Title	Nonlinear oscillation of a gyroscope
Sub Title	
Author	佐藤, 力(Sato, Chikara)
Publisher	慶応義塾大学藤原記念工学部
Publication year	1969
Jtitle	Proceedings of the Fujihara Memorial Faculty of Engineering Keio University (慶応義塾大学藤原記念工学部研究報告). Vol.22, No.89 (1969.) ,p.82(8)- 102(28)
JaLC DOI	
Abstract	A differential equation of an axially symmetrical gyroscope with high center of gravity is analyzed when its axis makes a small oscillation around a perpendicular line. The differential equation has the following form : [function] where functions f1 and f2 are generally nonlinear functions. According to the forms of the nonlinear functions, the solutions are: (1) stable (converging to zero), (2) unstable (oscillatory diverging to infinity), (3) periodic with a limit cycle, and (4) almost periodic. Mathematical treatments depend on the linear transformation with real parameters and averaging method given by I. G. Malkin.
Notes	
Genre	Departmental Bulletin Paper
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=KO50001004-00220089-0008

Nonlinear Oscillation of a Gyroscope

(Received October 15, 1969)

Chikara SATO*

Abstract

A differential equation of an axially symmetrical gyroscope with high center of gravity is analyzed when its axis makes a small oscillation around a perpendicular line. The differential equation has the following form:

$$\ddot{u}-k\dot{v}-u=\mu f_1(u, \dot{u}, v, \dot{v}),$$
$$\ddot{v}+k\dot{u}-v=\mu f_2(u, \dot{u}, v, \dot{v}),$$
$$0<\mu\ll 1, \quad k>2,$$

where functions f_1 and f_2 are generally nonlinear functions. According to the forms of the nonlinear functions, the solutions are: (1) stable (converging to zero), (2) unstable (oscillatory diverging to infinity), (3) periodic with a limit cycle, and (4) almost periodic. Mathematical treatments depend on the linear transformation with real parameters and averaging method given by I. G. Malkin.

I. Introduction

Since there are many papers(1)~(6) concerning a gyroscope in the field of physics or engineering, the analysis of them is not a new topic in these fields. However, there are not many papers which analyze a gyroscope applying the theory of nonlinear oscillation in the field of applied mathematics. This paper presents an analysis of of gyroscope by solving a nonlinear differential equation with small perturbation terms. In order to derive the nonlinear differential equation in a quasi-linear form, we start from Euler's equation using a fixed coordinate. We use Euler's equation because the other differential equations of a gyroscope usually involve nonlinear functions such as $\sin\theta$ or $\cos\theta$ that cannot be put to θ or 1 even when the oscillation is small around an equilibrium line, and thus it is difficult to derive the nonlinear differential equation in a quasi-linear form from them.

I. G. Malkin,(7) N. Minorsky,(8) and Mitropoliskii(9) have studied a gyroscope by anlyzing a nonlinear differential equation in a quasi-linear form. Two(7)(8) of the papers have presented some analysis and stability conditions for the equation having a special nonliner function. However, the stability conditions can be applicable for a restricted case, and not for a symmetric gyroscope of this paper. One(9) of the papers has treated an asymptotic solution, and not a stability condition.

In this paper a spining top is considered as a special type of axially symmetrical

*佐 藤 力 Professor, Faculty of Engineering, Keio University.

gyroscope with high center of gravity. The oscillation behavior and the stability conditions of the axially symmetrical spinning top is presented using a linear transformation with real parameters and an averaging method.

II. Fundamental equation

The main object of this section is to give a fundamental equation of a spinning top by using Euler's equation. Two basic assumptions are made: (1) the top is axially symmetric, and (2) the motion of the center of gravity is neglected.

We consider rectangular coordinates (u, v, z). The z-axis coincides with the opposite direction of gravity. Using this coordinate we describe the movement of a top by the following:

A	:	a moment of inertia around the u-axis,
B	:	a moment of inertia around the v-axis,
C	:	a moment of inertia around the z-axis,
$\theta_1, \theta_2, \theta_3$	:	angles around the u-, v-, z-axis,
m	:	total mass,
l	:	distance between the center of gravity and the supporting point,
e	:	a rotating torque for a top,
β	:	a coefficient of damping,
τ	:	time.

By Euler's equation:

$$\begin{aligned} A\theta_1'' &= (B-C)\theta_2'\theta_3' + mgl\sin\theta_1 + \mu f_1^*(\theta_1, \theta_1', \theta_2, \theta_2'), \\ B\theta_2'' &= (C-A)\theta_1'\theta_3' + mgl\sin\theta_2 + \mu f_2^*(\theta_1, \theta_1', \theta_2, \theta_2'), \\ C\theta_3'' &= (A-B)\theta_1'\theta_2' - \beta\theta_3' - e, \end{aligned} \tag{1}$$

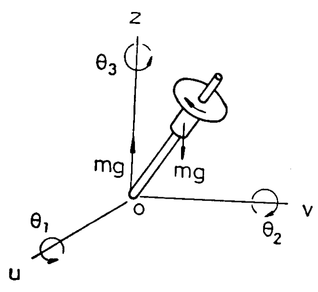


Fig. 1.

where f_1^* and f_2^* are generally nonlinear functions of torque, $0<\mu\ll 1$, and $\theta_1' = \frac{d\theta_1}{d\tau}$. When we consider a small oscillation around the z-axis, the following assumption correctly holds,

$$A=B=\text{const.}$$

Then from equation (1) we have

$$\begin{aligned} A\theta_1'' &= -(C-A)\theta_2'\theta_3' + mgl\,\theta_1 + \mu f_1^*(\theta_1,\ \theta_1',\ \theta_2,\ \theta_2'),\\ A\theta_2'' &= \ \ (C-A)\theta_1'\theta_3' + mgl\,\theta_2 + \mu f_2^*(\theta_1,\ \theta_1',\ \theta_2,\ \theta_2'),\\ C\theta_3'' &= -\beta\theta_3' - e. \end{aligned} \tag{2}$$

From the third equation of (2) θ_3' is given by the following form under stationary condition:

$$\theta_3' = \text{const.} \equiv -\omega.$$

The first and the second equations of (2) then become

$$\begin{aligned} \theta_1'' &= \frac{C-A}{A}\omega\theta_2' + \frac{mgl}{A}\theta_1 + \frac{\mu}{A} f_1^*(\theta_1,\ \theta'_1,\ \theta_2,\ \theta'_2),\\ \theta_2'' &= -\frac{C-A}{A}\omega\theta_1' + \frac{mgl}{A}\theta_2 + \frac{\mu}{A} f_2^*(\theta_1,\ \theta'_1,\ \theta_2,\ \theta'_2). \end{aligned} \tag{3}$$

