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Optimal Satisfactory Control: Formulation and Analysis by Lagrangian Technique

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Abstract

The optimal satisfactory control of systems subject to external disturance inputs is studied. The systems represented are essentially deterministic nonlinear, continuous-type processes which are normally operating in the steady-state. The objective is on-line static optimization according to an economic performance criterion.

The optimal satisfactory control is formulated by combining concept of satisfactory control and concept of optimal control. In order to obtain the optimal satisfactory control, several theories and techniques of decision making under uncertainties are developed to obtain necessary conditions for maximizing system performance. The Lagrange multiplier techniques for game solving under inequality constraints are derived to solve the formulated optimal satisfactory control.

1. Introduction

An objective is the control of industrial processes, subject to external disturbances, to an optimal performance based on an economic criterion. An essential characteristic of the control engineering problem is the presence of uncertainty (disturbances) regarding the future operation of the control system. While many theories have been developed for the optimization of deterministic systems, the optimal control of processes subject to disturbances has been studied relatively little.

The control decision problem under disturbances is attacked using the concept of satisfactory control [1, 2]. Specifically, an optimal satisfactory approach is formulated.

The satisfactory control was proposed by Mesarovic et al. as one approach to decision making under uncertainty. In this approach, the control is determined not necessarily to maximize performance but to ensure that it always exceeds a specified minimum for all possible disturbances in a given disturbance set (satisfaction condition).

In the satisfaction approach, the problem is treated as follows. We know the

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present value of the disturbances and the range of values the disturbances may take on over the future time interval of concern. It is required to choose a control value such that the performance remains satisfactory, i.e., satisfies the satisfaction condition. Moreover, the most desirable one from the class of satisfactory controls is required. Under this requirement, the "optimal satisfactory control" is formulated so that maximization of the performance index is carried out by assigning a proper weight to the present value of the disturbance and treating the satisfaction condition as an inequality constraint.

In order to synthesize the optimal satisfactory control by applying a game-theoretic method and mathematical programming technique, some fundamental theorems are established. Among these is a result which allows us to solve the game under inequality constraints by application of Lagrange multipliers.

2. Formulation of the Problem

We restrict our attention to the control problem associated with continuous-type processes. The following assumptions are made [3]:

(i) the optimizing system is designed for normal operation of the process in the steady-state,

(ii) the process is subject to a variety of disturbances; however, the regulating control actions are capable of maintaining the process reasonably close to the specified steady-state when the desired state is determined by static optimization,

(iii) the average frequency with which the optimizing controller recalculates a new steady-state operating level is low relative to the inditial response speed of the process.

As a result of the above assumptions, the dynamic properties of the process response are neglected except with reference to the direct control function. Accordingly, the performance index and process model are assumed to be described by the following static relations:

$$P(t)=P_1(\underline{y}(t), \underline{m}(t), \underline{u}(t))$$

$$\underline{y}(t)=f(\underline{m}(t), \underline{u}(t))$$

where $\underline{y}(t)$ is an output vector, $\underline{m}(t)$ a controllable input vector and $\underline{u}(t)$ is a disturbance input vector.

The control $\underline{m}(t)$ will be updated periodically with the period T_0 so that the performance index P^* is maximized.

$$P^*=\int_0^T P_1(\underline{y}(t), \underline{m}(t), \underline{u}(t))\,dt=\int_0^T P(\underline{m}(t), \underline{u}(t))\,dt\,.$$

The control decision of $\underline{m}(t)=\underline{m}(nT_0)$, $nT_0 \leq t<(n+1)T_0$, $n=0, 1, \cdots,$ will be made based on the observation of $\underline{u}(nT_0)$, the disturbance value at $t=nT_0$. Since $\underline{u}(t)$ is subject to random variations over the interval $[nT_0, (n+1)T_0)$, the control decision which may be calculated from the static relations

$$\underline{y}=\underline{f}(\underline{m}, \underline{u}(nT_0)) \tag{1}$$

$$\max_{\underline{m}} P_1(\underline{y}, \underline{m}, \underline{u}(nT_0))=\max_{\underline{m}} P(\underline{m}, \underline{u}(nT_0)) \tag{2}$$

is not generally optimal over the interval $[nT_0, (n+1)T_0)$. Our purpose should rather be to determine the control $\underline{m}$ such that

$$P^*(nT_0)=\int_{nT_0}^{(n+1)T_0} P(\underline{m}, \underline{u}(t))dt \tag{3}$$

is maximized. But this is difficult since we do not know about the disturbance $\underline{u}(t)$, $nT_0 \leq t < (n+1)T_0$, beforehand.

In the present work we shall assume the existence of a subset $U^*(nT_0) \subset U$ (U is a domain of $\underline{u}$) such that $\underline{u}(t) \in U^*(nT_0)$, $\forall t \in [nT_0, (n+1)T_0)$, and that we can assign $U^*(nT_0)$ to the observation of $\underline{u}(nT_0)=u^*(nT_0)$. This assumption bounds the range of disturbances expected. With T_0 small, this range may be expected to be narrow. Then we have the same type of control problem in each interval $[nT_0, (n+1)T_0)$, $n=0, 1, 2, \cdots$.

Since the decision of the control $\underline{m}(nT_0)$ must apply over the future interval $[nT_0, (n+1)T_0)$ over which the disturbance is generally unknown, it is impossible to give the completely optimal control. The best we can expect is to make the performance loss as small as possible. In order to obtain a reasonable performance, concepts of the game control [4, 5] or of the satisfactory control [1, 2] have been proposed. One thing we can say is that if we implement the control policy which causes the process to be relatively insensitive to the disturbance level, then we can expect a reasonable performance when the disturbance changes are restricted to within a certain range.

It may be assumed that the rate of change of the disturbance level is bounded in most process systems. Hence, if the interval T_0 is properly chosen relative to the maximum rates of change of the disturbance inputs, it will be reasonable to assign a weight to the measured value of the disturbance $\underline{u}^*(nT_0)$ in determining the derived control $\underline{m}(nT_0)$.

Now we shall make use of the concept of satisfactory control proposed by Mesarovic by requiring that

$$P(\underline{m}^0, \underline{u}) \geq \alpha_s(U^*(nT_0)) \tag{4}$$

for all $\underline{u} \in U^*(nT_0)$, where $\alpha_s(U^*(nT_0))$ is a specified constant corresponding to $U^*(nT_0)$. But in addition to this, we shall also require that the performance index be maximum for the particular value $\underline{u}=\underline{u}^*(nT_0)$, i.e.,

$$\max_{\underline{m}} P(\underline{m}, \underline{u}^*(nT_0)) = P(\underline{m}^0, \underline{u}^*(nT_0)) \tag{5}$$

subject to Eq. (4). Therefore, our purose is to solve the optimal control under some satisfactory condition Eq. (4), giving weight to the measured disturbance $\underline{u}^*(nT_0)$. Relation (4) is used as a constraint in the optimization procedure, and from Eq. (5) it is seen that a large weight is given to $\underline{u}^*(nT_0)$ for which the maximization of the performance is required.

Fig. 1 illustrates the performance obtained with several arbitrary satisfactory controls fulfilling the requirement of Eq. (4). These are compared with the performance of the particular control, referred to as "*optimal satisfactory control*"

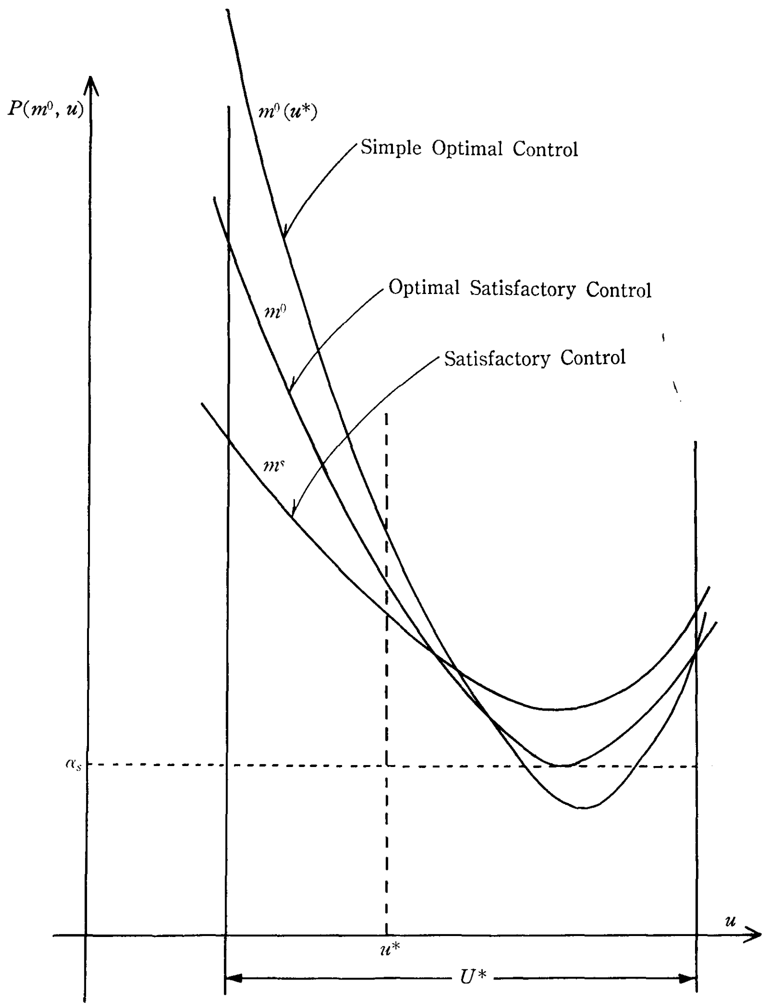


Fig. 1.

defined by the requirements of Eq. (4), (5).

For simplicity of notation, we wil denote $\underline{u}^*(nT_0)$, $U^*(nT_0)$ and $\alpha_s(U^*(nT_0))$ as $\underline{u}^*$, U^* and α_s, respectively, in the following development.

