Title	Vibration of a visco-elastic circular cylinder about its central axis
Sub Title	
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Publisher	慶応義塾大学藤原記念工学部
Publication year	1969
Jtitle	Proceedings of the Fujihara Memorial Faculty of Engineering Keio University (慶応義塾大学藤原記念工学部研究報告). Vol.22, No.90 (1969.) ,p.137(23)- 148(34)
JaLC DOI	
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Notes	
Genre	Departmental Bulletin Paper
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=KO50001004-00220090-0023

Vibration of a Visco-elastic Circular Cylinder about its Central Axis

(Received January 22, 1970)

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Abstract

In this paper, we take up the problem of vibration of hollow cylinder of visco-elastic material. Outer surface is attached to rigid wall, with no slip. To inner surface is connected, with no slip, a spindle which has fly-wheels at both ends. The whole dynamical system is subjected to action of external turning moment which varies with time t, in the form of $T_0 \sin \omega t$. The author discussed the solution of this problem of vibration, by using Laplace transformation. For the case of steady oscillation, the author has given expression which gives amplitude of vibration of this hollow cylinder of visco-elastic material. Some numerical values of coefficients appearing in this solution are also given, which may facilitate evaluation of individual cases.

I. Introduction

The author has, in the previous paper entitled "On the Vibration of an Elastic Circular cylinder about its Central Axis"[1], reported some results of analysis about small vibrations of an elastic circular cylinder which is arranged as shown in Fig. 1.

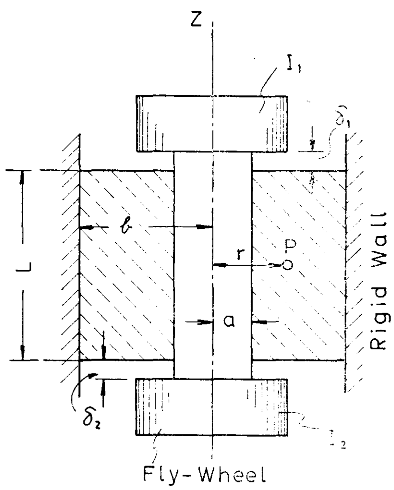

Fig. 1.

As shown in this Fig. 1., we considered an elastic hollow circular cylinder, whose inner and outer radii are a and b respectively. The inner surface of this cylinder is closely attached to a circular shaft, while the outside surface is connected to a rigid wall. To this shaft or spindle of radius a fly-wheels I_1 and I_2 are fixed at both ends. The connection of the elastic cylinder to outside rigid wall and central shaft are assumed to be made up in

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[1] This PROCEEDINGS, Vol. 21, No. 84, pp 12~20, 1968.

such a way that no tangential slip occurs at the contact surfaces. The problem studied in author's paper(1) was the analytical study about forced- and free-vibrations of this dynamical systeml for the case of small amplitudes of vibration. It was also assumed that the elastic material follows the law of perfect elasticity.

In the present paper, the author has made an analytical study of quite the same problem, only difference being that, in the case of present paper, the material composing the elastic cylinder is assumed to obey a very simple law of visco-elasticity. The tangential displacement u_θ only is taken into consideration, which is a function of radial distance r and time t. This inference is an approximate one, which holds good only when the cylinder is comparatively long, as was already pointed out in the previous paper.

II. Fundamental equation

In what follows, we shall use the following notations, which are almost the same as those in the author's previous paper:

a, b, L=inner- and outer-radius and length of the hollow visco-elastic cylinder; r=radial position of any point in the cylinder; I_1, I_2=moment of inertia of fly-wheel, which is attached to central shaft or spindle; u_θ(or, briefly u)=tangential displacement of small magnitude, ρ=density of the material composing the cylinder; G=modulus of rigidity; $T_\theta(t)$=bodily acting turning torque.