We transform variables τ, θ_1, θ_2, etc. to new forms as

$$\tau \equiv \sqrt{\frac{A}{mgl}}\,t, \quad \frac{C-A}{\sqrt{A}}\cdot\frac{\omega}{\sqrt{mgl}} \equiv k,$$

$$\frac{1}{mgl} f_1^*(\theta_1,\ \theta_1'\ \ \theta_2,\ \theta_2') \equiv f_1(\theta_1,\ \theta_1',\ \theta_2\ \ \theta_2'),$$

$$\frac{1}{mgl} f_2^*(\theta_1,\ \theta_1',\ \theta_2,\ \theta_2') \equiv f_2(\theta_1,\ \theta_1',\ \theta_2,\ \theta_2'),$$

$$\theta_1 \equiv u, \ \theta_2 \equiv v, \ \frac{du}{dt} = \dot{u}, \text{ etc.}$$

Using the new variables, the fundamental equation is given by

$$\boxed{\begin{aligned} \ddot{u} - k\dot{v} - u &= \mu f_1(u,\ \dot{u},\ v,\ \dot{v}),\\ \ddot{v} + k\dot{u} - v &= \mu f_2(u,\ \dot{u},\ v,\ \dot{v}). \end{aligned}} \qquad \begin{aligned} &k>2,\\ &0<\mu\ll 1. \end{aligned} \tag{4}$$

So far we have assumed a small oscillation around the z-axis in the process from equation (1) to (2). This assumption, however, can be loosened a little if we assume that the moments of inertia A and B are functions of θ_1 and θ_2 with small parameter μ such as

$$\begin{aligned} A(\theta_1,\ \theta_2) &= A_0 + \mu A_1(\theta_1,\ \theta_2),\\ B(\theta_1,\ \theta_2) &= B_0 + \mu B_1(\theta_1,\ \theta_2), \end{aligned}$$

The moment of inertia C is still assumed constant. Thus we have the following equation instead of equation (2)

$$\begin{aligned} A_0\theta_1'' &= -(C-A_0)\theta_2'\theta_3' + mgl\,\theta_1 + \mu f_1^{**}(\theta_1,\ \theta_1',\ \theta_2,\ \theta_2')\\ A_0\theta_2'' &= \ \ (C-A_0)\theta_1'\theta_3' + mgl\,\theta_2 + \mu f_2^{**}(\theta_1,\ \theta_1',\ \theta_2,\ \theta_2'), \end{aligned}$$

where functions f_1^{**} ane f_2^{**} involve not only the functions $A_1(\theta_1, \theta_2)$ and $B_1(\theta_1, \theta_2)$, but also the higher order terms in $\sin\theta_1=\theta_1-\frac{1}{3!}\theta_1^3+\cdots$, and $\sin\theta_2=\theta_2-\frac{1}{3!}\theta_2^3+\cdots$. The procedure after equations (2) to (4) is the same.

In the subsequent sections fundamental equation (4) will be investigated when the functions f_1 and f_2 have various forms. In the next section the very special case $f_1=f_2=0$ will be dealt with in order to derive a necessary transformation with real parameters. Then it will become clear that the condition $k>2$ is necessary for the solution is oscillatory. Generally f_1 and f_2 are arbitrary nonlinear functions, and differentiable at least once by each variable.

III. Basic oscillations with two modes

We consider the following differential equation,

$$\begin{aligned}&\ddot{u}-k\dot{v}-u=0,\\&\ddot{v}+k\dot{u}-v=0,\\&k>2.\end{aligned}\tag{5}$$

The characteristic equation for (5) is given by

$$S^4+(k^2-2)S^2+1=0,\tag{6}$$

where S is a characteristic root. If $k>2$, from equation (6) S^2 is given by

$$S^2=-\frac{k^2-2}{2}\pm\sqrt{\left(\frac{k^2-2}{2}\right)^2-1}\quad<0.\tag{7}$$

It is easily shown that the right side of equation (7) is always negative for both signs. Thus S is given by

$$S=\pm j\left(\frac{k}{2}\mp\sqrt{\left(\frac{k}{2}\right)^2-1}\right),\tag{8}$$

where $j=\sqrt{-1}$.

If we define ω_1 and ω_2 by the following

$$\begin{aligned}\omega_1&=\frac{k}{2}+\sqrt{\left(\frac{k}{2}\right)^2-1},\\\omega_2&=\frac{k}{2}-\sqrt{\left(\frac{k}{2}\right)^2-1},\end{aligned}\tag{9}$$

then the following relations hold,

$$\omega_1>\omega_2>0,\tag{10}$$

$$\omega_1^2-k\omega_1+1=0,\quad \omega_2^2-k\omega_2+1=0,\tag{11}$$

$$\omega_1=\frac{1}{\omega_2}.\tag{12}$$

Thus four distinct characteristic roots of pure imaginary form are given by

$$S=\pm j\omega_1,\ \pm j\omega_2. \tag{13}$$

The condition $k>2$ in equation (4) and (5) is necessary for all characteristic roots to be pure imaginary, otherwise some of the characteristic roots have positive real parts, and the solutions of equation (5) become unstable.

We can now transform variables u and v to the following:

$$u=x_1,\quad \dot{u}=x_2,\quad v=x_3,\quad \dot{v}=x_4. \tag{14}$$

Using these transformations (14), equation (5) can be written by matrix,

$$\dot{x}=Ax, \tag{15}$$

where

$$x=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},\quad A=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & k \\ 0 & 0 & 0 & 1 \\ 0 & -k & 1 & 0 \end{pmatrix}. \tag{16}$$

Here we use the following linear transformation:

$$x=Py, \tag{17}$$

where

$$y=\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix},\quad P=\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & \omega_1 & 0 & \omega_2 \\ 0 & 1 & 0 & 1 \\ -\omega_1 & 0 & -\omega_2 & 0 \end{pmatrix}. \tag{18}$$

Using the transformation (17) and equation (11), equation (15) can be rewritten by

$$\dot{y}=By, \tag{19}$$

where

$$B=P^{-1}AP=\begin{pmatrix} 0 & \omega_1 & 0 & 0 \\ -\omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_2 \\ 0 & 0 & -\omega_2 & 0 \end{pmatrix}. \tag{20}$$

The matrix P^{-1} exists since

$$\det P=-(\omega_1-\omega_2)^2\neq 0.$$

In short using transformation P, equation (5) or (15) can be written by the form (19). Equation (19) is equivalent to the form

$$\boxed{\begin{aligned} \ddot{y}_1+\omega_1{}^2 y_1=0, \\ \ddot{y}_3+\omega_2{}^2 y_3=0. \end{aligned}} \tag{21}$$

Equations (21) show two simple harmonic motions; one is y_1 with a large angular frequency ω_1, while the other is y_3 with a small angular frequency ω_2, since $\omega_1>\omega_2>0$. We define "nutation" for an oscillation with mode of ω_1 and "precession" for mode of ω_2. Matrix P^{-1} is given by

$$P^{-1}=\frac{1}{\omega_1-\omega_2}\begin{pmatrix} -\omega_2 & 0 & 0 & -1 \\ 0 & 1 & -\omega_2 & 0 \\ \omega_1 & 0 & 0 & 1 \\ 0 & -1 & \omega_1 & 0 \end{pmatrix}. \tag{22}$$