3. Analysis of the Formulated Problem, Part 1

The optimal satisfactory control was formulated as

$$\max_{\underline{m}} P(\underline{m}, \underline{u}^*) = P(\underline{m}^0, \underline{u}^*) \tag{5}$$

subject to the "*satisfaction condition*"

$$\forall \underline{u} \in U^*, \qquad P(\underline{m}, \underline{u}) - \alpha_s \geq 0 . \tag{6}$$

We assume that the property $\underline{u} \in U^*$ can be expressed as an inequality constraint $\underline{b}(\underline{u}) \geq \underline{0}$, for instance $(\underline{u} - \underline{u}^* + \underline{U}_L)(\underline{u}^* + \underline{U}_R - \underline{u}) \geq \underline{0}$, where $\underline{U}_L$, $\underline{U}_R$ are constants defining bounds of the disturbance set U^*. It must be noted that since $\forall$ is a quantifier, Eq. (6) cannot be simply written as

$$\underline{b}(\underline{u}) \geq \underline{0} \tag{7}$$

$$P(\underline{m}, \underline{u}) - \alpha_s \geq 0 \tag{8}$$

as the constraints for the maximization of Eq. (5).

For further analysis, however, let us suppose that $\underline{u}$ is acting to minimize the performance $P(\underline{m}, \underline{u})$ under constraints (7), (8). Then we have the *maximization and minimization problem which must be considered simultaneously*:

$$\begin{cases} \max_{\underline{m}} P(\underline{m}, \underline{u}^*) & (9) \\ \min_{\underline{u}} P(\underline{m}, \underline{u}) & (10) \end{cases}$$

subject to Eq. (7) and (8).

Now let us consider the (artificial) performance index

$$J(\underline{m}, \underline{u}; \gamma, \underline{u}^*) = \gamma P(\underline{m}, \underline{u}^*) + (1-\gamma) P(\underline{m}, \underline{u}), \qquad 0 \leq \gamma \leq 1 . \tag{11}$$

Using the extended performance index (11) we define the optimal satisfactory control as follows again.

$$\max_{\underline{m}} J(\underline{m}, \underline{u}; \gamma, \underline{u}^*)$$

$$\text{subject to} \qquad \forall \underline{u} \in U^*, \qquad P(\underline{m}, \underline{u}) - \alpha_s \geq 0 .$$

After this, we mean by the optimal satisfactory control a system of (11), (6). In order to continue analysis let us consider that $\underline{m}$ tries to maximize $J(\underline{m}, \underline{u}; \gamma, \underline{u}^*)$ and $\underline{u}$ tries to minimize it under constraints (7) and (8). This game theoretic formulation is the most pessimistic case of the satisfactory control. Clearly Eq. (11) implies both functions of Eq. (9) and (10). Furthermore by combining two functions into one performance index for both $\underline{m}$ and $\underline{u}$, we can apply a Lagrangian formulation and synthesize a solution by concave programming techniques.

The parameter γ is a weight factor to $\underline{u}^*$ which can take on any value between 0 and 1.0. If γ is taken close to 1.0, the first term $\gamma P(\underline{m}, \underline{u}^*)$ has a large weight in the performance index J in comparison to the second term $P(\underline{m}, \underline{u})$. Therefore, maximizing J with respect to $\underline{m}$ is almost equivalent to maximizing $P(\underline{m}, \underline{u}^*)$. Since the first term of J does not include $\underline{u}$, J is exactly equivalent to $P(\underline{m}, \underline{u})$ as far as minimizing with respect to $\underline{u}$ is concerned. If γ is chosen to be 0, we will have the usual game whose performance function is $P(\underline{m}, \underline{u})$. If $\gamma=1.0$, then we have a simple maximum problem, i.e., $\max_{\underline{m}} P(\underline{m}, \underline{u}^*)$.

Our problem is now restated as follows: To determine the optimal control $\underline{m}^0$ such that the performance index $J(\underline{m}, \underline{u}; \gamma, \underline{u}^*)$ is maximized under the satisfaction constraint Eq. (6). In order to solve this problem by methods of game theory, we will consider that the disturbance $\underline{u}$ acts to minimize J, thus including the quantifier effect.

4. A Method for Solving Games by Lagrange Multiplier Technique

In order to analyze and synthesize the formulated problem of the optimal satisfactory control applying game theoretic methods, we will develop a Lagrangian technique to solve the general game under constraints.

Problem 1. Let $J(\underline{m}, \underline{u})$ be a performance index (or a pay-off function of the game) and let

$$\underline{g}(\underline{m}, \underline{u})=\begin{pmatrix} g_1(\underline{m}, \underline{u}) \\ g_2(\underline{m}, \underline{u}) \\ \vdots \\ g_k(\underline{m}, \underline{u}) \end{pmatrix} \geq \underline{0} \tag{12}$$

be inequality constraints. Let us assume that $J(\underline{m}, \underline{u})$ and $\underline{g}(\underline{m}, \underline{u})$ are continuous differentiable functions with respect to $\underline{m}$ and $\underline{u}$. The problem is to find the optimal control $\underline{m}^0$ and the anti-optimal disturbance $\underline{u}^0$ such that

$$J(\underline{m}, \underline{u}^0) \leq J(\underline{m}^0, \underline{u}^0) \leq J(m^0, \underline{u}), \qquad \underline{g}(\underline{m}^0, \underline{u}^0) \geq \underline{0}$$

for any $\underline{m}$ satisfying $\underline{g}(\underline{m}, \underline{u}^0) \geq \underline{0}$ and any $\underline{u}$ satisfying $\underline{g}(\underline{m}^0, \underline{u}) \geq \underline{0}$.

Constraint Qualification

Let $\{\underline{m}^0, \underline{u}^0\}$ belong to the boundary of the constraint set of points $\{\underline{m}, \underline{u}\}$ satisfying $\underline{g}(\underline{m}, \underline{u}) \geq \underline{0}$. Let the inequalities $\underline{g}(\underline{m}^0, \underline{u}^0) \geq \underline{0}$ be separated into

$$g^1(\underset{\sim}{m}^0, \underset{\sim}{u}^0)=\underset{\sim}{0} \quad \text{and} \quad g^2(\underset{\sim}{m}^0, \underset{\sim}{u}^0)>\underset{\sim}{0}.$$

It will be assumed for each $\{\underset{\sim}{m}^0, \underset{\sim}{u}^0\}$ of the boundary of the constraint set that any vector differential $d\underset{\sim}{m}$, $d\underset{\sim}{u}$ satisfying the homogeneous linear inequalities

$$\frac{\partial g^1}{\partial \underset{\sim}{m}}(\underset{\sim}{m}^0, \underset{\sim}{u}^0)d\underset{\sim}{m} \geqslant \underset{\sim}{0} \tag{13}$$

$$\frac{\partial g^1}{\partial \underset{\sim}{u}}(\underset{\sim}{m}^0, \underset{\sim}{u}^0)d\underset{\sim}{u} \geq \underset{\sim}{0} \tag{14}$$

is tangent to an arc contained in the constraint set. This assumption is designed to rule out singularities on the boundary of the constraint set, such as an outward pointing "cusp" [6].

THEOREM 1. *In order that* $\underset{\sim}{m}^0$ *be the optimal control and* $\underset{\sim}{u}^0$ *the anti-optimal disturbance of Problem* 1, *it is necessary that* $\underset{\sim}{m}^0$, $\underset{\sim}{u}^0$ *and some* $\underset{\sim}{\lambda}^0$ *satisfy conditions*

$$\frac{\partial \Phi_1}{\partial \underset{\sim}{m}}(\underset{\sim}{m}^0, \underset{\sim}{u}^0, \underset{\sim}{\lambda}^0)=\underset{\sim}{0} \tag{15}$$

$$\frac{\partial \Phi_2}{\partial \underset{\sim}{u}}(\underset{\sim}{m}^0, \underset{\sim}{u}^0, \underset{\sim}{\lambda}^0)=\underset{\sim}{0} \tag{16}$$

$$\frac{\partial \Phi_1}{\partial \underset{\sim}{\lambda}}(\underset{\sim}{m}^0, \underset{\sim}{u}^0, \underset{\sim}{\lambda}^0) \geq \underset{\sim}{0}, \qquad \frac{\partial \Phi_1}{\partial \underset{\sim}{\lambda}}(\underset{\sim}{m}^0, \underset{\sim}{u}^0, \underset{\sim}{\lambda}^0)\lambda^0=0, \qquad \underset{\sim}{\lambda}^0 \geq \underset{\sim}{0} \tag{17}$$

where

$$\Phi_1(\underset{\sim}{m}, \underset{\sim}{u}, \underset{\sim}{\lambda})=J(\underset{\sim}{m}, \underset{\sim}{u})+\underset{\sim}{\lambda}^T g(\underset{\sim}{m}, \underset{\sim}{u}), \qquad \underset{\sim}{\lambda} \geq \underset{\sim}{0} \tag{18}$$

$$\Phi_2(\underset{\sim}{m}, \underset{\sim}{u}, \underset{\sim}{\lambda})=J(\underset{\sim}{m}, \underset{\sim}{u})-\underset{\sim}{\lambda}^T g(\underset{\sim}{m}, \underset{\sim}{u}), \qquad \underset{\sim}{\lambda} \geq \underset{\sim}{0}. \tag{19}$$

Proof. Given in Appendix A.

REMARK. These theorems on the game under inequality constraints developed in this section are an extension of the Kuhn-Tucker Theorems for the simple maximum problem. The result obtained here for the more complicated simultaneous maximum-minimum problem may be regarded as a Kuhn-Tucker Theorem for a constrained game. But we must consider a system of two Lagrangian functions corresponding to the maximization and minimization problems, respectively. This makes the difference from the simple constrained maximization problem. The technique used for proving the theorems follows that of Kuhn-Tucker [6].

Applying Theorem 1 to an ordinary game we have the following problem statement. Corollary 1 to Theorem 1 then provides a Lagrange multiplier technique for solving the usual game.

