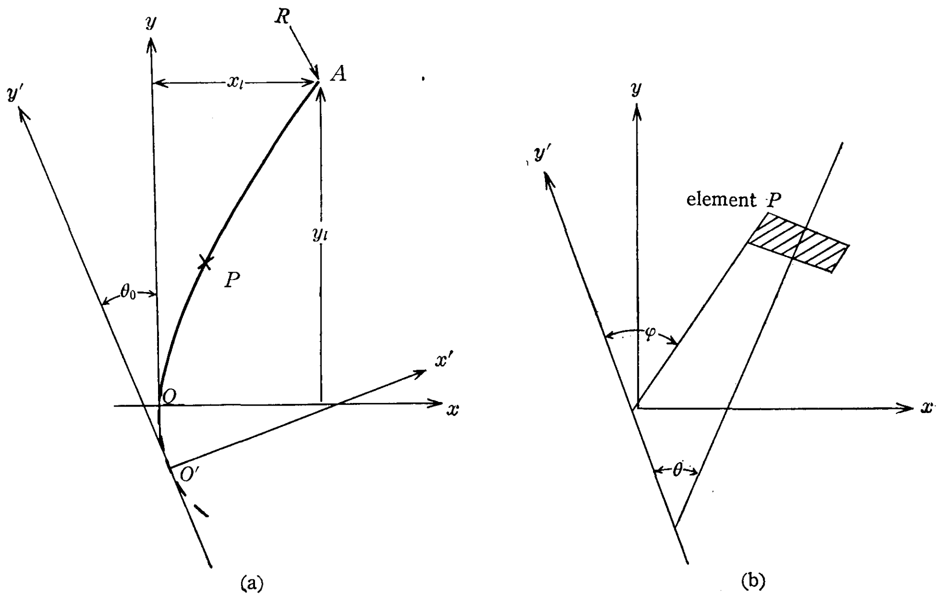

Fig. 6.

let θ_0 denote the angle of the acting direction of the load R. It should be noted that the tangent of the deflection curve at point O does not coincide with the y-axis. The discrepancy between them is caused by the shearing strain at O. Then we extend the deflection curve as a broken line shows, and draw another line tangent to the curve and parallel to the load R. Then we take this line as y'-axis and the line perpendicular to it as x'-axis. Now we obtain the same deflection curve as that of Fig. 1·a. As we have discussed this problem in Chap. 2, we can use those results obtained in order to solve the oblique load problem.

3·2. *Induction of Equations*

For simplicity, let the cross section of wire be circular. From Eq. (2·4), we have

$$s_0=\int_\theta^s \frac{ds}{1-\lambda\cos\theta}=\frac{l}{\sqrt{\mu}}\int_{\theta_0}^{\theta}\frac{1}{\sqrt{p(\alpha^2-\theta^2)-\dfrac{1}{2}q(\alpha^4-\theta^4)}}d\theta.$$

Integrating from point O to point A, we get

$$\frac{l_0}{2}=\sqrt{\frac{2}{\mu q}}\,l\int_{\theta_0}^{\alpha}\frac{1}{\sqrt{(\alpha^2-\theta^2)(\beta^2-\theta^2)}}d\theta$$

$$=\frac{l}{\beta}\sqrt{\frac{2}{\mu q}}\left\{F\left(\frac{\alpha}{\beta},\ \frac{\pi}{2}\right)-F\left(\frac{\alpha}{\beta},\ \sin^{-1}\frac{\theta_0}{\alpha}\right)\right\}, \quad \cdots\cdots\cdots\cdots\cdots(3\cdot1)$$

where

$$\beta^2=2(p/q)-\alpha^2. \quad \cdots\cdots (2\cdot g)$$

Substituting Eq. (2·f)′ into Eq. (3·1), we find

$$\frac{l_0}{r}=\frac{4}{\beta\sqrt{\lambda q\{(G/E)+1\}}}\left\{F\left(\frac{\alpha}{\beta},\ \frac{\pi}{2}\right)-F\left(\frac{\alpha}{\beta},\ \sin^{-1}\frac{\theta_0}{\alpha}\right)\right\}$$

or

$$\frac{1}{\beta}\left\{F\left(\frac{\alpha}{\beta},\ \frac{\pi}{2}\right)-F\left(\frac{\alpha}{\beta},\ \sin^{-1}\frac{\theta_0}{\alpha}\right)\right\}$$

$$=\frac{1}{4}\sqrt{\lambda q\{1+2(G/E)\}}\left(\frac{l_0}{r}\right). \quad \cdots\cdots (3\cdot 2)$$