The relationship between x and y can be given by scalar forms,

$$\begin{aligned} y_1&=\frac{1}{\omega_1-\omega_2}(-\omega_2x_1-x_4), \\ y_2&=\frac{1}{\omega_1-\omega_2}(x_2-\omega_2x_3), \\ y_3&=\frac{1}{\omega_1-\omega_2}(\omega_1x_1+x_4), \\ y_4&=\frac{1}{\omega_1-\omega_2}(-x_2+\omega_1x_3), \end{aligned} \tag{23}$$

and

$$\begin{aligned} x_1&=y_1+y_3, \\ x_2&=\omega_1y_2+\omega_2y_4, \\ x_3&=y_2+y_4, \\ x_4&=-\omega_1y_1-\omega_2y_3. \end{aligned} \tag{24}$$

Equations (21) have the following general solutions

$$y_1=A_1\cos(\omega_1t+\alpha_1), \quad y_3=A_2\cos(\omega_2t+\alpha_2),$$

and

$$y_2=-A_1\sin(\omega_1t+\alpha_1), \quad y_4=-A_2\sin(\omega_2t+\alpha_2).$$

By relation (24) the original equation (5) has the following solution:

$$u=x_1=A_1\cos(\omega_1t+\alpha_1)+A_2\cos(\omega_2t+\alpha_2),$$

$$v=x_3=-A_1\sin(\omega_1t+\alpha_1)-A_2\sin(\omega_2t+\alpha_2).$$

In this section we have investigated the very special case of $f_1=f_2=0$ in order to obtain the linear transformation (17). Using this linear transformation (17) the general case of f_1 and f_2 will be treated in the following sections.

IV. General analysis of the fundamental equation

In this section the general nonlinear case will be treated. The fundamental equation (4) with nonlinear functions f_1 and f_2 is given by the matrix form

$$\dot{x}=Ax+\mu f, \quad 0<\mu\ll 1, \tag{25}$$

where

$$f=\begin{pmatrix}0\\ f_1(x_1,\ x_2,\ x_3,\ x_4)\\ 0\\ f_2(x_1,\ x_2,\ x_3,\ x_4)\end{pmatrix} \tag{26}$$

Using transform (17), equation (25) is rewritten by $P\dot{y}=APy+\mu f$, and then

$$\dot{y}=P^{-1}APy+\mu P^{-1}f.$$

By matrix (20) the following equation holds

$$\dot{y}=By+\mu F(y), \tag{27}$$

where

$$F(y)=P^{-1}g, \tag{28}$$

$$g=\begin{pmatrix}0\\ g_1(y_1,\ y_2,\ y_3,\ y_4)\\ 0\\ g_2(y_1,\ y_2,\ y_3,\ y_4)\end{pmatrix}, \tag{29}$$

$$\begin{aligned}g_1(y_1,\ y_2,\ y_3,\ y_4)&=f_1(y_1+y_3,\ \omega_1y_2+\omega_2y_4,\ y_2+y_4,\ -\omega_1y_1\ -\omega_2y_3),\\ g_2(y_1,\ y_2,\ y_3,\ y_4)&=f_2(y_1+y_3,\ \omega_1y_2+\omega_2y_4,\ y_2+y_4,\ -\omega_1y_1\ -\omega_2y_3),\end{aligned} \tag{30}$$

Function g is exactly equal to f if variables x_1, x_2, x_3, and x_4 in f are transformed by y_1, y_2, y_3, and y_4. Using equations (22) and (28), function $F(y)$ is given by the following :

$$\begin{aligned}F(y)&=\frac{1}{\omega_1-\omega_2}\begin{pmatrix}-\omega_2 & 0 & 0 & -1\\ 0 & 1 & -\omega_2 & 0\\ \omega_1 & 0 & 0 & 1\\ 0 & -1 & \omega_1 & 0\end{pmatrix}\cdot\begin{pmatrix}0\\ g_1\\ 0\\ g_2\end{pmatrix}\\ &=\frac{1}{\omega_1-\omega_2}\begin{pmatrix}-g_2\\ g_1\\ g_2\\ -g_1\end{pmatrix}.\end{aligned} \tag{31}$$

Equation (27) can be written in scalar form as

$$\begin{aligned}\dot{y}_1&=\omega_1y_2-\frac{\mu}{\omega_1-\omega_2}g_2(y_1,\ y_2,\ y_3,\ y_4),\\ \dot{y}_2&=\omega_1y_1+\frac{\mu}{\omega_1-\omega_2}g_1(y_1,\ y_2,\ y_3,\ y_4),\\ \dot{y}_3&=\omega_2y_4+\frac{\mu}{\omega_1-\omega_2}g_2(y_1,\ y_2,\ y_3,\ y_4),\\ \dot{y}_4&=-\omega_2y_3-\frac{\mu}{\omega_1-\omega_2}g_1(y_1,\ y_2,\ y_3,\ y_4).\end{aligned} \tag{32}$$

Equation (32) can be written by the variables y_1, $\dot{y}_1$, y_3, and $\dot{y}_3$ as

$$\ddot{y}_1+\omega_1^2 y_1=\frac{\mu}{\omega_1-\omega_2}\left[\omega_1 g_1\left(y_1, \frac{\dot{y}_1}{\omega_1}, y_3, \frac{\dot{y}_3}{\omega_2}\right)-\dot{g}_2\left(y_1, \frac{\dot{y}_1}{\omega_1}, y_3, \frac{\dot{y}_3}{\omega_2}\right)\right]+0(\mu^2),$$
$$\ddot{y}_3+\omega_2^2 y_3=-\frac{\mu}{\omega_1-\omega_2}\left[\omega_2 g_1\left(y_1, \frac{\dot{y}_1}{\omega_1}, y_3, \frac{\dot{y}_3}{\omega_2}\right)-\dot{g}_2\left(y_1, \frac{\dot{y}_1}{\omega_1}, y_3, \frac{\dot{y}_3}{\omega_2}\right)\right]+0(\mu^2), \tag{33}$$

where function $\dot{g}_2\left(y_1, \frac{\dot{y}_1}{\omega_1}, y_3, \frac{\dot{y}_3}{\omega_2}\right)$ is the time derivative of $g_2(y_1, y_2, y_3, y_4)$, i.e., $\frac{\partial g_2}{\partial y_1}\dot{y}_1+\frac{\partial g_2}{\partial y_2}\dot{y}_2+\frac{\partial g_2}{\partial y_3}\dot{y}_3+\frac{\partial g_2}{\partial y_4}\dot{y}_4$, in which $y_2=\frac{\dot{y}_1}{\omega_1}$, $\dot{y}_2=-\omega_1 y_1$, $y_4=\frac{\dot{y}_3}{\omega_2}$, and $\dot{y}_4=-\omega_2 y_3$ have been substituted after differentiation. If the terms in the orders higher than μ, i.e. $0(\mu^2)$, are neglected the following equations are derived:

$$\boxed{\begin{aligned}\ddot{y}_1+\omega_1^2 y_1&=\mu G_1(y_1, \dot{y}_1, y_3, \dot{y}_3),\\ \ddot{y}_3+\omega_2^2 y_3&=\mu G_2(y_1, \dot{y}_1, y_3, \dot{y}_3),\end{aligned}} \tag{34}$$

where

$$G_1(y_1, \dot{y}_1, y_3, \dot{y}_3)=\frac{1}{\omega_1-\omega_2}\left[\omega_1 g_1\left(y_1, \frac{\dot{y}_1}{\omega_1}, y_3, \frac{\dot{y}_3}{\omega_2}\right)-\dot{g}_2\left(y_1, \frac{\dot{y}_1}{\omega_1}, y_3, \frac{\dot{y}_3}{\omega_2}\right)\right],$$
$$G_2(y_1, \dot{y}_1, y_3, \dot{y}_3)=-\frac{1}{\omega_1-\omega_2}\left[\omega_2 g_1\left(y_1, \frac{\dot{y}_1}{\omega_1}, y_3, \frac{\dot{y}_3}{\omega_2}\right)-\dot{g}_2\left(y_1, \frac{\dot{y}_1}{\omega_1}, y_3, \frac{\dot{y}_3}{\omega_2}\right)\right]. \tag{35}$$