Problem 2. Let us suppose that control domain M and disturbance domain U

are expressed as $g_1(\underline{m})\geq\underline{0}$ and $g_2(\underline{u})\geq\underline{0}$, respectively. The performance function $J(\underline{m},\underline{u})$ and the inequality constraints $g_1(\underline{m})$, $g_2(\underline{u})$ are continuous differentiable functions in $\underline{m}$ and $\underline{u}$. The problem is to find a saddle point solution of the game $\{\underline{m}^0,\underline{u}^0\}$ such that

$$\min_{\underline{u}\in U}\max_{\underline{m}\in M} J(\underline{m},\underline{u})=\max_{\underline{m}\in M}\min_{\underline{u}\in U} J(\underline{m},\underline{u})=J(\underline{m}^0,\underline{u}^0) \tag{20}$$

COROLLARY 1. *In order that $\{\underline{m}^0,\underline{u}^0\}$ be a saddle point solution of Problem 2, it is necessary that*

$$\frac{\partial\Phi_1}{\partial\underline{m}}(\underline{m}^0,\underline{u}^0,\underline{\lambda}_1^0)=\underline{0} \tag{21}$$

$$\frac{\partial\Phi_2}{\partial\underline{u}}(\underline{m}^0,\underline{u}^0,\underline{\lambda}_2^0)=\underline{0} \tag{22}$$

$$\frac{\partial\Phi_1}{\partial\underline{\lambda}_1}(\underline{m}^0,\underline{u}^0,\underline{\lambda}_1^0)\geq\underline{0}\,,\qquad \frac{\partial\Phi_1}{\partial\underline{\lambda}_1}(\underline{m}^0,\underline{u}^0,\underline{\lambda}_1^0)\underline{\lambda}_1^0=0\,,\qquad \underline{\lambda}_1^0\geq\underline{0} \tag{23}$$

$$\frac{\partial\Phi_2}{\partial\underline{\lambda}_2}(\underline{m}^0,\underline{u}^0,\underline{\lambda}_2^0)\geq\underline{0}\,,\qquad \frac{\partial\Phi_2}{\partial\underline{\lambda}_2}(\underline{m}^0,\underline{u}^0,\underline{\lambda}_2^0)\underline{\lambda}_2^0=0\,,\qquad \underline{\lambda}_2^0\geq\underline{0} \tag{24}$$

where

$$\Phi_1(\underline{m},\underline{u},\underline{\lambda}_1)=J(\underline{m},\underline{u})+\underline{\lambda}_1^T g_1(\underline{m})\,,\qquad \underline{\lambda}_1\geq\underline{0} \tag{25}$$

$$\Phi_2(\underline{m},\underline{u},\underline{\lambda}_2)=J(\underline{m},\underline{u})-\underline{\lambda}_2^T g_2(\underline{u})\,,\qquad \underline{\lambda}_2\geq\underline{0} \tag{26}$$

Proof. Given in Appendix B

THEOREM 2. *In order that $\{\underline{m}^0,\underline{u}^0\}$ be a solution of Problem* 1 *it is sufficient that*
(i) *$\underline{m}^0$, $\underline{u}^0$ and some $\underline{\lambda}^0\geq\underline{0}$ satisfy conditions* (15), (16) *and* (17)
(ii) *$\Phi_1(\underline{m},\underline{u}^0,\underline{\lambda}^0)$ be concave in $\underline{m}$ and $\Phi_2(\underline{m}^0,\underline{u},\underline{\lambda}^0)$ be convex in $\underline{u}$.*

Proof. Given in Appendix C

COROLLARY 2. *In order that $\{\underline{m}^0,\underline{u}^0\}$ be a saddle point solution for Problem 2, it is sufficient that*
(i) *$\underline{m}^0$, $\underline{u}^0$ and some $\underline{\lambda}^0\geq\underline{0}$ satisfy conditions* (21) *through* (24) *and*
(ii) *$\Phi_1(\underline{m},\underline{u}^0,\underline{\lambda}_1^0)$ and $\Phi_2(\underline{m}^0,\underline{u},\underline{\lambda}_2^0)$ defined by Eq.* (25) *and* (26) *be concave in $\underline{m}$ and convex in $\underline{u}$, respectively.*

Problem 3. (A Saddle Value Problem for the Game).
To find $\underline{m}^0$, $\underline{u}^0$, and $\underline{\lambda}^0\geq\underline{0}$ such that

$$\begin{cases}\Phi_1(\underline{m},\underline{u}^0,\underline{\lambda}^0)\leq\Phi_1(\underline{m}^0,\underline{u}^0,\underline{\lambda}^0)\leq\Phi_1(\underline{m}^0,\underline{u}^0,\underline{\lambda}) & (27)\\ \Phi_2(\underline{m}^0,\underline{u}^0,\underline{\lambda})\leq\Phi_2(\underline{m}^0,\underline{u}^0,\underline{\lambda}^0{}_0\leq\Phi_2(\underline{m}^0,\underline{u},\underline{\lambda}^0) & (28)\end{cases}$$

are satisfied simultaneously for all $\underline{m}$, $\underline{u}$ and $\underline{\lambda}\geq\underline{0}$, where $\Phi_1(\underline{m},\underline{u},\underline{\lambda})$ and $\Phi_2(\underline{m},\underline{u},\underline{\lambda})$

are given by Eq. (18) and (19), respectively.

REMARK. If $\{\underline{m}^0, \underline{u}^0, \underline{\lambda}^0\}$ is a solution of a system of saddle value functions Φ_1 and Φ_2, Eq. (27) and (28) may be expressed as

$$\begin{cases} \max\limits_{\underline{m}} \min\limits_{\underline{\lambda}\geq \underline{0}} \Phi_1(\underline{m}, \underline{u}^0, \underline{\lambda}) = \min\limits_{\underline{\lambda}\geq 0} \max\limits_{\underline{m}} \Phi_1(\underline{m}, \underline{u}^0, \underline{\lambda}) \\ \min\limits_{\underline{u}} \max\limits_{\underline{\mu}\geq \underline{0}} \Phi_2(\underline{m}^0, \underline{u}, \underline{\mu}) = \max\limits_{\underline{\mu}\geq \underline{0}} \min\limits_{\underline{u}} \Phi_2(\underline{m}^0, \underline{u}, \underline{\mu}) \end{cases}$$

LEMMA 1. *Let $\Phi_1(\underline{m}, \underline{u}, \underline{\lambda})$ and $\Phi_2(\underline{m}, \underline{u}, \underline{\lambda})$ be continuous differentiable functions with respect to $\underline{m}$, $\underline{u}$ and $\underline{\lambda}$. Then the conditions*

$$\frac{\partial \Phi_1}{\partial \underline{m}}(\underline{m}^0, \underline{u}^0, \underline{\lambda}^0) = \underline{0} \tag{29}$$

$$\frac{\partial \Phi_2}{\partial \underline{u}}(\underline{m}^0, \underline{u}^0, \underline{\lambda}^0) = \underline{0} \tag{30}$$

$$\frac{\partial \Phi_1}{\partial \underline{\lambda}}(\underline{m}^0, \underline{u}^0, \underline{\lambda}^0) \geq \underline{0}\,, \qquad \frac{\partial \Phi_1}{\partial \underline{\lambda}}(\underline{m}^0, \underline{u}^0, \underline{\lambda}^0)\underline{\lambda}^0 = 0\,, \qquad \underline{\lambda}^0 \geq \underline{0}\,, \tag{31}$$

are necessary for $\{\underline{m}^0, \underline{u}^0, \underline{\lambda}^0\}$ to provide a saddle point solution for Problem 3.

LEMMA 2. *In order that $\underline{m}^0, \underline{u}^0$ and some $\underline{\lambda}^0 \geq \underline{0}$ provide a saddle point solution for Problem 3, it is sufficient that*

(i) *Eq.* (29), (30) *hold and*

(ii) *$\Phi_1(\underline{m}, \underline{u}^0, \underline{\lambda}^0)$ be concave in $\underline{m}$, $\Phi_1(\underline{m}^0, \underline{u}^0, \underline{\lambda})$ be convex in $\underline{\lambda}$, $\Phi_2(\underline{m}^0, \underline{u}, \underline{\lambda}^0)$ be convex in $\underline{u}$ and $\Phi_2(m^0, \underline{u}^0, \underline{\lambda})$ be concave in $\underline{\lambda}$.*

Lemma 1, 2 can be easily proved by the method analogous to Kuhn-Tucker [6].

THEOREM 3. *Let $J(\underline{m}, \underline{u})$ be concave in $\underline{m}$ and convex in $\underline{u}$ as well as continuous and differentiable. Let $g_1(\underline{m}, \underline{u}), \cdots, g_k(\underline{m}, \underline{u})$ be concave in $\underline{m}$ and $\underline{u}$ as well as continuous and differentiable. Then $\{\underline{m}^0, \underline{u}^0\}$ is a solution for Problem* 1 *if and only if $\underline{m}^0, \underline{u}^0$ and some $\underline{\lambda}^0 \geq \underline{0}$ give a solution of the saddle value problem for $\Phi_1(\underline{m}, \underline{u}, \underline{\lambda})$ and $\Phi_2(\underline{m}, \underline{u}, \underline{\lambda})$ defined in Theorem* 1.

Proof. Given in Appendix D.

Let us now consider a game which has a state vector and equality constraints.

Problem 4. Let $J(\underline{y}, \underline{m}, \underline{u})$ be a pay-off function and let

$$\underline{f}(\underline{y}, \underline{m}, \underline{u}) = \underline{0} \tag{32}$$

$$\underline{g}(\underline{m}, \underline{u}) \geq \underline{0} \tag{33}$$

be process equations and inequality constraints. Let us assume that J, $\underline{f}$, $\underline{g}$ be continuous differentiable functions with respect to $\underline{y}$, $\underline{m}$ and $\underline{u}$ and $\partial \underline{f}/\partial \underline{y}$ a nonsingular matrix. We assume there exist pure strategies (a saddle point solution) for $\underline{m}$ and $\underline{u}$. Then the problem is to find the optimal control $\underline{m}^0$ and the anti-optimal disturbance $\underline{u}^0$ and the corresponding $\underline{y}^0$.