The tangential displacement u_θ is considered to be a function of r and t. The modulus of rigidity G, for the case of visco-elastic material, must be regarded as a function of time t, corresponding to so called effect of heredity. The equation of motion for the case of small oscillation, of our dynamical system of Fig. 1, may be written in the following form:

$$\rho \frac{\partial^2 u_\theta(r, t)}{\partial t^2} + T_\theta(t)$$

$$= \frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}\cdot r\right\}\int_0^t G(t-\xi)\frac{\partial u_\theta(r, \xi)}{\partial \xi}d\xi\,. \tag{1}$$

This equation (1) is not rigorous, but an approximate equation which holds good only for a slender cylinder. In the following discussion of this equation (1), further simplifications are made as follows:

(*a*) We write simply $u(r, t)$, instead of $u_\theta(r, t)$

(*b*) Assuming that no body-force (or moment) is acting, we put

$$T_\theta(t) \equiv 0$$

(*c*) We assume that the modulus of rigidity $G(t)$ is given by the formula

$$G(t) = a_1 + b_1 \exp(-\mu t) \tag{2}$$

where a_1; b_1 and μ are positive constants which are determined by the visco-elastic property of the material. The similar argument as given below may be made for a more complicated law of visco-elasticity, for example,

$$G(t)=a_1+\sum_{i=1}^{n} b_i \exp(-\lambda_i t).$$

(*d*) Restricting ourselves to the case of start (at $t=0$) from the state of resting, we have, at $t=0$,

$$u(r,\ t)=u(r,\ 0)=0\ ;\ \frac{\partial u(r,\ t)}{\partial t}=\dot{u}(r,\ t)=0.$$

But the solutions for other cases can be treated in the similar way as given below, arriving at more complicated results.

Taking Laplace transforms of both sides of our equation (1), that is, by making an operation

$$\int_0^\infty e^{-st}[\cdots]\,dt$$

on both sides of equation (1). we obtain

$$\begin{aligned}&\rho[-\dot{u}(r,\ 0)-su(r,\ 0)+s^2\bar{u}(r,\ s)]\\&=\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}\cdot r\right\}\Big[G(0)\bar{u}(r,\ s)\\&\quad-\bar{G}(s)u(r,\ 0)+\bar{H}(s)\bar{u}(r,\ s)\Big].\end{aligned}\tag{3}$$

In this equation $\bar{u}(r,\ s)$ and $\bar{G}(s)$ are Laplace transforms of $u(r,\ t)$ and $G(t)$ respectively. $\bar{H}(s)$ is the Laplace transforms $G'(t)$.

Taking into account the above mentioned condition (*d*), and writing for shortness

$$h(s)=G(0)+\bar{H}(s)$$

the above equation (3) is reduced into the following form:

$$\rho s^2\bar{u}(r,\ s)=\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}\cdot r\right\}\Big[h(s)\bar{u}(r,\ s)\Big]$$

or

$$\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}\cdot r\right\}\bar{u}(r,\ s)+\lambda^2\bar{u}(r,\ s)=0\tag{4}$$

where we have put

$$\lambda^2=-\rho s^2/h(s).\tag{5}$$

Thus, we see that $P=\bar{u}(r, s)$, regarded as a function of r, must satisfy the following equation, which is the differential equation of Bessel functions:

$$\frac{d^2P}{dr^2}+\frac{1}{r}\frac{dP}{dr}+\left(\lambda^2-\frac{1}{r^2}\right)P=0.$$

Hence, we have

$$P=\bar{u}(r,\ s)=AJ_1(\lambda r)+BX_1(\lambda r) \tag{6}$$

where A and B are arbitrary constants. Contrary to the case of the author's previous paper, we must here regard λ as a complex variable which is connected to the variable s by equation (5). From this view point, it would be natural to represent the solution (6) in terms of $H^{(1)}$, $H^{(2)}$ instead of J_1, Y_1. Nevertheless we choose the expression (6) in order to facilitate comparison with the results of the previous paper.

III. Boundary conditions

In order to obtain the solution to our dynamical problem, we impose the following boundary conditions to the tangentical displacement u or u_θ:

(A) At the outer radius $r=b$, $u=0$. According to this condition, we must have at $r=b$,

$$P=\bar{u}(r,\ s)=0,$$

this is

$$AJ_1(\lambda b)+BY_1(\lambda b)0.$$

Thus, the solution (6) may be written as

$$P=A[J_1(\lambda r)-MY_1(\lambda r)] \tag{7}$$

where we put

$$M=J_1(\lambda b)/Y_1(\lambda b).$$

(B) At the inner radius $r=a$, the total amount of turning torque exerted by the cylinder is considered to be balanced by the turning moment caused by effect of inertia of fly-wheels I_1 and I_2. The turning moment T_a exerted by the cylinder is given by

$$T_a=2\pi a^2L[\tau_{r\theta}(\text{at } r=a)]$$

$$=2\pi a^2L[\mu]\cdot\left[\frac{\partial u(r,\ t)}{\partial r}-\frac{u(r,\ t)}{r}\right](r=a)$$

where $[\mu]$ is an operator giving effect of visco-elasticity. Thus, for a function $F(t)$, by $[\mu]F(t)$ we mean that

$$[\mu]\cdot F(t)=\int_0^t G(t-\xi)\frac{\partial F}{\partial \xi}d\xi.$$

Thus, we have

$$T_a(t)=2\pi a^2L\int_0^t G(t-\xi)\frac{\partial Q(a,\ \xi)}{\partial \xi}d\xi$$

where we put for shortness

$$Q(a,\ t)=\left[\frac{\partial u(r,\ t)}{\partial r}-\frac{u(r,\ t)}{r}\right](r=a).$$

Taking Laplace-transform of $T_a(t)$, we have

$$\overline{T}_a(s)=2\pi a^2 L\left[h(s)\overline{Q}(a,\ s)-\overline{G}(s)Q(a.\ 0)\right]$$

where

$$\overline{Q}(a,\ s)=\left[\frac{\partial \bar{u}(r,\ s)}{\partial r}-\frac{\bar{u}(r,\ s)}{r}\right](r=a).$$

If we confine ourselves to the case of start from the state of resting, we have

$$Q(a,\ 0)=0.$$

Also, we have, by (7),

$$\begin{aligned}&\left[\frac{\partial \bar{u}(r\ s)}{\partial r}-\frac{\bar{u}(r,\ s)}{r}\right](r=a)\\&=A\left[\lambda J_1'(\lambda a)-\frac{1}{a}J_1(\lambda a)\right]\\&-MA\left[\lambda Y_1'(\lambda a)-\frac{1}{a}Y_1(\lambda a)\right].\end{aligned}$$

Thus, the expression for $\overline{T}_a(s)$ becomes

$$\begin{aligned}\overline{T}_a(s)=2\pi a^2 Lh(s)A&\left[\left\{\lambda J_1'(\lambda a)-\frac{1}{a}J_1(\lambda a)\right\}\right.\\&\left.-M\left\{\lambda Y_1'(\lambda a)-\frac{1}{a}Y_1(\lambda a)\right\}\right].\end{aligned}\tag{8}$$

(*C*) The equation of turning moment at $r=a$, being given by

$$I_r\frac{\partial^2}{\partial t^2}\left[\frac{u(r,\ t)}{r}\right](r=a)-T_a=T_0\sin\omega t,\tag{9}$$

its Laplace-transform is :

$$\begin{aligned}&\frac{I_r}{a}\left[-\dot{u}(r,\ 0)-su(r,\ 0)+s^2\bar{u}(r,\ s)\right](r=a)\\&-\overline{T}_a(s)=\frac{T_0}{s^2+\omega^2}.\end{aligned}\tag{10}$$

Or, taking the case of $\dot{u}(r,\ 0)=0$, $u(r,\ 0)=0$, and putting the above expression (8) into this equation (10), we obtain

$$\begin{aligned}&\frac{T_0}{A}\frac{\omega}{s^2+\omega^2}\\&=\left[\frac{I_r}{a}s^2+2\pi aLh(s)\right]\left[J_1(\lambda a)-MY_1(\lambda a)\right]\\&-2\pi a^2L\lambda h(s)\left[J_1'(\lambda a)-MY_1'(\lambda a)\right].\end{aligned}\tag{11}$$

From this equation, the unknown constant A is given in the following form:

$$A=\frac{\omega T_0}{s^2+\omega^2}\cdot\frac{1}{\Phi(\lambda)} \tag{12}$$

where we put for convenience

$$\Phi(\lambda)=\left[\frac{I_r}{a}s^2+2\pi aLh(s)\right]\cdot\left[J_1(\lambda a)-MY_1(\lambda a)\right]$$
$$-2\pi a^2L\lambda h(s)[J_1'(\lambda a)-MY_1'(\lambda a)]. \tag{13}$$