We assume that

$$m_1\omega_1+m_2\omega_2 \neq 0, \tag{36}$$

where m_1 and m_2 are integers. In order to use the averaging method we express solutions of the nonlinear equation (34) by

$$\begin{aligned} y_1&=A_1\cos(\omega_1 t+\alpha_1),\\ \dot{y}_1&=-\omega_1 A_1\sin(\omega_1 t+\alpha_1),\\ y_3&=A_2\cos(\omega_2 t+\alpha_2),\\ \dot{y}_3&=-\omega_2 A_2\sin(\omega_2 t+\alpha_2).\end{aligned} \tag{37}$$

Averaged equations are following,

$$\begin{aligned} \dot{A}_1&=-\frac{\mu}{\omega_1}\lim_{T\to\infty}\frac{2}{T}\int_0^T G_1^*\sin(\omega_1 t+\alpha_1)\,dt,\\ \dot{A}_2&=-\frac{\mu}{\omega_2}\lim_{T\to\infty}\frac{2}{T}\int_0^T G_2^*\sin(\omega_2 t+\alpha_2)\,dt,\\ \dot{\alpha}_1&=-\frac{\mu}{\omega_1 A_1}\lim_{T\to\infty}\frac{2}{T}\int G_1^*\cos(\omega_1 t+\alpha_1)\,dt,\\ \dot{\alpha}_2&=-\frac{\mu}{\omega_2 A_2}\lim_{T\to\infty}\frac{2}{T}\int G_2^*\cos(\omega_2 t+\alpha_2)\,dt,\end{aligned} \tag{38}$$

where G_1^* and G_2^* are given by relations (35), in which relations (37) have been substituted. Thus G_1^* and G_2^* are not functions of y_1, $\dot{y}_1$, y_3, $\dot{y}_3$, etc., but functions of A_1, A_2, α_1, α_2, and t. Putting the right sides equal to zero in equation (38), we have solutions A_1^*, A_2^*, α_1^*, and α_2^*. From relation (37), solutions y_1 and y_3 can be determined. From equation (32), y_2 and y_4 are easily obtained neglecting μ or higher order terms, since $y_2=\frac{1}{\omega_1}\dot{y}_1+0(\mu)$ and $y_4=\frac{1}{\omega_2}\dot{y}_3+0(\mu)$. Thus we have y_1, y_2, y_3, and y_4 as follows:

$$\begin{aligned} y_1&=A_1^*\cos(\omega_1 t+\alpha_1^*),\\ y_2&=-A_1^*\sin(\omega_1 t+\alpha_1^*),\\ y_3&=A_2^*\cos(\omega_2 t+\alpha_2^*),\\ y_4&=-A_2^*\sin(\omega_2 t+\alpha_2^*). \end{aligned} \tag{39}$$

Therefore, using relations (24), the fundamental nonlinear equation (14) or (25) has the following solution:

$$\begin{aligned} u=x_1&=A_1^*\cos(\omega_1 t+\alpha_1^*)+A_2^*\cos(\omega_2 t+\alpha_2^*),\\ v=x_3&=-A_1^*\sin(\omega_1 t+\alpha_1^*)-A_2^*\sin(\omega_2 t+\alpha_2^*). \end{aligned} \tag{40}$$

This solution (40) has the same form given in section 2 where $f_1=f_2=0$, except for the fact that parameters A_1, A_2, α_1, and α_2 in section 2 are determined by initial condition, while parameters A_1^*, A_2^*, α_1^* and α_2^* in this section are determined by the differential equation. Stability of the solution (40) can be investigated by equation (38).

V. A linear symmetric top

A linear axially symmetric top is dealt with in this section. When spinning top has the assumptions of linearity and axial symmetry, the functions f_1 and f_2 are related with

$$f_1(u,\ \dot{u},\ v,\ \dot{v})=f_2(v,\ \dot{v},\ -u,\ -\dot{u}),$$

and thus they have the forms:

$$f_1(u,\ \dot{u},\ v,\ \dot{v})=\delta_1 u+\delta_2\dot{u}+\delta_3 v+\delta_4\dot{v},$$

and
$$f_2(u,\ \dot{u},\ v,\ \dot{v})=-\delta_3 u-\delta_4\dot{u}+\delta_1 v+\delta_2\dot{v}.$$

From the fundamental equation (4) and the above relations, we can put $\delta_1=\delta_4=0$ without loss of generality. Thus we have $f_1(u,\ \dot{u},\ v,\ \dot{v})=\delta_2\dot{u}+\delta_3 v$, $f_2(u,\ \dot{u},\ v,\ \dot{v})=-\delta_3 u+\delta_2\dot{v}$. When we use parameters μ and b instead of δ_2 and δ_3 with the relations $\delta_2=-\mu b$ and $\delta_3=\mu$, functions f_1 and f_2 are given by

$$\begin{aligned} &\mu f_1(u,\ \dot{u},\ v,\ \dot{v})=-\mu(b\dot{u}-v),\\ &\mu f_2(u,\ \dot{u},\ v,\ \dot{v})=-\mu(b\dot{v}+u),\\ &0<\mu\ll 1,\ b>0. \end{aligned} \tag{41}$$

We consider the stability problem only under the above mentioned conditions of $0<\mu\ll 1$ and $b>0$, because, otherwise no stable solutions are given.