THEOREM 4. *In order that* $\{\underline{m}^0, \underline{u}^0, \underline{y}^0\}$ *be a saddle point solution of Problem* 4, *it is necessary that*

$$\frac{\partial \Phi_1}{\partial \underline{m}}(\underline{y}^0, \underline{m}^0, \underline{u}^0, \underline{\psi}^0, \underline{\lambda}^0)=\underline{0} \tag{34}$$

$$\frac{\partial \Phi_2}{\partial \underline{u}}(\underline{y}^0, \underline{m}^0, \underline{u}^0, \underline{\psi}^0, \underline{\lambda}^0)=\underline{0} \tag{35}$$

$$\frac{\partial \Phi_1}{\partial \underline{y}}(\underline{y}^0, \underline{m}^0, \underline{u}^0, \underline{\psi}^0, \underline{\lambda}^0)=\underline{0} \tag{36}$$

$$\frac{\partial \Phi_1}{\partial \underline{\psi}}(\underline{y}^0, \underline{m}^0, \underline{u}^0, \underline{\psi}^0, \underline{\lambda}^0)=\underline{0} \qquad \underline{\psi}: \textit{unrestricted} \tag{37}$$

$$\frac{\partial \Phi_1}{\partial \underline{\lambda}}(\underline{y}^0, \underline{m}^0, \underline{u}^0, \underline{\psi}^0, \underline{\lambda}^0)\geqslant\underline{0}\,, \qquad \frac{\partial \Phi_1}{\partial \underline{\lambda}}(\underline{y}^0, \underline{m}^0, \underline{u}^0, \underline{\psi}^0, \underline{\lambda}^0)\underline{\lambda}^0=0\,, \qquad \underline{\lambda}^0\geq\underline{0} \tag{38}$$

where

$$\Phi_j(\underline{\lambda}, \underline{m}, \underline{u}, \underline{\psi}, \underline{\lambda})=J(\underline{y}, \underline{m}, \underline{u})+\underline{\psi}^T \underline{f}(\underline{y}, \underline{m}, \underline{u})+(-1)^{j-1}\underline{\lambda}^T \underline{g}(\underline{m}, \underline{u})\,, \qquad j=1, 2\,.$$

This theorem can be proved in the manner analogous to the proof of Theorem 1 by using Farkas's lemma containing an equality equation also. Here constraint qualification corresponding to $\underline{f}(\underline{y}, \underline{m}, \underline{u})=\underline{0}$ is also assumed in addition to the constraint qualification on $\underline{g}(\underline{m}, \underline{u})\geq\underline{0}$, i.e., there exist differentials $d\underline{y}$, $d\underline{m}$, $d\underline{u}$ such that $\partial \underline{f}/\partial \underline{y}\cdot d\underline{y}=\underline{0}$, $\partial \underline{f}/\partial \underline{m}\cdot d\underline{m}=\underline{0}$, $\partial \underline{f}/\partial \underline{u}\cdot d\underline{u}=\underline{0}$.

REMARK. It is noticed that Theorem 4 holds only when inequality constraints $\underline{g}$ does not include $\underline{y}$.

5. Duality Theorem for the Game

We shall define a dual problem of a game as well as a duality theorem of nonlinear programming [7]. Let $J(\underline{m}, \underline{u})$ be a concave differentiable function with respect to $\underline{m}$ and a convex differentiable one with respect to $\underline{u}$. Let $\underline{g}(\underline{m}, \underline{u})$ be a concave differentiable function with respect to both $\underline{m}$ and $\underline{u}$. Defining Lagrange functions

$$\Phi_1(\underline{m}, \underline{u}, \underline{\lambda})=J(\underline{m}, \underline{u})+\underline{\lambda}^T \underline{g}(\underline{m}, \underline{u}) \tag{39}$$

$$\Phi_2(\underline{m}, \underline{u}, \underline{\mu})=J(\underline{m}, \underline{u})-\underline{\mu}^T \underline{g}(\underline{m}, \underline{u}) \tag{40}$$

let us define a system of problems:

Primal Problem:
$$\begin{cases} \max\limits_{\underline{m}} J(\underline{m}, \underline{u}) & (41) \\ \min\limits_{\underline{u}} J(\underline{m}, \underline{u}) & (42) \\ \text{subject to } \underline{g}(\underline{m}, \underline{u}) \geq \underline{0} & (43) \end{cases}$$

Dual Problem:
$$\begin{cases} \min\limits_{\underline{\lambda}} [\Phi_1(\underline{m}, \underline{u}, \underline{\lambda}) = J(\underline{m}, \underline{u}) + \underline{\lambda}^T \underline{g}(\underline{m}, \underline{u})] & (44) \\ \text{subject to } \dfrac{\partial \Phi_1}{\partial \underline{m}}(\underline{m}, \underline{u}, \underline{\lambda}) = \dfrac{\partial J}{\partial \underline{m}}(\underline{m}, \underline{u}) + \underline{\lambda}^T \dfrac{\partial \underline{g}}{\partial \underline{m}}(\underline{m}, \underline{u}) = \underline{0} & (45.\text{a}) \\ \qquad \underline{\lambda} \geq \underline{0} & (45.\text{b}) \\ \max\limits_{\underline{\mu}} [\Phi_2(\underline{m}, \underline{u}, \underline{\mu}) = J(\underline{m}, \underline{u}) + \underline{\mu}^T \underline{g}(\underline{m}, \underline{u})] & (46) \\ \text{subject to } \dfrac{\partial \Phi_2}{\partial \underline{u}}(\underline{m}, \underline{u}, \underline{\mu}) = \dfrac{\partial J}{\partial \underline{u}}(\underline{m}, \underline{u}) - \underline{\mu}^T \dfrac{\partial \underline{g}}{\partial \underline{u}}(\underline{m}, \underline{u}) = \underline{0} & (47.\text{a}) \\ \qquad \underline{\mu} \geq \underline{0} & (47.\text{b}) \end{cases}$$

The Duality Theorem: The constraint qualification for the game will be assumed from now on. From the Kuhn-Tucker Theorem for the game (Theorem 3) under the constraint qualification and the assumptions on J and g, we can say as follows. (The point $\{\underline{m}^0, \underline{u}^0\}$ is a solution of the primal problem if, and only if $\{\underline{m}^0, \underline{u}^0\}$ and $\underline{\lambda}^0 \geq \underline{0}$ constitute a saddle point solution for a pair of Lagrangians Φ_1 and Φ_2 given by Eq. (39) and (40). That is, for any $\underline{m}$, $\underline{u}$ and $\underline{\lambda} \geq \underline{0}$, $\underline{\mu} \geq 0$, a system of

$$\begin{cases} \Phi_1(\underline{m}, \underline{u}^0, \underline{\lambda}^0) \leq \Phi_1(\underline{m}^0, \underline{u}^0, \underline{\lambda}^0) \leq \Phi_1(\underline{m}^0, \underline{u}^0, \underline{\lambda}), \qquad \underline{\lambda}^0 \geq \underline{0} & (48) \\ \Phi_2(\underline{m}^0, \underline{u}^0, \underline{\mu}) \leq \Phi_2(\underline{m}^0, \underline{u}^0, \underline{\mu}^0) \leq \Phi_2(\underline{m}^0, \underline{u}, \underline{\mu}^0), \qquad \underline{\mu}^0 \geq \underline{0} & (49) \end{cases}$$

is satisfied simultaneously and it becomes that $\underline{\mu} \equiv \underline{\lambda}^0$).

Now our duality theorem for the game is:

THEOREM 5 (The Duality Theorem for game). *Jf $\{\underline{m}^0, \underline{u}^0\}$ is a solution of the primal problem, there exists $\underline{\lambda}^0 \equiv \underline{\mu}^0$ such that $\{\underline{m}^0, \underline{u}^0, \underline{\lambda}^0\}$ solves a solution of the dual, and the extrema are equal.*

Proof. Given in Appendix E.

6. Analysis of the Formulated Problem, Part 2

Let us now return to analysis of the optimal satisfactory control described in Section 3. A domain of the feasible $\underline{m}$ such that the satisfaction condition $\forall \underline{u} \in U^*$, $P(\underline{m}, \underline{u}) - \alpha_s \geq 0$ is satisfied is given by

$$P(\underline{m}, h(m)) - \alpha_s \geq 0 \tag{50}$$

where $\underline{h}(\underline{m})$ is the disturbance minimizing $P(\underline{m}, \underline{u})$ with respect to $\underline{u} \in U^*$. The control $\underline{m}$ satisfying the above equation surely satisfies $P(\underline{m}, \underline{u}) \geqslant \alpha_s$ for any $\underline{u} \in U^*$. Thus applying Corollary 1 to the performance index (11) and constraints (50) and $\underline{b}(\underline{u}) \geq \underline{0}$, we define Lagrangian functions as follows.

$$\Phi_1(\underline{m}, \underline{u}, \lambda_1; \gamma, \underline{u}^*) = J(\underline{m}, \underline{u}; \gamma, \underline{u}^*) + \lambda_1(P(\underline{m}, \underline{h}(\underline{m})) - \alpha_s) \tag{51}$$

$$\Phi_2(\underline{m}, \underline{u}, \underline{\lambda}_2; \gamma, \underline{u}^*) = J(\underline{m}, \underline{u}; \gamma, \underline{u}^*) - \underline{\lambda}_2^T \underline{b}(\underline{u}) \tag{52}$$

where $J(\underline{m}, \underline{u}; \gamma, \underline{u}^*) = \gamma P(\underline{m}, \underline{u}^*) + (1-\gamma) P(\underline{m}, \underline{u})$. Then we have the necessary conditions for the optimal satisfactory control from Corollary 1 as follows.

COROLLARY 3. *Suppose that $P(\underline{m}, \underline{u})$ and $\underline{b}(\underline{u})$ are continuous differentiable functions in $\underline{m}$ and $\underline{u}$. In order that $\underline{m}^0$ be an optimal control and $\underline{u}^0$ be an anti-optimal disturbance for the problem of the optimal satisfactory control (it is assumed that there exists a saddle point solution $\{\underline{m}^0, \underline{u}^0\}$ in the sense of the statement in Problem* 1), *it is necessary that $\underline{m}^0$, $\underline{u}^0$ and some λ_1^0, $\underline{\lambda}_2^0 \geq \underline{0}$ satisfy the conditions of Eq.* (21)~(24) *for Φ_1 and Φ_2 defined by Eq.* (52) *and* (52) *respectively.*

Discussion: The value $\underline{u}^*$ does not necessarily have to be the observed disturbance $\underline{u}^*(nT_0)$. It may be better to take some estimated mean value of $\underline{u}(t)$, $nT_0 \leq t < (n+1)T_0$, based on the observations $\underline{u}(nT_0 - kT_1)$, $k = 0, 1, \cdots$, where $T_1 < T_0$. We choose γ arbitrary depending on the amount of weight we wish to assign to $\underline{u}^*$, in other words, depending on the importance we attach to the performance in the neighborhood of $\underline{u}^*$. Thus, we may choose γ such that $\max_{\gamma} \underset{\underline{u}}{E}\, P(\underline{m}(\gamma, \underline{u}), \underline{u})$, where $\underline{m}(\gamma, \underline{u})$ is obtained by maximizing $J(\underline{m}, \underline{u}; \gamma, \underline{u}^*)$ with respect to $\underline{m}$, based on information of past disturbances $\underline{u}(t)$ or the probability distribution of the disturbances estimated from the past observations. The proper choice of α_s is dependent on U^*. One way to determine α_s is to use the performance value $P(\mathrm{m}^0(\underline{u}^0), \underline{u}^0)$ which is the minimum value of the optimal performance $P(\underline{m}^0(\underline{u}), \underline{u})$ over all $\underline{u} \in U^*$. The performance can never be greater than $P(\underline{m}^0(\underline{u}^0), \underline{u}^0)$ when the worst disturbance $\underline{u}^0$ occurs. Therefore, α_s must be smaller than that value.