In a special case in which

$$a_1=G_{00}, \qquad b_1=0,$$

that is, in the case of perfect elasticity, we have

$$h(s)=G_{00}, \qquad \lambda^2=-\frac{\rho}{G_{00}}s^2, \qquad s^2=-\lambda^2\frac{G_{00}}{\rho}.$$

Hence, in that case

$$\Phi(\lambda)=\frac{1}{Y_1(\lambda b)}\begin{vmatrix} J_1(\lambda b), & Y_1(\lambda b) \\ U, & V \end{vmatrix}$$

$$U=\frac{I_r}{a}\frac{G_{00}}{\rho}\lambda^2J_1(\lambda a)$$
$$+2\pi aLG_{00}[\lambda aJ_1'(\lambda a)-J_1(\lambda a)].$$
$$V=\frac{I_r}{a}\frac{G_{00}}{\rho}\lambda^2Y_1(\lambda a)$$
$$+2\pi aLG_{00}[\lambda aY_1'(\lambda a)-Y_1(\lambda a)].$$

Thus, the expression $Y_1(\lambda b)\Phi(\lambda)$ corresponds to the determinantal expression D given in the author's previous paper.

Lastly, we remark that for value of $G(t)$ given by (2), we have

$$h(s)=G(0)+\bar{\bar{H}}(s)$$
$$=a_1+\frac{s}{s+\mu}b_1,$$
$$\lambda^2=-\rho\frac{s^2}{\left[a_1+\left(\frac{s}{s+\mu}\right)b_1\right]}.$$

We may write, for convenience,

$$\lambda^2=-\frac{\rho}{G_{00}}s^2+k$$

or

$$-s^2=\frac{G_{00}}{\rho}(\lambda^2-k).$$

IV. Solution of our problem

Since the unknown constant A is obtained in the form of (12), the value of $\bar{u}(r, s)$ is given by the equation (7). Actual displacement $u(r, t)$ is to be obtained from $\bar{u}(r, s)$ by applying so-called inversion formula. If we write, for convenience,

$$F(s)=\frac{T_0\omega}{\Phi(\lambda)}[J_1(\lambda r)-MY_1(\lambda r)]$$

with

$$M=J_1(\lambda b)/Y_1(\lambda b),$$

we may write

$$u(r, t)=\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\frac{F(s)}{s^2+\omega^2}e^{st}ds, \tag{14}$$

where the integration is to be carried out about a path of integration which consists of a straight line parallel to the imaginary axis of complex s-plane. We note that, when the absolute value of complex variable s becomes very great, we have

$$|\lambda a| \text{ very great}, \quad |\lambda r| \text{ very great},$$

$$h(s)\to a_1+b_1=G_{00}$$

$$\lambda^2=-\frac{\rho}{a_1+b_1}s^2.$$

Therefore, we have for a very large value of $|s|$,

$$\Phi(\lambda)=\left[\frac{I_r}{a}s^2+2\pi aG_{00}\right]\frac{1}{\sqrt{\lambda a}}-2\pi aL\frac{1}{\sqrt{\lambda a}}$$

that is, if $I_r\neq 0$,

$$\Phi(\lambda)\to\frac{I_r}{a}s^{3/2}.$$

Thus, we see that for a very large value of $|s|$, we have

$$F(s)=0\left[s^{-3/2}\frac{1}{\sqrt{\lambda r}}\right]=0[s^{-2}],$$

while, if $I_r=0$, we have

$$\Phi(\lambda)=0\left[\frac{1}{\sqrt{\lambda a}}\right]=0\left[\frac{1}{\sqrt{s}}\right]$$

$$F(s)=0\left[\sqrt{s}\frac{1}{\sqrt{s}}\right]=0[1].$$

In any way, we have

$$|F(s)|<\text{a const. as } |s|\to\infty.$$

From this inference, we remark that we may use Jordan's lemma for the integrand of equation (14), and so when $R \to \infty$ we have,

$$\int_c \frac{F(s)}{s^2+\omega^2} e^{st}\, ds \to 0$$

where c is a contour line made up of semi-circle of radius R, which lies in the left half of the complex plane s, and whose center is situated at the origin. Thus we see that the contour integral of eq. (14) can be evaluated by means of calculus of residues.