Functions g_1 and g_2 are given by

$$
\begin{aligned}
&g_1(y_1,\ y_2,\ y_3,\ y_4)=(1-\omega_1 b)\,y_2+(1-\omega_2 b)\,y_4,\\
&g_2(y_1,\ y_2,\ y_3,\ y_4)=(\omega_1 b-1)\,y_1+(\omega_2 b-1)\,y_3.
\end{aligned}
\tag{42}
$$

From equation (35), functions G_1 and G_2 are given by

$$
\begin{aligned}
G_1&=\frac{1}{\omega_1-\omega_2}\left[2\left(1-\omega_1 b\right)\dot{y}_1+\left(1+\frac{\omega_1}{\omega_2}-\omega_1 b-\omega_2 b\right)\dot{y}_3\right],\\
G_2&=-\frac{1}{\omega_1-\omega_2}\left[\left(1+\frac{\omega_2}{\omega_1}-\omega_1 b-\omega_2 b\right)\dot{y}_1+2\left(1-\omega_2 b\right)\dot{y}_3\right].
\end{aligned}
\tag{43}
$$

Using expression (37), functions $G_1{}^*$ and $G_2{}^*$ are given by

$$
\begin{aligned}
G_1{}^*=&-\frac{1}{\omega_1-\omega_2}\Big[2\left(1-\omega_1 b\right)\omega_1 A_1\sin\left(\omega_1 t+\alpha_1\right)\\
&+\left(1+\frac{\omega_1}{\omega_2}-\omega_1 b-\omega_2 b\right)\omega_2 A_2\sin\left(\omega_2 t+\alpha_2\right)\Big],\\
G_2{}^*=&\frac{1}{\omega_1-\omega_2}\Big[\left(1+\frac{\omega_2}{\omega_1}-\omega_1 b-\omega_2 b\right)\omega_1 A_1\sin\left(\omega_1 t+\alpha_1\right)\\
&+2\left(1-\omega_2 b\right)\omega_2 A_2\sin\left(\omega_2 t+\alpha_2\right)\Big].
\end{aligned}
\tag{44}
$$

Averaged equations are followings,

$$
\begin{aligned}
&A_1=\mu\frac{2(1-\omega_1 b)}{\omega_1-\omega_2}A_1,\quad \dot{A}_2=-\mu\frac{2(1-\omega_2 b)}{\omega_1-\omega_2}A_2,\\
&\dot{\alpha}_1=0,\quad \dot{\alpha}_2=0.
\end{aligned}
\tag{45}
$$

Stability condition of both A_1 and A_2 is given by

$$1-\omega_1 b<0,\ \text{for}\ A_1,$$

and

$$1-\omega_2 b>0,\ \text{for}\ A_2.$$

Hence we have

$$\frac{1}{\omega_1}<b<\frac{1}{\omega_2}.$$

Using relation (12), we have

$$\omega_2<b<\omega_1. \tag{46}$$

Using relation (9), we have

$$\frac{k}{2}-\sqrt{\left(\frac{k}{2}\right)^2-1}<b<\frac{k}{2}+\sqrt{\left(\frac{k}{2}\right)^2-1}.$$

This inequality is equivalent to the following,

$$\left|\,b=\frac{k}{2}\,\right|<\sqrt{\left(\frac{k}{2}\right)^2-1}.$$

Squaring both sides, we have

$$k>b+\frac{1}{b}\,. \tag{47}$$

Thus, condition (47) gives the necessary and sufficient stability condition of A_1 and A_2 in equation (45). This inequality (47) is therefore the necessary and sufficient condition that the fundamental equation (4) with (41) has stable solutions $\boldsymbol{u}$ and v. The boundary curve from stability to instability is shown on the $(k,\ b)$ plane in Fig. 2, by the critical condition:

$$k=b+\frac{1}{b}\,.$$

There are three distinct cases where each mode becomes stable or unstable:

$$\begin{aligned}
b<\omega_2 &: \begin{cases}\text{nutation (mode } \omega_1) \rightarrow \text{diverge,}\\ \text{precession (mode } \omega_2) \rightarrow 0,\end{cases}\\
\omega_2<b<\omega_1 &: \begin{cases}\text{nutation (mode } \omega_1) \rightarrow 0,\\ \text{precession (mode } \omega_2) \rightarrow 0,\end{cases}\\
\omega_1<b &: \begin{cases}\text{nutation (mode } \omega_1) \rightarrow 0,\\ \text{precession (mode } \omega_2) \rightarrow \text{diverge.}\end{cases}
\end{aligned} \tag{48}$$

The same stabitity condition as (47) is given by the stability criterion of Hurwitz. Stability regions for distinctive modes are shown in Fig. 2 by condition (48). Although region 3 for $k\leqq 2$ in this figure is not the region of our interest, it is easily shown from characteristic roots (8) that the solution is unstable since one of the roots has a positive real part.

If the condition $b>0$ alone is replaced by $b\leqq 0$ using the same condition $0<\mu\ll 1$ as before, then no stable solution is obtained, because parameter b cannot satisfy both $b\leqq 0$ and $\omega_2<b<\omega_1$ since $\omega_2>0$. If the conditions $0<\mu\ll 1$ and $b>0$ are replaced by $0<-\mu\ll 1$ and $b\gtreqless 0$, then from equations (45) stability condition is that $1-\omega_1 b>0$ for A_1, and $1-\omega_2 b<0$ for A_2. However, it is impossiple for b to satisfy both $1/\omega_1>b$ and $1/\omega_2<b$, since $1/\omega_1<1/\omega_2$.

Thus the only form of functions f_1 and f_2 given by (41) with the conditions $0<\mu\ll 1$ and $b>0$ and with the condition (47) gives the stable solution when the linearity and axial symmetry are satisfied.

So far the functions (41) with conditions $0<\mu\ll 1$ and $b>0$ are all given from mathematical standpoint so that a stable converging solutions are derived. On the other hand, from physical standpoint an actual symmetric top satisfys the condition (41) with $0<\mu\ll 1$ and $b>0$ if the following physical assumptions are made: (1) the top spins on a blunt peg slipping with friction between the blunt peg and a supporting plane, (2) the bottom of the peg has a form of sphere whose center is located at a lower position than the center of gravity of the whole top, (3) there is a positive damping force such as an air damping force.

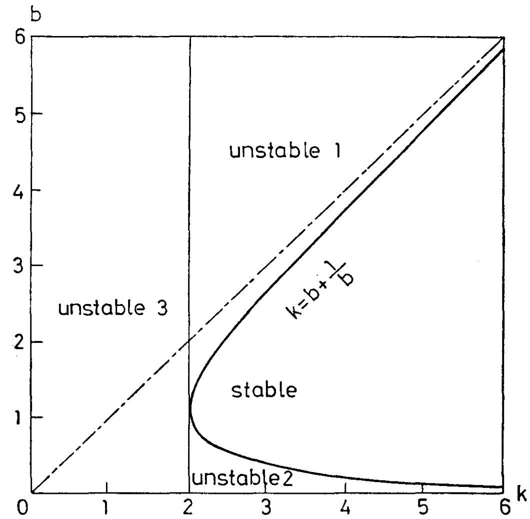


Fig. 2. Stability regions in equation
$\ddot{u}-k\dot{v}-u=\mu(-b\dot{u}+v)$,
$\ddot{v}+k\dot{u}-v=\mu(-b\dot{v}-u)$,
$0<\mu\ll 1,\ b>0$
In the resion 1 pression is unstable,
while in the region 2 nutation is unstable.