7. Analysis of the Formulated Problem, Part 3, A Heuristic

In general, it is difficult to solve the problem formulated as a game since sufficiently powerful game-solving methods are not available. Therefore, we consider a heuristic approach to solve the optimal satisfactory control as an approximation. The optimal satisfactory control formulated in Section 3 can be restated as

$$\max_{\underline{m} \in M^*} J(\underline{m}, \underline{u}; \gamma, \underline{u}^*)$$

where

$$M^*=\{\underline{m}: P(\underline{m}, \underline{u}) \geq \alpha_s, \ \forall \underline{u} \in U^*\}$$

$$U^*=\{\underline{u}: \underline{b}(\underline{u}) \geq \underline{0}\} .$$

Then the saddle point solution $\{\underline{m}^0, \underline{u}^0\}$ is given by

$$\max_{\underline{m} \in M^*} \min_{\underline{u} \in U^*} J(\underline{m}, \underline{u} ; \gamma, \underline{u}^*)=\min_{\underline{u} \in U^*} \max_{\underline{m} \in M^*} J(\underline{m}, \underline{u} ; \gamma, \underline{u}^*)=J(\underline{m}^0, \underline{u}^0 ; \gamma, \underline{u}^*) . \tag{53}$$

Let us assume the U^* consists of only a finite number of discrete values, $\underline{u}^1, \underline{u}^2, \cdots, \underline{u}^q$. Then M^* is defined by a set of inequality equations

$$P(\underline{m}, \underline{u}^i)-\alpha_s \geq 0 , \qquad i=1, 2, \cdots, q . \tag{54}$$

Let us consider assigning a weight to each quantum level $\underline{u}^i$, $i=1, 2, \cdots, q$. Suppose that we measured the k-th quantum level at nT_0, i.e., $\underline{u}^*(nT_0)=\underline{u}^k$. Then it seems reasonable, in practice, to give a relatively large weight to $\underline{u}^k$; then the weight to be assigned to each quantum level will decrease with increasing distance from $\underline{u}^k$. Therefore, our problem may be stated as follows:

To find the optimal control $\underline{m}^0$ such that

$$\sum_{i=1}^{q} P(\underline{m}, \underline{u}^i)\omega(\underline{u}^i) \tag{55}$$

is maximized subject to the satisfaction constraints

$$P(\underline{m}, \underline{u}^i)-\alpha_s \geq 0 , \qquad i=1, 2, \cdots, q \tag{56}$$

where $\omega(\underline{u}^i)$ is a weighting function. The weight factor $\omega(\underline{u}^i)$ will be replaced by the probability distribution if the distribution is known. The optimal satisfactory control has a meaning also for stochastic optimization under the satisfaction constraints.

This formulated problem can be solved by directly applying the Kuhn-Tucker Theorem for the simple maximization problem, since the disturbance is not treated here as a variable but as a finite set of constants $\underline{u}^i$'s. Furthermore, we can include additional constraints which further limit the control domain, such as $\underline{d}(\underline{m}) \geq \underline{0}$.

8. Numerical Example

Let us consider some numerical example to demonstrate the meaning of the optimal satisfactory control and to verify that Theorem 1 can be applied to obtain the numerical solution. The process equation and the performance index are given as follows:

$$y=m-2u$$

$$P_1(y, m, u)=-y^2-2m^2+(y-2)m+5u^2$$

$$\therefore \quad P(m, u)=-2m^2+2m(u-1)+u^2 \tag{57}$$

where m is the unconstrained control and u is the disturbance. The performance $P(m, u)$ is concave in m and convex in u, respectively. Suppose that we have $u^*=2$ to which the disturbance interval $U^*=[-2, 3]$ is assigned. Then U^* can be described as $U^*=\{u: 2.5^2-(u-0.5)^2\geq 0\}$.

The worst disturbance $\hat{u}$ minimizing $P(\underline{m}, \underline{u})$ with respect to $\underline{u}\in U^*$ is given as

$$h(\underline{m})=\begin{cases}-2, & 2<m\\ -m, & -3\leqslant m\leqslant 2\\ 3, & m<-3.\end{cases}$$

Thus, the domain of feasible m satisfying the satisfaction condition is given by $P(m, u)-\alpha_s=-3m^2-2m-\alpha_s\geq 0$.

Let us define

$$\Phi_1(m, u, \lambda_1)=J(m, u; \gamma, u^*)+\lambda_1(-3m^2-2m-\alpha_s)$$

$$\Phi_2(m, u, \lambda_2)=J(m, u; \gamma, u^*)-\lambda_2(2.5^2-(u-0.5)^2).$$

Where γ is a weight factor and α_s is a specified constant. From Corollary 3 we have the following necessary conditions.

$$\frac{\partial\Phi_1}{\partial m}=-2\gamma(m-1)+2(1-\gamma)(-2m+u-1)+\lambda_1(-6m-2)=0 \tag{58}$$

$$\frac{\partial\Phi_2}{\partial u}=2(1-\gamma)(m+u)-\lambda_2(-2(u-0.5))=0 \tag{59}$$

$$-3m^2-2m-\alpha_s\geq 0, \quad \lambda_1(-3m^2-2m-\alpha_s)=0, \quad \lambda_1\geq 0 \tag{60}$$

$$2.5^2-(u-0.5)^2\geq 0, \quad \lambda_2(2.5^2-(u-0.5)^2)=0, \quad \lambda_2\geq 0. \tag{61}$$

CASE I. $\lambda_1^0=0$, $\lambda_2^0=0$.

The saddle point, in this case, is not on the boundary of the constraint equation, therefore

$$\frac{\partial J}{\partial m}=-2\gamma(2m-1)+2(1-\gamma)(-2m+u-1)=0$$

$$\frac{\partial J}{\partial u}=2(1-\gamma)(m+u)=0$$

$$\therefore \quad u^0=-m^0, \qquad m^0=\frac{1-2\gamma}{\gamma-3}. \tag{62}$$

The minimum performance corresponding to the saddle point of CASE I can be calculated from Eq. (57) (62), i.e.,

$$P(m^0, u^0)=\frac{(1-2\gamma)(4\gamma+3)}{(\gamma-3)^2}=\alpha_{s\,\text{limit}}$$

and this is the maximum P against the worst u when J is maximized without considering the satisfaction constraint. Therefcre CASE I occurs as $\alpha_s<\alpha_{s\,\mathrm{limit}}$. But from $-3m^2-2m-\alpha_s\geq 0$, it must be that

$$\frac{-1-1\sqrt{1-3\alpha_s}}{3}\leq m^0(\gamma)\leq\frac{-1+\sqrt{1-3\alpha_s}}{3}.$$

For example when $\alpha_s=0.28$, it must be that $\gamma<0.2$. If $\gamma=0.2$, for example, $m^0=-0.214$.

CASE II. $\lambda_1^0\neq 0$, $\lambda_2^0=0$.

This is the case when the saddle point exists on the boundary of the satisfaction constraint set. CASE II happens as $\alpha_s\geq\alpha_{s\,\mathrm{limit}}$. A set of equations to be solved is given:

$$-3m^2-2m-\alpha_s=0$$

$$\frac{\partial\Phi_1}{\partial m}=-2\gamma(2m-1)+2(1-\gamma)(-2m+u-1)+2\lambda_1(-3m-1)=0$$

$$\frac{\partial\Phi_2}{\partial u}=2(1-\gamma)(m+u)=0\,,\qquad 2.5^2-(u-0.5)^2\geq 0\,.$$

From the first equation

$$m^0=\frac{-1\pm\sqrt{1-3\alpha_s}}{3}\,,\qquad \alpha_s\leq\frac{1}{3}\,.$$

Since $\lambda_1\geq 0$, we obtain a condition

$$\lambda_1=\gamma\frac{m^0+2}{3m^0+1}-1\geq 0$$

$$\therefore\quad \gamma\geq\frac{3m^0+1}{m^0+2}=3\frac{-1\pm\sqrt{1-3\alpha_s}+1}{-1\pm\sqrt{1-3\alpha_s}+6}\,.$$

For example, when $\alpha_s=0$ it must be that $\gamma\geq 0.2$, and if $\gamma=0.5$ for example, it becomes that $\lambda_1^0=\frac{5}{4}\geq 0$.

CASE III. $\lambda_1^0=0$, $\lambda_2^0\neq 0$.