Poles of the function

$$\frac{F(s)}{s^2+\omega^2}$$

of a complex variable s are as follows:

(1) There are two poles

$$s = \pm i\omega.$$

(2) In the case of $b_1 = 0$ in the expression (2), that is, in the case of perfectly elastic material, we have. as shown in the previous paper, an infinite number of roots $k_1, k_2 \cdots$ of the equation $\Phi(k) = 0$. For the case of the present paper, we observe that the function $\Phi(\lambda)$ is an analytical expression about b_1. Therefore. we infer that, at least for a sufficiently small value of $|b_1|$, there exists an infinite number of roots $K_1, K_2 \cdots$ of the equation

$$\Phi(\lambda) = 0,$$

which approaches to the roots $k_1, k_2, \cdots$, when we make $|b_1| \to 0$.

V. Case of steady oscillation

For the case of steady oscillation under the action of external oscillatory torque of

$$T_0 \sin \omega t,$$

with the effect of initial conditions being disregarded, we take only two poles

$$s = \pm i\omega$$

of the function

$$\frac{F(s)}{s^2+\omega^2}$$

into account. Thus, we arrive at the following expression for $u(r, t)$:

$$u(r,t) = \frac{1}{2i\omega}[F(i\omega)\exp(i\omega t) - F(-i\omega t)\exp(-i\omega t)]. \tag{15}$$

We note also that

$$\text{for} \quad s=i\omega \qquad \lambda a=\lambda_0 a(1+i\varepsilon),$$

$$\text{for} \quad s=-i\omega \quad \lambda a=\lambda_0 a(1-i\varepsilon),$$

we put

$$\lambda_0=\sqrt{\rho/G_{00}}\ \omega$$

$$\varepsilon=\frac{1}{2}\frac{\mu}{\omega}\frac{b_1}{a_1+b_1}.$$

Especially, if the value $|\varepsilon|$ is very small in comparison with unity, we may approximately put

$$F(i\omega)=\mathring{F}(\lambda_0 a)+(\lambda_0 a)\mathring{F}'(\lambda_0 a)\,\varepsilon i$$

$$F(-i\omega)=\mathring{F}(-\lambda_0 a)+(\lambda_0 a)\mathring{F}'(-\lambda_0 a)\,\varepsilon i \text{ [see Note below].}$$

(I) The first parts correspond to the case of perfectly elastic material, and the vibration is taking place in phase with the external moment

$$T_0 \sin \omega t.$$

The second part gives effect of visco-elasticity, and consists of oscillation with the amplitude

$$\frac{\lambda_0 a}{\omega}\mathring{F}(\lambda_0 a)\ \varepsilon$$

the vibration taking place, with phase angle lagging by 90° to that of the external moment $T_0 \sin \omega t$.

NOTE: $\mathring{F}(\lambda_0 a)$, $\mathring{F}'(\lambda_0 a)$ represent $F(i\omega)$ and $F'(i\omega)$, wherein we put $\lambda_0 a(1+i\varepsilon)$ instead of λa,

$$\mathring{F}(-\lambda_0 a)=\mathring{F}(\lambda_0 a)$$

$$-(\lambda_0 a)\mathring{F}'(-\lambda_0 a)=(\lambda_0 a)\mathring{F}'(\lambda_0 a).$$

Therefore, the result of (15) may also be given in the following form:

$$\begin{aligned} u(r,\ t)=&\frac{1}{2i\omega}[\mathring{F}(\lambda_0 a)\{\exp(i\omega t) \\ &-\exp(-i\omega t)\} \\ &+(\lambda_0 a)\mathring{F}'(\lambda_0 a)\{\exp(i\omega t) \\ &+\exp(-\omega t)\}] \\ =&\frac{1}{\omega}[\mathring{F}(\lambda_0 a)\sin\omega t \\ &+(\lambda_0 a)\mathring{F}'(\lambda_0 a)\,\varepsilon\cos\omega t]. \end{aligned} \tag{16}$$

This solution of (16) shows us that the steady oscillation consists of two parts. The first part of amplitude

$$\frac{1}{\omega}\mathring{F}(\lambda_0 a)$$

corresponds to oscillation in the case of previous paper.