VI. Some nonlinear example

In this section a spinning symmetric top with nonlinear perturbation forces will be treated. Functions f_1 and f_2 are assumed to be of the form:

$$\begin{aligned}
&\mu f_1(u,\ \dot{u},\ v,\ \dot{v})=-\mu[b\dot{u}-v(u^2+v^2)],\\
&\mu f_2(u,\ \dot{u},\ v,\ \dot{v})=-\mu[b\dot{v}+u(u^2+v^2)],\\
&0<\mu\ll 1,\ b>0.
\end{aligned}\tag{49}$$

Physically speaking, this relation corresponds to a spinning top with a linear damping force and a nonlinear frictional force between the peg and the supporting plane. The nonlinear frintional force is very small when the oscillation is small, while the frictional force becomes larger when the oscillation increases. There are no other deep reason for taking this nonlinear form. With y_1 and y_3 in the form (37), we use an averaging method. We then have the following two averaged equations for A_1 and A_2:

$$\begin{aligned}
A_1&=\mu\frac{2}{\omega_1-\omega_2}\left(-b\omega_1+A_1{}^2+\frac{6\omega_1+\omega_2}{2\omega_1}A_2{}^2\right)A_1,\\
\dot{A}_2&=-\mu\frac{2}{\omega_1-\omega_2}\left(-b\omega_2+A_2{}^2+\frac{\omega_1+6\omega_2}{2\omega_2}A_1{}^2\right)A_2.
\end{aligned}\tag{50}$$

It is easily shown that equations for α_1 and α_2 are given by $\dot{\alpha}_1=0$ and $\dot{\alpha}_2=0$. Hence the solutions α_1 and α_2 are constants. Putting the right side of equation (50) equal to zero, we have

$$\text{for } A_1=0: \quad A_1=0 \quad \text{or} \quad \frac{A_1^2}{b\omega_1}+\frac{A_2^2}{\frac{2b\omega_1^2}{6\omega_1+\omega_2}}=1, \tag{51}$$

$$\text{for } A_2=0: \quad A_2=0 \quad \text{or} \quad \frac{A_1^2}{\frac{2b\omega_2^2}{\omega_1+6\omega_2}}+\frac{A_2^2}{b\omega_2}=1. \tag{52}$$

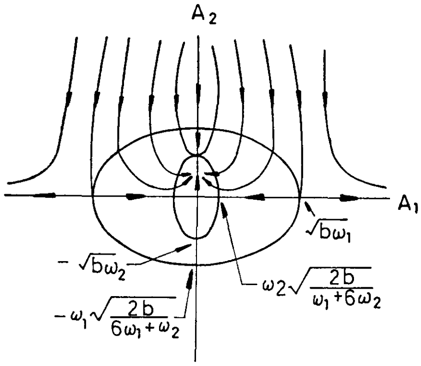


Fig. 3.
Phase trajectories in the (A_1, A_2) plane for equation (50) when $k>2.34$

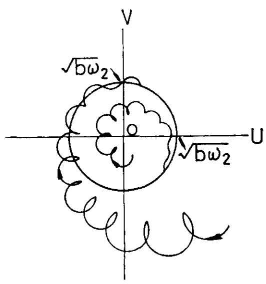


Fig. 4.
Phase trajectories in the (u, v) plane for equation:
$\ddot{u}-k\dot{v}-u=-\mu[b\dot{u}-v(u^2+v^2)]$,
$\ddot{v}+k\dot{u}-v=-\mu[b\dot{v}+u(u^2+v^2)]$,
when $k>2.34$

Equations (51) and (52) correspond to two lines and two ellipses in the (A_1, A_2) plane. We must now compare the radii of the two ellipses in order to know whether the two ellipses intersect each other or not. First we compare the two radii of ellipses (51) and (52) on the A_1 axis. The radius of (51) on the A_1 axis is $\sqrt{b\omega_1}$, and the radius of (52) on the A_1 axis is $\omega_2\sqrt{\frac{2b}{\omega_1+6\omega_2}}$. Comparing these two, we have the following inequality,

$$\omega_2\sqrt{\frac{2b}{\omega_1+6\omega_2}}<\omega_2\sqrt{\frac{2b}{\omega_2+6\omega_2}}=\sqrt{\frac{2\omega_2 b}{7}}<\sqrt{b\omega_2}<\sqrt{b\omega_1}. \tag{53}$$

Accordingly the radius of (51) on the A_1 axis is larger than that of (52). Next we compare the two radii on the A_2 axis. The radius of ellipse (51) on the A_2 axis is $\omega_1\sqrt{\frac{2b}{6\omega_1+\omega_2}}$, and the radius of (52) on the A_2 axis is $\sqrt{b\omega_2}$. To compare these two, we take the difference after squaring; i.e.,

$$\frac{2b\omega_1^2}{6\omega_1+\omega_2}-b\omega_2=b\,\frac{2\omega_1^2-6\omega_1\omega_2-\omega_2^2}{6\omega_1+\omega_2}$$

$$=2b\,\frac{\left(\omega_1-\frac{3+\sqrt{11}}{2}\omega_2\right)\left(\omega_1+\frac{-2+\sqrt{11}}{2}\omega_2\right)}{6\omega_1+\omega_2}.$$

Since $\omega_1>\omega_2>0$, we have

$$2b\,\frac{\omega_1+\frac{-3+\sqrt{11}}{2}\omega_2}{6\omega_1+\omega_2}>0\,.$$

Therefore we have

$$\text{if}\quad \omega_1>\frac{3+\sqrt{11}}{2}\omega_2\quad\text{then}\quad \omega_1\sqrt{\frac{2b}{6\omega_1+\omega_2}}>\sqrt{b\omega_2}\,, \tag{54}$$

$$\text{if}\quad \omega_1<\frac{3+\sqrt{11}}{2}\omega_2\quad\text{then}\quad \omega_1\sqrt{\frac{2b}{6\omega_1+\omega_2}}<\sqrt{b\omega_2}\,. \tag{55}$$

Thus there are two cases where one of the radii of (51) and (52) on the A_2 axis is larger than the other, and vice versa. We consider these two cases separately below.

The case of $\omega_1>\frac{3+\sqrt{11}}{2}\omega_2$.

From relation (9) this condition is equivalent to the following:

$$k<\sqrt{\frac{1+3\sqrt{11}}{2}}=2.340 \tag{56}$$

In this case the ellipse (52) is located inside of the ellipse (51). From equation (50) there are two kinds of singular points, one is a saddle point, and the other is a stable nodal point. That is,

$$\begin{aligned} &A_1=\pm\sqrt{b\omega_1},\ A_2=0:\ \text{saddle point},\\ &A_1=0,\ A_2=\pm\sqrt{b\omega_2}:\ \text{stable nodal point}. \end{aligned} \tag{57}$$

The phase trajectories of the present system are shown in Fig. 3. In this figure the lower half is abbreviated because of the symmetry. Among four averaged equations (38). two equations for α_1 and α_2 have constant or slowly varying solutions α_1* and α_2*. Using α_2* the following stable solutions y_1 and y_3 are obtained:

$$y_1=0,\ y_3=\sqrt{b\omega_2}\ \cos\ (\omega_2 t+\alpha_2{}^*). \tag{58}$$

Thus solutions u and v in the fundamental equation (4) with (49) and (56) are given by

$$u=\sqrt{b\omega_2}\ \cos\ (\omega_2 t+\alpha_2{}^*),\ v=-\sqrt{b\omega_2}\ \sin\ (\omega_2 t+\alpha_2{}^*). \tag{59}$$

The phase trajectories in the $(u,\ v)$ plane are shown in Fig. 4. According to the initial conditions some trajectories converge to a stable limit cycle which is a circle with radius $\sqrt{b\omega_2}$, and others diverge to infinity. Although these trajectories must be shown in the four dimensional space $(u,\ \dot{u},\ v,\ \dot{v})$, the trajectories in Fig. 4 show only a project to the $(u,\ v)$ plane. Therefore the trajectories in this plane seem to cross each other, while the actual trajectories in the four dimensional space never cross each other except at the singular points.