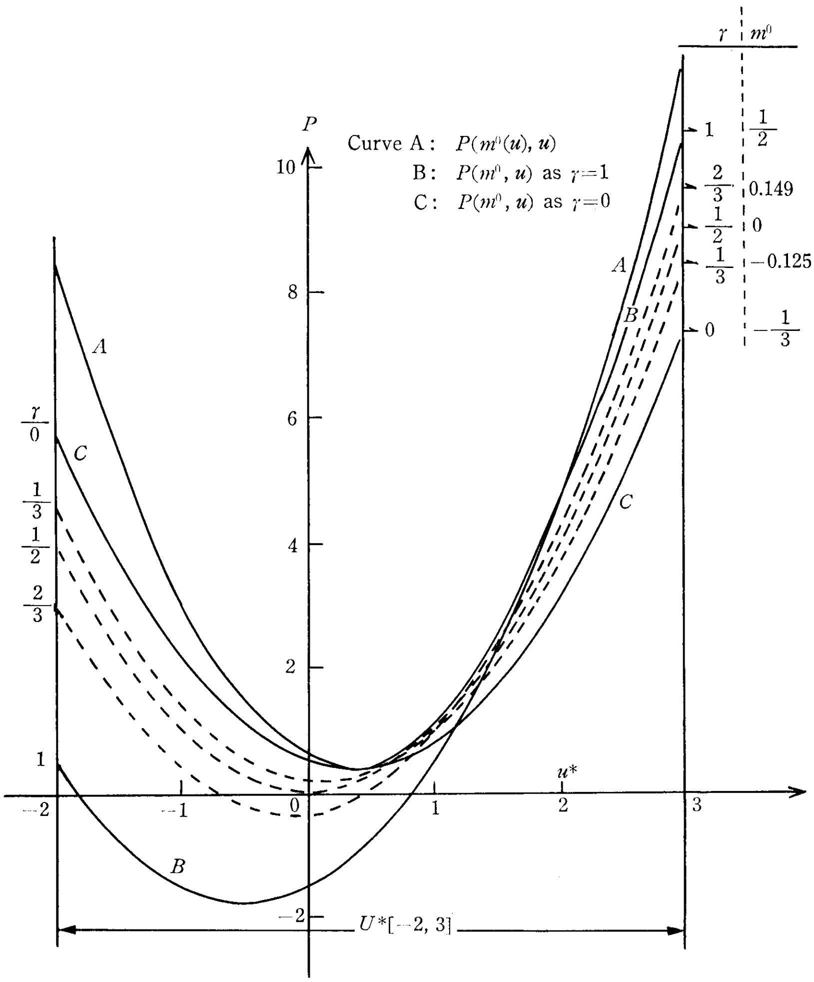
This is the case when u^0 occurs on the boundary point -2 or 3 of U^*. If we write it as $u_e=-2$ or 3 denoting the boundary point, we obtain from Eq. (68) $m^0=[\gamma(2-u_e)+u_e-1]/2$ since $\lambda_1^0=0$. On the other hand since $\lambda_2\geq 0$, we have a condition $[\gamma(2-u_e)+3u_e-1]/(u_e-0.5)\geq 0$. Furthermore substituting the above m^0 into $-3m^2-2m-\alpha_s>0$ we obtain a condition for γ.

CASE IV. $\lambda_1^0\neq 0$, $\lambda_2^0\neq 0$.

This is the case when $\{m^0, u^0\}$ is on the boundary of both U^* and the satisfaction condition. However, if we check the condition $\lambda_1, \lambda_2 \geq 0$, we can easily see that this case can not occur to be a solution of the optimal satisfactory control for the specified $U^*=[-2, 3]$.

Fig. 2 represents curves of $P(m^0(\gamma), u)$, $u \in U^*$ for various values of γ, as the solution of the optimal satisfactory control is an inner point of the constraints set. Fig. 3 shows curves of $P(m^0(\alpha_s), u)$, $u \in U^*$ as m^0 occurs on the boundary of satisfaction condition.

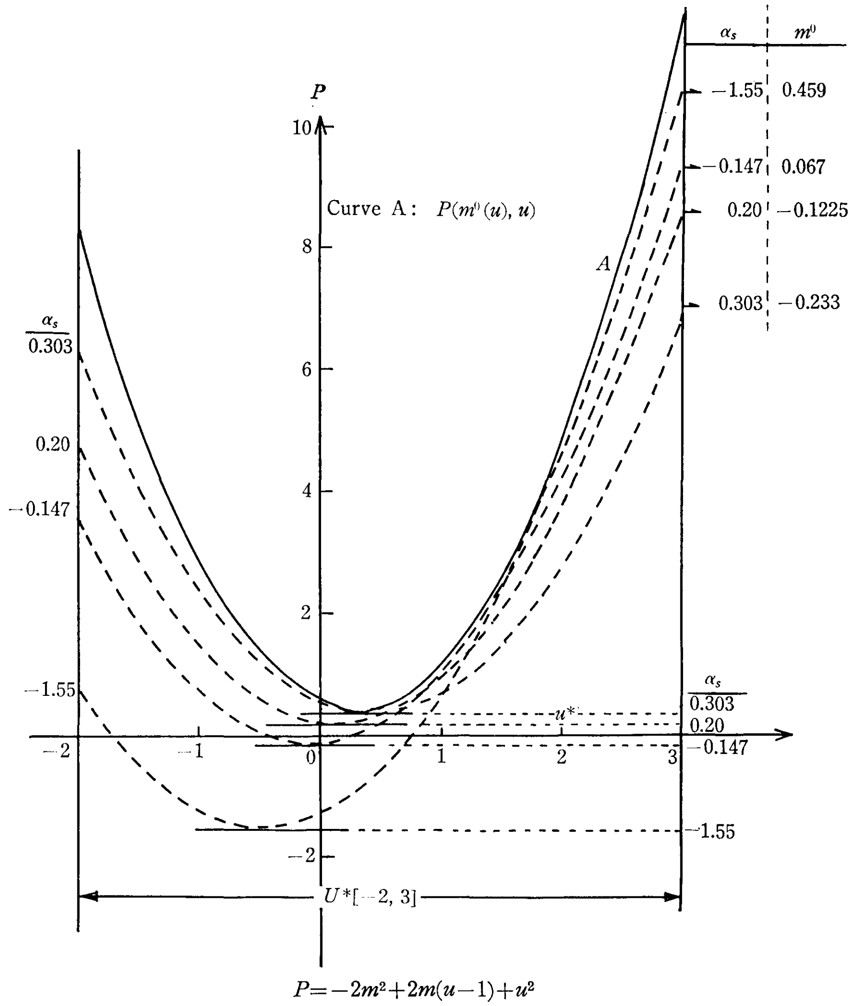
In Fig. 2 and 3, Curve A represents the performance $P(m^0(u), u)$ where m is controlled to maximize P for each u. Therefore, Curve A represents the best performance we can obtain and all other curves of $P(m, u)$ must lie below Curve A. It



$P=-2m^2+2m(u-1)+u^2$

Case I $\alpha_s < \alpha_{s\,\text{limit}}$.

Fig. 2. The Optimal Satisfactory Control



$P = -2m^2 + 2m(u-1) + u^2$

Case II $\alpha_s \geq \alpha_{s\,\mathrm{limit}}$.

Fig. 3. The Optimal Satisfactory Control

is natural that $P(m^0(\gamma), u)$ with $\gamma = 1$ coincides at $u = u^*$ with $\max_m P(m, u^*)$ and smaller values of γ yield smaller $P(m^0(\gamma), u^*)$. The optimal control m^0 takes on values between $\frac{1}{2}$ (corresponding to $\gamma = 1$) and $\frac{-1}{3}$ (corresponding to $\gamma = 0$). When $\gamma = 0$ (Curve C) we have the original performance index (67) and the usual min-max problem for which there exists the saddle point $\left(m^0 = \frac{-1}{3},\ u^0 = \frac{1}{3}\right)$. Implementing this m^0 is the most conservative solutive of the game, but it is very pessimistic. Empirically we may be able to detect more frequent occurance of u arround u^*, thus we should give a proper weight to u^* for practice. By giving more weight to u^*, an average performance in regions removed from u^* is obliged to be small.

9. Conclusion

In this paper, the optimal satisfactory control is formulated as one method of decision making under uncertainties based on a small amount of information about the uncertainties. The optimal satisfactory control is considered to be carried out in the context of a multi-layer approach [3], so that the control value is updated periodically by the second layer control function.

The formulated problem is analyzed by application of game-theoretic method. The general method of solving a game which is constrained by inequality equations has been developed using Lagrange multiplier technique. We have established several theorems which are analogous to the Kuhn-Tucker theorem for a simple maximization problem.

The optimal satisfactory control can be extended to a multi-stage system and we shall obtain necessary conditions expressed with Hamiltonian form.

The approach formulated in this paper may be applied to stochastic optimizing control. When we determine the optimizing control algorithm to maximize the expected performance with respect to the predicted probability distribution of disturbances, our concept is used so that the expected value of the performance is not less than the specified constant against the uncertainty of the probability distribution.

The optimal satisfactory control is one practical approach to decision making under uncertainty. Since it is quite general in its formulation, this concept may be applied to various engineering problems and be modified depending on the type of application.

Appendix A. Proof of Theorem 1

It is assumed that there exists a saddle point $\{\underline{m}^0, \underline{u}^0\}$ that is a solution to the game, i.e., (optimum, anti-optimum). Let $\underline{m}^0$ maximize $J(\underline{m}, \underline{u}^0)$ constrained $\underline{g}(\underline{m}, \underline{u}^0) \geq \underline{0}$ and $\underline{u}^0$ minimize $J(\underline{m}^0, \underline{u})$ constrained $\underline{g}(\underline{m}^0, \underline{u}) \geq \underline{0}$ (subject to the above constraint qualification of Eq. (13) and (14)). Then the inequalities

$$\frac{\partial J}{\partial \underline{m}}(\underline{m}^0, \underline{u}^0) d\underline{m} \leq 0 \tag{A. 1}$$

$$\frac{\partial J}{\partial \underline{u}}(\underline{m}^0, \underline{u}^0) d\underline{u} \geq 0 \tag{A. 1}$$

must hold for all vector differentials $d\underline{m}$ satisfying Eq. (13) and $d\underline{u}$ satisfying Eq. (14), respectively (see Fig. 4).

Then applying Farkas' Lemma* to Eq. (13) (A. 1) and Eq. (14) (A. 2), respectively,

* (Farkas' Lemma) An inequality $\underline{b}^T\underline{x} \geq 0$ holds for all n-vectors $\underline{x}$ satisfying a system of m inequalities $A\underline{x} \geq \underline{0}$ where A is a matrix, only if $\underline{b} = A^T\underline{t}$ for some m-vector $\underline{t} \geq \underline{0}$.

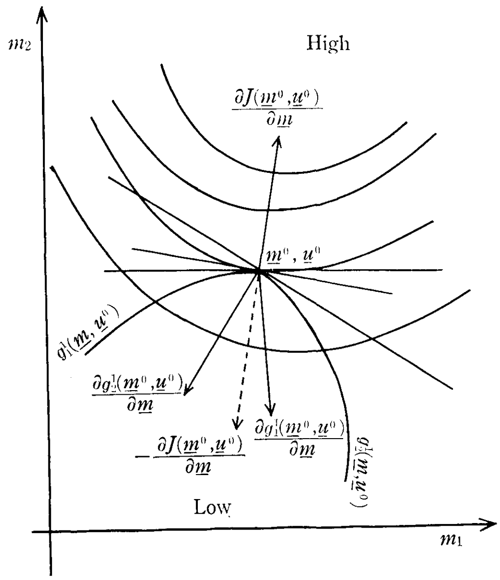


Fig. 4a.