VI. Numerical example

In order to obtain numerical values of amplitude of oscillation, we must know numerical values of the function

$$F(s)=\frac{T_0\omega}{\Phi(\lambda)}[J_1(\lambda r)-MY_1(\lambda r)].$$

Especially, at the inside radius $r=a$, we have

$$F_a(s)=\frac{T_0\omega}{\Phi(\lambda)}[J_1(\lambda a)-MY_1(\lambda a)]$$

and corresponding value of $u(a, t)$ may be obtained by eq. (15), wherein we put $F_a(i\omega)$ instead of $F(i\omega)$. In other words, we obtain the value of $u(a, t)$ from eq. (16) by putting $r=a$ into it.

Thus, it will be seen that the numerical value of $u(a, t)$ may be obtained easily if we know values of the following expression:

$$F_a(a)=\frac{T_0\omega}{\Phi(\lambda)}[J_1(\lambda a)-MY_1(\lambda a)]$$

or

$$\begin{aligned}\frac{T_0\omega}{F_a(s)}&=\frac{\Phi(\lambda)}{J_1(\lambda a)-MY_1(\lambda a)}\\&=2\pi aG_{00}L-\frac{G_{00}}{\rho}\frac{I_r}{a}(\lambda^2-k)\\&\quad-2\pi a^2\lambda G_{00}L\left\{\frac{J_1'(\lambda a)-MY_1'(\lambda a)}{J_1(\lambda a)-MY_1(\lambda a)}\right\}\\&=2\pi aG_{00}L-\frac{G_{00}}{\rho}\frac{I_r}{a}(\lambda^2-k)\\&\quad-2\pi aG_{00}L\frac{R-MS}{N-M}\end{aligned}\tag{17}$$

where we put, for shortness,

$$M=J_1(\lambda b)/Y_1(\lambda b)$$
$$N=J_1(\lambda a)/Y_1(\lambda a)$$
$$R=J_1'(\lambda a)/Y_1(\lambda a)$$
$$S=Y_1'(\lambda a)/Y_1(\lambda a).$$

Thus, if we know numerical values of M, N, R and S for given values of a and b, it will be an easy mater to obtain numerical values of expression (17) for given values of G_{00}, L, I_r, etc.

Some results of numerical estimation of values of M, N, R and S are reported below. They correspond to the case at which we have

$$\lambda_0 = 1.00$$

$$\lambda_0 a = 0.10,\ 0.20,\ 0.30,\ 0.40,\ 0.50.$$

It is assumed that we may put

$$\lambda b = \lambda_0 b\,[1 + i\varepsilon]$$

$$\lambda a = \lambda_0 a\,[1 + i\varepsilon]$$

where $|\varepsilon|$ is very small in comparison with unity. In that case we have

$$J_1(\lambda b) = J_1(\lambda_0 b) + i\varepsilon(\lambda_0 b) J_1'(\lambda_0 b)$$

$$Y_1(\lambda b) = Y_1(\lambda_0 b) + i\varepsilon(\lambda_0 b) Y_1'(\lambda_0 b)$$

$$J_1(\lambda a) = J_1(\lambda_0 a) + i\varepsilon(\lambda_0 a) J_1'(\lambda_0 a)$$

$$Y_1(\lambda a) = Y_1(\lambda_0 a) + i\varepsilon(\lambda_0 a) Y_1'(\lambda_0 a)$$

$$J_1'(\lambda a) = J_1'(\lambda_0 a) + i\varepsilon(\lambda_0 a) J_1''(\lambda_0 a)$$

$$Y_1'(\lambda a) = Y_1'(\lambda_0 a) + i\varepsilon(\lambda_0 a) Y_1''(\lambda_0 a).$$

We could obtain numerical values by applying known formula below and making use of table of Bessel functions.

$$J_1'(z) = \frac{1}{2}[J_0(z) - J_2(z)]$$

$$J_1''(z) = \left(\frac{1}{z^2} - 1\right) J_1(z) - \frac{1}{z} J_1'(z)$$

$$Y_1'(z) = \frac{1}{2}[Y_0(z) - Y_2(z)]$$

$$Y_1''(z) = \left(\frac{1}{z^2} - 1\right) Y_1(z) - \frac{1}{z} Y_1'(z).$$

The results obtained are given in Table 1.