Next, let us multiply Eq. (2·4) by $l\sin(\varphi-\theta_0)$,

$$dx=\frac{\left\{(1-\lambda)+\frac{1}{2}\lambda\theta^2\right\}l}{\sqrt{\mu\left\{(\alpha^2-\theta^2)-\frac{1}{2}q(\alpha^4-\theta^4)\right\}}}\left(\frac{p\theta}{1-\lambda}-\theta_0\right)d\theta$$

and integrate it from O to P, then we get

$$x=\sqrt{\frac{2}{\mu p}}\,l\int_{\theta_0}^{\theta}\frac{p\theta-(1-\lambda)\theta_0}{\sqrt{(\alpha^2-\theta^2)(\beta^2-\theta^2)}}\,d\theta. \quad \cdots\cdots (3\cdot 3)$$

If we take α as the upper limit of the integration, we have

$$x_l=\sqrt{\frac{2}{\mu q}}\,l\left\{\int_{\theta_0}^{\alpha}\frac{p\theta\, d\theta}{\sqrt{(\alpha^2-\theta^2)(\beta^2-\theta^2)}}\right.$$

$$\left.-(1-\lambda)\theta_0\int_{\theta_0}^{\alpha}\frac{d\theta}{\sqrt{(\alpha^2-\theta^2)(\beta^2-\theta^2)}}\right\}$$

$$=\sqrt{\frac{2}{\mu q}}\,l\left[p\ln\frac{\sqrt{\alpha^2-\theta_0^2}+\sqrt{\beta^2-\theta_0^2}}{\sqrt{\beta^2-\alpha^2}}\right.$$

$$\left.-\frac{(1-\lambda)\theta_0}{\beta}\left\{F\left(\frac{\alpha}{\beta},\ \frac{\pi}{2}\right)-F\left(\frac{\alpha}{\beta},\ \sin^{-1}\frac{\theta_0}{\alpha}\right)\right\}\right].$$

Substituting Eq. (2·f)′ into this, we get

$$\frac{x_l}{l_0}=\frac{2p}{\sqrt{\lambda q\{1+2(G/E)\}}}\left(\frac{r}{l_0}\right)\ln\frac{\sqrt{\alpha^2-\theta_0^2}+\sqrt{\beta^2-\theta_0^2}}{\sqrt{\beta^2-\alpha^2}}-\frac{(1-\lambda)\theta_0}{2}.$$

$$\cdots\cdots (3\cdot 4)$$

Eq. (3·4) is the x-compornent of the point A.

Similarly, we find y_l, multiplying Eq. (2·4) by $l\cos(\varphi-\theta_0)$ and integrating from $\theta=\theta_0$ to $\theta=\alpha$:

$$y_l=\frac{l}{\sqrt{\mu}}\int_{\theta_0}^{\alpha}\frac{(1-\lambda)+\frac{1}{2}\left\{\lambda-\left(1+\frac{\nu}{1-\lambda}\right)^2\right\}\theta^2}{\sqrt{p(\alpha^2-\theta^2)-\frac{1}{2}q(\alpha^4-\theta^4)}}d\theta$$

$$+\frac{l}{\sqrt{\mu}}p\theta_0\int_{\theta_0}^{\alpha}\frac{\theta}{\sqrt{p(\alpha^2-\theta^2)-\frac{1}{2}q(\alpha^4-\theta^4)}}d\theta$$

$$-\frac{l\theta_0^2}{2\sqrt{\mu}}(1-\lambda)\int_{\theta_0}^{\alpha}\frac{1}{\sqrt{p(\alpha^2-\theta^2)-\frac{1}{2}q(\alpha^4-\theta^4)}}d\theta$$

$$=\sqrt{\frac{2}{\mu q}}l\left[\left(\frac{1-\lambda}{\beta}+t\beta\right)\left\{F\left(\frac{\alpha}{\beta},\ \frac{\pi}{2}\right)-F\left(\frac{\alpha}{\beta},\ \sin^{-1}\frac{\theta_0}{\alpha}\right)\right\}\right.$$

$$+t\beta\left\{E\left(\frac{\alpha}{\beta},\ \frac{\pi}{2}\right)-E\left(\frac{\alpha}{\beta},\ \sin^{-1}\frac{\theta_0}{\alpha}\right)\right\}$$

$$+p\theta_0\ln\frac{\sqrt{\alpha^2-\theta_0^2}+\sqrt{\beta^2-\theta_0^2}}{\sqrt{\beta^2-\alpha^2}}$$