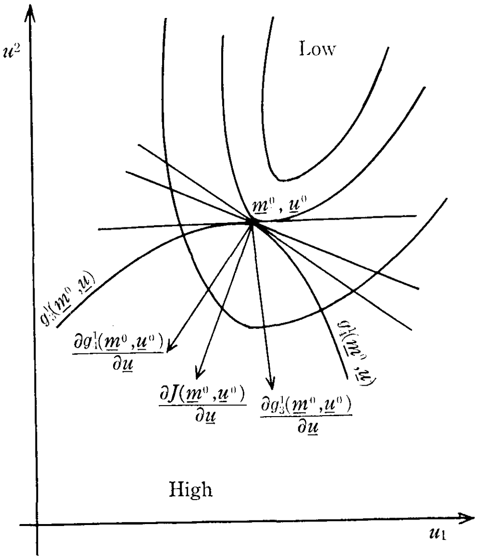


Fig. 4b.

we have

$$-\frac{\partial J}{\partial \underline{m}}(\underline{m}^0, \underline{u}^0)^T = \frac{\partial \underline{g}^1}{\partial \underline{m}}(\underline{m}^0, \underline{u}^0)^T \underline{\lambda}^{10} \qquad (\text{for some } \underline{\lambda}^{10} \geq \underline{0}) \qquad \text{(A. 3)}$$

$$\frac{\partial J}{\partial \underline{u}}(\underline{m}^0, \underline{u}^0)^T = \frac{\partial \underline{g}^1}{\partial \underline{u}}(\underline{m}^0, \underline{u}^0)^T \underline{\mu}^{10} \qquad (\text{for some } \underline{\mu}^{10} \geq \underline{0})\,. \qquad \text{(A. 4)}$$

If $\{\underline{m}^0, \underline{u}^0\}$ is an interior point of the constraint set then $\frac{\partial \underline{g}^1}{\partial \underline{m}}(\underline{m}^0, \underline{u}^0)$, $\frac{\partial \underline{g}^1}{\partial \underline{u}}(\underline{m}^0, \underline{u}^0)$ are both null. In this case, the point $\{\underline{m}^0, \underline{u}^0\}$ is obtained from $J(\underline{m}, \underline{u})$ independent of the constraints, so $\frac{\partial J}{\partial \underline{m}}(\underline{m}^0, \underline{u}^0)=\underline{0}$, $\frac{\partial J}{\partial \underline{u}}(\underline{m}^0, \underline{u}^0)=\underline{0}$. Then conditions (15) (16) (17) hold for $\underline{\lambda}^0=\underline{0}$.

Eq. (A. 3) (A. 4) may be written as

$$-\frac{\partial J}{\partial \underline{m}}(\underline{m}^0, \underline{u}^0)^T = \frac{\partial \underline{g}}{\partial \underline{m}}(\underline{m}^0, \underline{u}^0)^T \underline{\lambda}^0 \qquad (\text{for some } \underline{\lambda}^0 \geq \underline{0})$$

$$\frac{\partial J}{\partial \underline{u}}(\underline{m}^0, \underline{u}^0)^T = \frac{q\underline{g}}{\partial \underline{u}}(m^0, \underline{u}^0)^T \underline{\mu}^0 \qquad (\text{for some } \underline{\mu}^0 \geq \underline{0})$$

by adding zeros as components to $\underline{\lambda}^{10}$ and $\underline{\mu}^{10}$ corresponding to non-binding constraints in order to form $\underline{\lambda}^0$ and $\underline{\mu}^0$. Consequently,

$$\frac{\partial \Phi_1}{\partial \underline{m}}(\underline{m}^0, \underline{u}^0, \underline{\lambda}^0) = \frac{\partial J}{\partial \underline{m}}(\underline{m}^0, \underline{u}^0) + \underline{\lambda}^{0T} \frac{\partial \underline{g}}{\partial \underline{m}}(\underline{m}^0, \underline{u}^0) = \underline{0} \qquad (\text{for some } \underline{\lambda}^0 \geq \underline{0}) \qquad \text{(A. 5)}$$

$$\frac{\partial \Phi_2'}{\partial \underline{u}}(\underline{m}^0, \underline{u}^0, \underline{\mu}^0)=\frac{\partial J}{\partial \underline{u}}(\underline{m}^0, \underline{u}^0)-\underline{\mu}^{0T}\frac{\partial \underline{g}}{\partial \underline{u}}(\underline{m}^0, \underline{u}^0)=\underline{0} \qquad \text{(for some } \underline{\mu}^0\geq\underline{0}) \tag{A. 6}$$

where

$$\Phi_2'(\underline{m}, \underline{u}, \underline{\mu})=J(\underline{m}, \underline{u})-\underline{\mu}^T\underline{g}(\underline{m}, \underline{u})\,.$$

Meanwhile we must have from the original inequality constraint

$$\frac{\partial \Phi_1}{\partial \underline{\lambda}}(\underline{m}^0, \underline{u}^0, \underline{\lambda}^0)^T=\underline{g}(\underline{m}^0, \underline{u}^0)\geq\underline{0} \tag{A. 7}$$

$$-\frac{\partial \Phi_2'}{\partial \underline{\mu}}(\underline{m}^0, \underline{u}^0, \underline{\mu}^0)^T=\underline{g}(\underline{m}^0, \underline{u}^0)\geq\underline{0}\,. \tag{A. 8}$$

Moreover,

$$\frac{\partial \Phi_1}{\partial \underline{\lambda}}(\underline{m}^0, \underline{u}^0, \underline{\lambda}^0)\underline{\lambda}^0=\underline{\lambda}^{10T}\underline{g}^1(\underline{m}^0, \underline{u}^0)+\underline{\lambda}^{20T}\underline{g}^2(\underline{m}^0, \underline{u}^0)=0 \tag{A. 9}$$

$$-\frac{\partial \Phi_2'}{\partial \underline{\mu}}(\underline{m}^0, \underline{u}^0, \underline{\mu}^0)\underline{\mu}^0=-\underline{\mu}^{10T}\underline{g}^1(\underline{m}^0, \underline{u}^0)-\underline{\mu}^{20T}\underline{g}^2(\underline{m}^0, \underline{u}^0)=0\,. \tag{A. 10}$$

As necessary conditions of Problem 1, Eqs. (A. 5) through (A. 10) must be satisfied simultaneously. It is noticed, however, that Eq. (A. 7) and (A. 8) yield the same equation as well as Eq. (A. 9) and (A. 10), since $\underline{\lambda}$ and $\underline{\mu}$ are Lagrange multipliers corresponding to the same constraint $\underline{g}(\underline{m}, \underline{u})\geq\underline{0}$. We may consider two special cases for the above. If $\{\underline{m}^0, \underline{u}^0\}$ is an interior point of the constraint set, i.e., the set g^1 is empty, Theorem 1 holds for $\underline{\lambda}^0=\underline{\mu}^0=\underline{0}$. On the other hand, if the set g^2 is empty, that is $\underline{g}(\underline{m}^0, \underline{u}^0)=\underline{0}$, then Eq. (A. 5) (A. 6) and $\underline{g}(\underline{m}^0, \underline{u}^0)=\underline{0}$ must be solved simultaneously. There are, in total, (dim $\underline{m}$) plus (dim $\underline{u}$) plus (dim $\underline{g}$) equations for (dim $\underline{m}$) plus (dim $\underline{u}$) plus two times (dim $\underline{g}$) variables to be solved. But we can set $\underline{\mu}\equiv\underline{\lambda}$ in general. The fact that there exists $\underline{\mu}$ being equal to $\underline{\lambda}$ is proved as follows:

Let us consider $\begin{pmatrix} d\underline{m} \\ d\underline{u} \end{pmatrix}=\left(d\begin{pmatrix} \underline{m} \\ \underline{u} \end{pmatrix}\right)$ as one column vector. Since $-\frac{\partial J}{\partial \underline{m}}(\underline{m}^0, \underline{u}^0)d\underline{m}\geq 0$ and $\frac{\partial J}{\partial \underline{u}}(\underline{m}^0, \underline{u}^0)\geq\underline{0}$ from (A. 1), (A. 2) respectively, we can write

$$\left[-\frac{\partial J}{\partial \underline{m}}(\underline{m}^0, \underline{u}^0),\ \frac{\partial J}{\partial \underline{u}}(\underline{m}^0, \underline{u}^0)\right]\begin{pmatrix} d\underline{m} \\ d\underline{u} \end{pmatrix}\geq\underline{0}\,. \tag{A. 11}$$

If we consider the constraint qualification in $M\times U$, where M and U are the control space and the disturbance space, respectively, we must have

$$\left[\frac{\partial \underline{g}^1}{\partial \underline{m}}(\underline{m}^0, \underline{u}^0),\ \frac{\partial \underline{g}^1}{\partial \underline{u}}(\underline{m}^0, \underline{u}^0)\right]\begin{pmatrix} d\underline{m} \\ d\underline{u} \end{pmatrix}\geq\underline{0} \tag{A. 12}$$

instead of Eq. (13), (14). Eq. (A. 11) must hold for all differentials $\begin{pmatrix} d\underline{m} \\ d\underline{u} \end{pmatrix}$ satisfying Eq. (A. 12). Therefore, by Farkas' Lemma,

$$\left[-\frac{\partial J}{\partial \underline{m}}(\underline{m}^0, \underline{u}^0),\ \frac{\partial J}{\partial \underline{u}}(\underline{m}^0, \underline{u}^0)\right]^T = \left[\frac{\partial \underline{g}^1}{\partial \underline{m}}(\underline{m}^0, \underline{u}^0),\ \frac{\partial \underline{g}^1}{\partial \underline{u}}(\underline{m}^0, \underline{u}^0)\right]^T \underline{\lambda}^{10}$$

for some $\underline{\lambda}^{10} \geq \underline{0}$. By adding zero components to $\underline{\lambda}^{10}$, we have

$$\left[-\frac{\partial J}{\partial \underline{m}}(\underline{m}^0, \underline{u}^0),\ \frac{\partial J}{\partial \underline{u}}(\underline{m}^0, \underline{u}^0)\right] = \left[\frac{\partial \underline{g}}{\partial \underline{m}}(\underline{m}^0, \underline{u}^0),\ \frac{\partial \underline{g}}{\partial \underline{u}}(\underline{m}^0, \underline{u}^0)\right]^T \underline{\lambda}^0 \qquad \underline{\lambda}^0 \geq \underline{0}. \tag{A. 13}$$

The dimension of $\underline{\lambda}$ is equal to the number of columns of the transposed matrix $\left(\frac{\partial \underline{g}}{\partial \underline{m}}, \frac{\partial \underline{g}}{\partial \underline{u}}\right)^T$, which is the same as the dimension of $\underline{g}(\underline{m}, \underline{u})$. Eq. (A. 13) may be written separately as Eq. (A. 5) and (A. 6) in which $\underline{\mu}$ is replaced by $\underline{\lambda}$. Derivation of condition (17) is made in the $M \times U$ space in the manner analogous to derivation of Eq. (A. 9), (A. 10).