Table 1.

Values of M, N, R and S

(*A*) Value of M

$\lambda_0 b = 1.00$	$M = -0.663 - 1\,233\,(i\varepsilon)$

(*B*) Value of N

$\lambda_0 a = 0.10$	$N = -0.00757\,[1 + 1.957(i\varepsilon)]$
0.20	$-0.0299\ [1 + 1.926(i\varepsilon)]$
0.30	$-0.0646\ [1 + 1.892(i\varepsilon)]$
0.40	$-0.111\ [1 + 1.823(i\varepsilon)]$
0.50	$-0.1645\ [1 + 1.951(i\varepsilon)]$

(*C*) Values of R

$\lambda_0 a = 0.10$	$R = -0.0747\,[1 + 0.950(i\varepsilon)]$

0.20	$-0.1482\,[1+0.903(\varepsilon i)]$
0.30	$-0.2150\,[1+0.806(i\varepsilon)]$
0.40	$-0.2643\,[1+0.739(i\varepsilon)]$
0.50	$-0.316\ \ [1+0.821(i\varepsilon)]$

(*D*) **Values of** S

$\lambda_0 a=0.10$	$-9.55\ \ [1-1.053(i\varepsilon)]$
0.20	$-4.68\ \ [1-1.10\ (i\varepsilon)]$
0.30	$-2.975\,[1-1.126(i\varepsilon)]$
0.40	$-2.157\,[1-1.107(i\varepsilon)]$
0.50	$-2.025\,[1-0.718(i\varepsilon)]$

From these values, numerical values of the coefficient

$$f=f_1+(i\varepsilon)\,f_2=\frac{R-M}{N-M}$$

which is contained in the formula (17) have been estimated, their results being shown in Table 2, and also shown as graphs in Fig. 2. We also here remark that, in the present state, we are to take

$$-s^2=\frac{G_0}{\rho}\,(\lambda^2-k)$$

where $-s^2=\omega^2$. That is, we are to take in the present stage

$$\lambda^2-k=\frac{\rho}{G_{00}}\,\omega^2$$

in the formula (17).

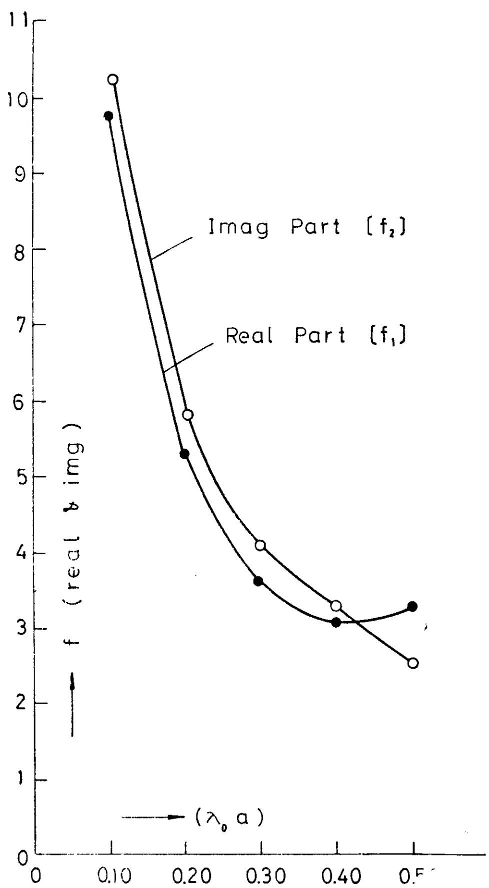

Fig. 2. **Value of** f_1 **and** f_2 **in coefficient** $f=f_1+(i\varepsilon)f_2$

Table 2. (for $\lambda_0 b=1.00$)

$\lambda_0 a=0.10$	$f=-9.77+10.26\,(i\varepsilon)$
0.20	$-5.29+5.83\,(i\varepsilon)$
0.30	$-3.645+4.04\,(i\varepsilon)$
0.40	$-3.085+3.36\,(i\varepsilon)$
0.50	$-3.33+2.1514\,(i\varepsilon)$