The case of $\omega_1<\dfrac{3+\sqrt{11}}{2}\omega_2$

From relation (9) the above condition yields

$$2<k<\sqrt{\frac{1+3\sqrt{11}}{2}}=2.340 \tag{60}$$

In this case two ellipses (51) and (52) intersect at four points, as shown in Fig. 5. Equation (50) has singular points of saddle given by

$$A_1=\pm\sqrt{b\omega_1},\ A_2=0:\ \text{saddle point},$$

and for singular points given by the intersections of two ellipses (51) and (52) as

$$A_1=\pm A_1{}^*,\ A_2=\pm A_2{}^*,$$

where $A_1{}^*$, $A_2{}^*>0$. The character of these singular points depends on the charcteristic root S given by characteristic equation

$$\begin{vmatrix} 2A_1{}^*-S & \dfrac{6\omega_1+\omega_2}{\omega_1}A_2{}^* \\ -\dfrac{\omega_1+6\omega_2}{\omega_2}A_1{}^* & -2A_2{}^*-S \end{vmatrix}=0.$$

The phase trajectories for the averaged equation are shown in Fig. 5. In this figure the lower half is abbreviated because of the symmetry. Solutions y_1 and y_3 are given by

$$y_1=A_1{}^*\cos\ (\omega_1 t+\alpha_1{}^*),\ y_3=A_2{}^*\cos\ (\omega_2 t+\alpha_2{}^*).$$

Thus solutions u and v of the fundamental equation (4) with (49) and (60) are given by

$$\begin{aligned} u&=A_1{}^*\cos\ (\omega_1 t+\alpha_1{}^*)+A_2{}^*\cos\ (\omega_2 t+\alpha_2{}^*),\\ v&=-A_1{}^*\sin\ (\omega_1 t+\alpha_1{}^*)-A_2{}^*\sin\ (\omega_2 t+\alpha_2{}^*). \end{aligned} \tag{61}$$

The phase trajectories in the (u, v) plane are shown in Fig. 6. Depending on the initial conditions some trajectories converge to a region between two circles, and others diverge to infinity. Since ω_1 and ω_2 are assumed to have no rational relation, there is no limit cycle. The solutions u and v are almost periodic solutions.

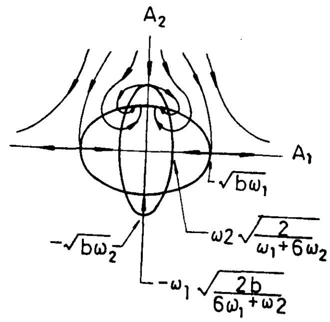


Fig. 5.
Phase trajectories in the (A_1, A_2) plane for equation (50) when $2<k<2.34$.

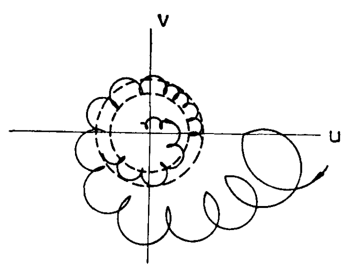
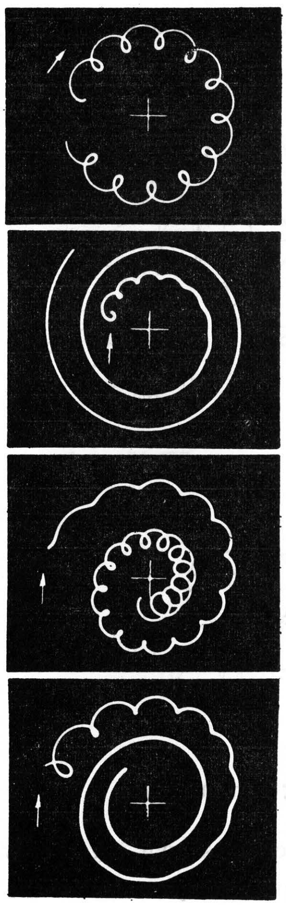


Fig. 6.
Phase trajectories in the (u, v) plane for equation:
$$\ddot{u}-k\dot{v}-u=-\mu[b\dot{u}-v(u^2+v^2)],$$
$$\ddot{v}+k\dot{u}-v=-\mu[b\dot{v}+u(u^2+v^2)],$$
when $2<k<2.34$.

VII. Collective results with analog computation

In this section several results by an analog computer are shown, together with all the results in the preceding sections. As an example, phase trajectories for the linear case when $k=4.5$ are shown in Fig. 7. Variables u and v are taken as abscissa and ordinate respectively. In Fig. 7 (*a*) the case of $\mu f_1=\mu f_2=0$ is shown. Fig. 7 (*b*) is the case of $\mu f_1=-0.224\,\dot{u}$, $\mu f_2=-0.224\,\dot{v}$. The corresponding top has a small damping. Here nutation converges to zero, and precession diverges. Figure 7 (*c*) is the case of $\mu f_1=0.25\,v$, $\mu f_2=-0.25\,u$. The corresponding top has a frictional force between the peg and supporting plane. Here precession converges to zero, and nutation diverges. Figure 7 (*d*) is the case of $\mu f_1=-0.224\,\dot{u}+0.25\,v$, $\mu f_2=-0.224\,\dot{v}-0.25\,u$. A damping force and a frictional force exist between the peg and supporting plane. In this case both precession and nutation converge to zero. From these figures it is apparent that the top is stabilized by the combined actions of the damping force and the frictional force between the blunt peg and the supporting plane. The damping force suppresses nutation, while the frictional force suppresses precession.

All the results for both linear and nonlinear cases are listed in Table, together with some examples which are omitted in the preceding sections.



(a) $\mu f_1 = 0$, $\mu f_2 = 0$.
(No damping and frictional forces exist)

(b) $\mu f_1 = -0.224u$, $\mu f_2 = -0.224v$.
(Damping force exists)

(c) $\mu f_1 = 0.25v$, $\mu f_2 = -0.25u$.
(Frictional force exists)

(d) $\mu f_1 = -0.224u + 0.25v$, $u f_2 = -0.224v - 0.25u$.
(Both damping and frictional forces exist)