Since $\underline{\mu} \equiv \underline{\lambda}$, necessary conditions Eq. (A. 5) through (A. 10) are stated simply as Eq. (15) (16) and (17).

Appendix B. Proof of Corollary 1

Applying Theorem 1 to Lagrangian functions

$$\Phi_1'(\underline{m}, \underline{u}, \underline{\lambda}) = J(\underline{m}, \underline{u}) + \underline{\lambda}_1^T \underline{g}_1(\underline{m}) + \underline{\lambda}_2^T \underline{g}_2(\underline{u})$$

$$\Phi_2'(\underline{m}, \underline{u}, \underline{\lambda}) = J(\underline{m}, \underline{u}) - \underline{\lambda}_0^T \underline{g}_1(\underline{m}) - \underline{\lambda}_2^T \underline{g}_2(\underline{u})$$

we obtain necessary conditions Eq. (21) through (24) corresponding the Eq. (15) through (17). Since M and U are separated domains of $\underline{m}$ and $\underline{u}$, respectively, Φ_1 and Φ_2 are equivalently expressed in the simpler forms of Eq. (25) and (26).

Appendix C. Proof of Theorem 2

From the statement (ii), we have

$$\Phi_1(\underline{m}, \underline{u}^0, \underline{\lambda}^0) \leq \Phi_1(\underline{m}^0, \underline{u}^0, \underline{\lambda}^0) + \frac{\partial \Phi_1}{\partial \underline{m}}(\underline{m}^0, \underline{u}^0, \underline{\lambda}^0)(\underline{m} - \underline{m}^0) \tag{C. 1}$$

$$\Phi_2(\underline{m}^0, \underline{u}, \underline{\lambda}^0) \geq \Phi_2(\underline{m}^0, \underline{u}^0, \underline{\lambda}^0) + \frac{\partial \Phi_2}{\partial \underline{u}}(\underline{m}^0, \underline{u}^0, \underline{\lambda}^0)(\underline{u} - \underline{u}^0). \tag{C. 2}$$

Following the procedure used by Kuhn and Tucker [6] and applying Eq. (C. 1) (15) and (17) in turn, one has that

$$J(\underline{m}, \underline{u}^0) + \underline{\lambda}^{0T} \underline{g}(\underline{m}, \underline{u}^0) \leq J(\underline{m}^0, \underline{u}^0) \qquad (\text{for all } \underline{m}).$$

But $\underline{\lambda}^{0T} \underline{g}(\underline{m}, \underline{u}^0) \geq \underline{0}$ for all $\underline{m}$ satisfying $\underline{g}(\underline{m}, \underline{u}^0) \geq \underline{0}$. Hence, $J(m, \underline{u}^0) \leq J(\underline{m}^0, \underline{u}^0)$ for all $\underline{m}$ satisfying the constraint $\underline{g}(\underline{m}, \underline{u}^0) \geq \underline{0}$. By applying Eq. (C. 2) (16) and (17) in turn,

$$J(\underline{m}^0, \underline{u}) - \underline{\lambda}^{0T} \underline{g}(\underline{m}^0, \underline{u}) \geq J(\underline{m}^0, \underline{u}^0) \qquad (\text{for all } \underline{u}).$$

But $-\lambda^{0T}g(m^0, u)\leq 0$ for all u satisfying $g(m^0, u)\geq 0$. Hence $J(m^0, u)\geq J(m^0, u^0)$ for all u satisfying $g(m^0, u)\geq 0$. These prove Theorem 2.

Appendix D. Proof of Theorem 3

If $J(m, u)$ and $g(m, u)$ are concave with respect to m then $\Phi_1(m, u, \lambda)=J(m, u)+\lambda^T g(m, u)$ for $\lambda\geq 0$ must also be concave with respect to m. Thus, condition (C. 1) holds for any m^0 and m. If $g(m, u)$ is concave with respect to u, then $-g(m, u)$ is convex with respect to u. Since $J(m, u)$ is also convex with respect to u, then for any $\lambda\geq 0$ $\Phi_2(m, u, \lambda)=J(m, u)-\lambda^T g(m, u)$ must also be convex with respect to u. Thus, condition (C. 2) holds for all u^0 and u.

Therefore, Theorems 1 and 2 provide both necessary and sufficient conditions that $\{m^0, u^0\}$ be a solution for Problem 1.

Since $\Phi_1(m, u, \lambda)$ and $\Phi_2(m, u, \lambda)$ are linear with respect to λ, they are both convex and concave with respect to λ. Hence, convexity of Φ_1 and concavity of Φ_2 in λ are satisfied automatically.

Hence, Lemma 1 and Lemma 2 provide both necessary and sufficient conditions that $\{m^0, u^0, \lambda^0\}$ gives a solution for the saddle value problem. This completes the proof.

Appendix E. Proof of Theorem 5

$\Phi_1(m, u, \lambda)$ is concave in m for any $\lambda\geq 0$:

$$\Phi_1(z, u, \lambda)-\Phi_1(m, u, \lambda)\leq \frac{\partial \Phi_1}{\partial m}(m, u, \lambda)(z-m)\,.$$

Now $\frac{\partial \Phi_1}{\partial m}(m, u, \lambda)=0$ for any $\{m, u, \lambda\}$ satisfying Eq. (45. a, b), so that if $\{m^1, u, \lambda\}$ and $\{m^2, u, \lambda\}$ both satisfs Eq. (7. a, b) we have both $\Phi_1(m^1, u, \lambda)-\Phi_1(m^2, u, \lambda)\geq 0$ and the reverse inequality, whence $\Phi_1(m^1, u, \lambda)=\Phi_2(m^2, u, \lambda)$. In other words, $\Phi_1(m, u, \lambda)$ is independent of u for $\{m, u, \lambda\}$ satisfying (45. a, b).

Consequently, for fixed $u=u^0$

$$\begin{aligned}\Phi_1(m^0, u^0, \lambda^0)&=\min_{\lambda}\{\Phi_1(m^0, u^0, \lambda)\,|\,\lambda\geq 0\}\\ &\leq\min_{\lambda}\{\Phi_1(m^0, u^0, \lambda)\,|\,\{m^0, u^0, \lambda\}\text{ satisfying (45. a, b)}\}\\ &=\min_{\lambda}\{\Phi_1(m, u^0, \lambda)\,|\,\{m, u^0, \lambda\}\text{ satisfying (45. a, b)}\}\\ &\leq\Phi_1(m^0, u^0, \lambda^0)\end{aligned}$$

so that $\Phi_1(m^0, u^0, \lambda^0)$ is the minimal objective value of $\Phi_1(m, u^0, \lambda)$ for the dual problem (44) (45).

Next we can say

$$\Phi_1(\underline{m}^0, \underline{u}^0, \underline{\lambda}^0)=J(\underline{m}^0, \underline{u}^0)+\sum_{i=1}^{r}\lambda_i^0 g_i(\underline{m}^0, \underline{u}^0)=J(\underline{m}^0, \underline{u}^0)$$

because each $\lambda_i^0 g_i(\underline{m}^0, \underline{u}^0)=0$; if this last statement did not hold, we would have $\lambda_i^0>0$, $g_i(\underline{m}^0, \underline{u}^0)>0$ for some i, and then $\Phi_1(\underline{m}^0, \underline{u}^0, \underline{\lambda}^0)$ could be increased by increasing λ_i^0, in contradiction to the saddle point property that $\{\underline{m}^0, \underline{\lambda}^0\}$ is a saddle point of Eq. (49) (given $\underline{u}^0$).

Therefore, given $\underline{u}^0$ a solution to the dual problem (44) (45) agrees with a solution of the primal $J(\underline{m}^0, \underline{u}^0)$.

On the other hand, $\Phi_2(\underline{m}, \underline{u}, \underline{\mu})$ is convex in $\underline{u}$ for any $\underline{\mu}\geq\underline{0}$. Therefore, by the manner analogous to the above discussion, $\Phi_2(\underline{m}^0, \underline{u}^0, \underline{\mu}^0)$ is the maximal objective value of $\Phi_1(\underline{m}^0, \underline{u}, \underline{\mu})$ for the dual problem (46) (47) for fixed $\underline{m}=\underline{m}^0$. Furthermore

$$\Phi_2(\underline{m}^0, \underline{u}^0, \underline{\mu}^0)=J(\underline{m}^0, \underline{u}^0)-\sum_{i=1}^{r}\mu_i^0 g_i(\underline{m}^0, \underline{u}^0)=J(\underline{m}^0, \underline{u}^0)\,.$$

But by the Kuhn-Tucker Theorem for the game there exists $\underline{\mu}^0\equiv\underline{\lambda}^0$ such that if $\{\underline{m}^0, \underline{u}^0\}$ solves the primal problem, the system of (48) and (49) is satisfied simultaneously. Therefore, if $\{\underline{m}^0, \underline{u}^0\}$ solves the primal, there exists $\underline{\lambda}^0\equiv\underline{\mu}^0$ such that $\{\underline{m}^0, \underline{u}^0, \underline{\lambda}^0\}$ solves the dual consisting of a system of (44) (45) and (46) (47) and both extrema are equal.

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