$$\left.-\frac{\theta_0^2(1-\lambda)}{2\beta}\left\{F\left(\frac{\alpha}{\beta},\ \frac{\pi}{2}\right)-F\left(\frac{\alpha}{\beta},\ \sin^{-1}\frac{\theta_0}{\alpha}\right)\right\}\right],$$

further substituting Eq. (3·1) into this, we get

$$\frac{y_l}{l_0}=\frac{(1-\lambda)(1-\theta_0^2)-t\beta^2}{2}+\frac{2}{\sqrt{\lambda q\{1+2(G/E)\}}}\left[t\beta\left\{E\left(\frac{\alpha}{\beta},\ \frac{\pi}{2}\right)\right.\right.$$

$$\left.\left.-E\left(\frac{\alpha}{\beta},\ \sin^{-1}\frac{\theta_0}{\alpha}\right)\right\}-p\theta_0\ln\frac{\sqrt{\alpha^2-\theta_0^2}+\sqrt{\beta^2-\theta_0^2}}{\sqrt{\beta^2-\alpha^2}}\right]\left(\frac{r}{l_0}\right).\quad\cdots\cdots\cdots\cdots(3\cdot5)$$

Thus we obtain the coordinates of the point A.

Now, in the case $\theta_0=0$, Eq. (3·2), Eq. (3·4) and Eq. (3·5) should coincide with the corresponding equations in Chap. 2. The elementary culculations give the desired results:

Eq. $(3\cdot2)_{\theta_0=0}$=Eq. (2·7)$'$,
Eq. $(3\cdot4)_{\theta_0=0}$=Eq. (2·10),
Eq. $(3\cdot5)_{\theta_0=0}$=Eq. (2·11)$'$,

3·3. *Application to the Problem of Lateral Rigidity*

We will consider the spring compressed by parallel plates and bent by the

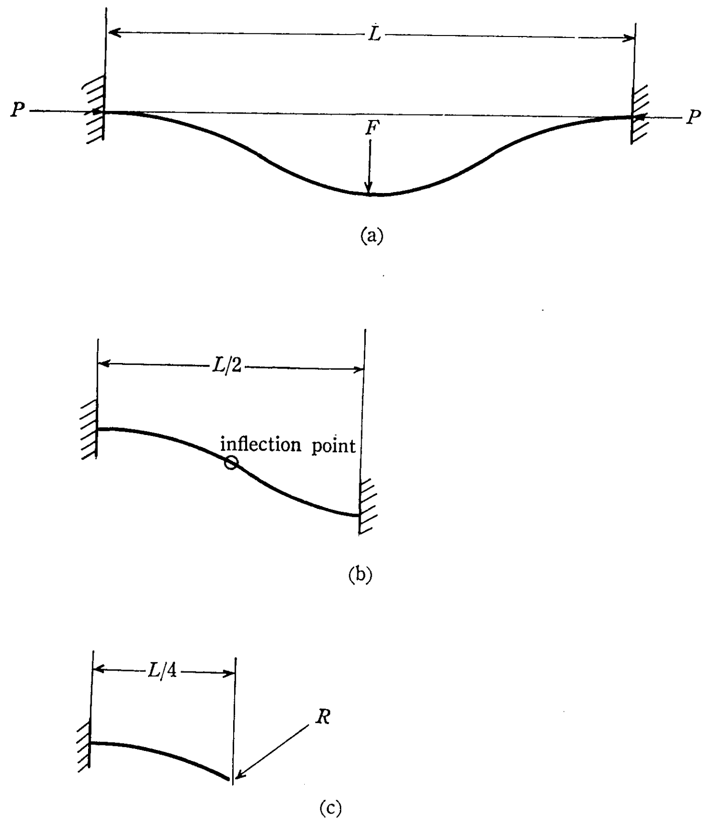

Fig. 7.

lateral load at its middle point (Fig. 7a). If we leave the distance between plates to be unchanged and vary the magnitude of F, then the horizontal load P should be varied. But as the current theory is based on the assumption that P=const., it is varid only in the case P varies little, in other words in the case of finite deflections. So it is of practical importance to apply our theory of large deformations to this problem.

Details will be reported in our next paper. So we will briefly sketch out its process here. The deflection curves (a), (b) and (c) in Fig. 7 are all the same, so the analysis of (a) is reduced to the problem of an oblique load (c). Then the additional equation under this condition is given as (y_l/l_0)=const. in Eq. (3·5).

4 Acknowledgement

We are debted to Mr. F. Murata and Mr. A. Ishiguchi for experiments and numerical calculations.

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