Fig. 7. Phase Trajectories in the (u, v) plane for equation:

$$\ddot{u} - k\dot{v} - u = \mu f_1,$$
$$\ddot{v} + k\dot{u} - v = \mu f_2$$

where $k = 4.5$

Table Character of solutions in equation:

$$\ddot{u}-k\dot{v}-u=\mu f_1(u,\ \dot{u},\ v,\ \dot{v}),$$

$$\ddot{v}+k\dot{u}-v=\mu f_2(u,\ \dot{u},\ v,\ \dot{v}),$$

$$0<\mu\ll 1,\ k>2.$$

$f_1(u, \dot{u}, v, \dot{v})$	$f_2(u, \dot{u}, v, \dot{v})$	Nutation <Mode ω_1>	Precession <Mode ω_2>	Condition
$-\dot{u}$	$-\dot{v}$	0	diverge	positive resistance
$\dot{u}$	$\dot{v}$	diverge	0	negative resistance
v	$-u$	diverge	0	blunt peg: radius of sphere at bottom$>l$
$-v$	u	0	diverge	blunt peg: radiov of sphere at bottom$>l$
$-b\dot{u}+v$	$-b\dot{v}-u$	diverge	0	$b<\omega_2$
		0	0	$\omega_2<b<\omega_1$
		0	diverge	$\omega_1<b$
$-b\dot{u}+v$	0	diverge	0	$b<\omega_2$
		0	0	$\omega_2<b<\omega_1$
		0	diverge	$\omega_1<b$
$-b\dot{u}$	$-u$	diverge	0	$b<\omega_2$
		0	0	$\omega_2<b<\omega_1$
		0	diverge	$\omega_1<b$
$-b\dot{u}+v(u^2+v^2)$	$-b\dot{v}-u(u^2+v^2)$	0	stable sol.	$k<2.34$, limit cycle
		stable sol.	stable sol.	$2<k<2.34$, almost periodic

VIII. Conclusion

An axially symmetric spinning top is analyzed when the axis of the top makes a small angle with perpendicular direction. The differential equation is nonlinear with small perturbation terms, that is, a quasilinear differential equation. By linear transformation the differential equation is separated into two equations with two modes ; nutation and precession. Stability and character of solutions for these equations are investigated. According to the results obtained herein, the solutions are: (1) stable and converging to zero, (2) unstable and diverging to infinity, (3) periodic with a limit cycle, and (4) almost periodic.

Especially when the spinning top is assumed to be linear and axially symmetric, a combination of two forces results in the stabilization of the top. One is a damping force and the other is a frictional force between a blunt peg and supporting plane. The damping force suppresses nutation, while the frictional force suppresses precession. If either one of these two causes does not take place, stability of the top cannot be maintained. There are no other causes (that are mathematically different) for the stabilization of a symmetric linear top.

An example of a nonlinear case with small perturbation terms is used to obtain conditions when (1) solution is periodic with a limit cycle, and (2) solution is almost periodic.

Reference

(1) F. Klein, A. Sommerfeld: Ueber die Theorie des Kreisels, I, II, III, IV, Druck und Verlag von B. G. Teubner, 1910.

(2) H. Crabtree: Spinning Tops and Gyroscopic Motion, Longmans, 1914.

(3) A. Gray: A Treatise on Gyrostatics and Rotational Motion, London, 1918.

(4) J. B. Scarborough · The Gyroscope, Theory and Applications, Interscience Publishers, New York, 1958.

(5) B. H. Okunev: Svobodnoe dvijenie Giroskopa (Free Motion of Gyroscope), Moscow, 1951.

(6) B. V. Bulgakov: Prikladnaya teoriya giroskopov (Applied Theory of Gyroscopes), Moscow, 1955.

(7) I. G. Malkin: Nekotorye zadachi teorii nelineinyix kolebanii (Certain Problems of Nonlinear Oscillations), Moscow, pp. 300–307, 1956.

(8) N. Minorsky: Nonlinear Oscillations, Van Nostrand, pp. 301–306, 1962.

(9) Yu. A. Mitropoliskii: Problemi asimptoticheskoi teorii nestachionarix kolebanii (Asymptotic Problems of Nonlinear Oscillation Theory), pp. 240–313, Moscow, 1964.

(10) P. Contensou: Couplage entre frottement de glissement et frottement de pivotement dans la théorie de la toupie, "Gyrodynamics," International Union of Theoretical and Applied Mechanics, Springer, 1963.

(11) A. Yu. Ishlinskii: Mechanics of Gyroscopic Systems, 1963, Translated from Russian, 1965, Available from the U. S. Department of Commerce.

(12) B. D. H. Tellegen: The Gyrator, A New Electric Network Element, Philips Res. Rep. Vol. 3, pp. 81 – 101, 1948.
(13) A. Bloch: A New Approach to Dynamics of Systems with Gyroscopic Coupling Terms, Phil. Mag. Vol. 35, pp. 315 – 334, 1944.
(14) C. Sato: Nonlinear Oscillation of Gyroscope, Research Report of Purdue University, School of Engineering, February 1967.

Appendix

Frictional force between the axis and its supporting plane:

The following assumptions are made in Section 4: the axis of the peg rotates in the same direction as the top slipping between the bottom of the peg and the supporting plane, (2) the bottom of the peg is a sphere, and (3) the center of the sphere is located at a lower position than the center of gravity of the whole top. Under these assumptions, the frictional force gives a torque around the center of gravity. As shown in Fig. 8 (*a*), the perpendicular axis z and the rotational axis make angle γ, and the distance between the center of gravity and the supporting point is equal to l. The top receives a force F at the contact point. This force F has two componets F_u and F_v as shown in Fig. 8 (*b*). If the angle γ is small, the following relation holds:

$$\tan \gamma = \sqrt{u^2+v^2}.$$

The bottom of the peg has the form of a sphere which is approximately given by

$$\eta = h\xi^2,$$

where h is a constant. In Fig. 8 (*d*) a tangential line can be drawn, such as

$$\tan \gamma = 2h\xi.$$

Thus the following relation holds:

$$\xi = \frac{1}{2h} \tan \gamma = \frac{1}{2h} \sqrt{u^2+v^2}.$$

The magnitude of the frictional force $|F|$ shown in Fig. 8 (*b*) is proportional to a frictional coefficient B', and also proportional to the velocity at the contact point. Thus,

$$|F| = B'\omega\xi,$$

or

$$F_u = |F| \sin \delta = |F| \cdot \frac{v}{\sqrt{u^2+v^2}} = \frac{B'\omega}{2h} v,$$

$$F_v = -|F| \cos \delta = -|F| \frac{u}{\sqrt{u^2+v^2}} = -\frac{B'\omega}{2h} u.$$

Thus the torque around the center of gravity has two components,

$$lF_u = \mu v, \qquad lF_v = -\mu u,$$

where $\mu = \frac{lB'\omega}{2h}$.

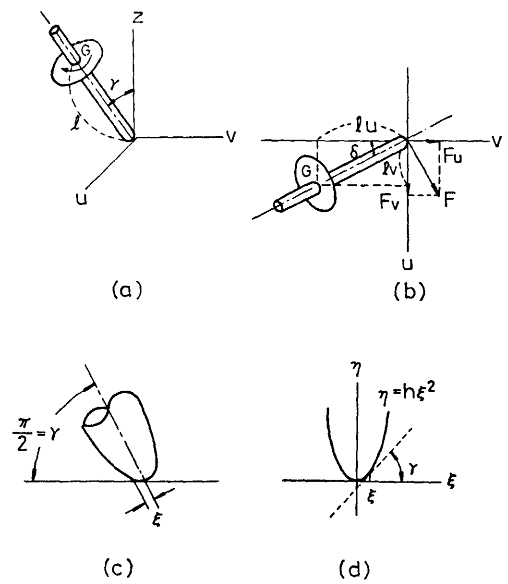


Fig. 8.