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Author	高橋, 進一(Takahashi, Shinichi) 杉浦, 禎彦(Sugiura, Tadahiko)
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Some Properties of Passive *RC* Network Giving Over-Unity Gain

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Shin-ich TAKAHASHI*
Tadahiko SUGIURA**

Abstract

It is possible to obtain a two terminal-pair passive resistor-capacitor network having a gain greater than unity over a band of frequencies. The present paper gives the results of a general investigation of such networks, so called *RC* step-up networks. It is conclusively shown the following; 1) the theoretical gain obtainable from an *RC* step-up network can be infinite; 2) in an *RC* step-up network, the absolute value of the open-circuit input impedance is less than that of the open-circuit output impedance at a certain frequency giving over-unity gain; 3) therefore, symmetrical quadripoles cannot be *RC* step-up networks; etc.

I. Introduction

This paper describes the properties of the passive *RC* networks which have gain greater than unity. The *RC* current or voltage transfer function $T(s)$ which is $|T(j\omega_0)|>1$ at some frequency w_0, is defined as transfer function of *RC* step-up network or network giving over-unity gain(1) (by duality voltage step-up network is also current step-up network, then we consider voltage transfer function only). Several considerations have been proposed for passive *RC* network giving over-unity gain(2)(3), but the literatures on the *RC* transfer functions only treat quite special networks which phase shift has π radians at some frequency. In the present paper, we don't restrict ourselves to networks of any special classes, but treat the general two terminal-pair *RC* networks, and present some theoretical results in the analysis of the passive *RC* transfer functions which have gain greater than unity.

II. Some examples of *RC* step-up networks

A theoretical investigation is first undertaken to know the possibility of the step-up in the simple and typical passive *RC* networks, for example, ladder, lattice, bridged-*T* and parallel-*T* networks.

*高橋進一 Lecturer, Faculty of Engineering, Keio University.
**杉浦禎彦 Graduate Student, Faculty of Engineering, Keio University.

(A): The gain of passive RC ladder, lattice networks is never greater than unity. (where the input interchanged network or complementary network of the original ladder is not called ladder).

It is quite simple to show the property (A). From Fig. 1 (a), the terminal relationship at ports a and b of the RC ladder network is

$$\left(\frac{e'_{out}}{e'_{in}}\right)=\frac{Z_a}{Z_a+Z_k} \tag{1}$$

where Z_a is RC driving point impedance looking to the right at a—a'.

From Cauer's results on RC impedances and Brune's residue conditions: RC driving point impedances Z_a, Z_k are written by

$$Z_k(jw)=\alpha_k(w)-j\beta_k(w)$$
$$Z_a(jw)=\alpha_a(w)-j\beta_a(w), \tag{2}$$

where α_k α_a, β_k, β_a are positive.
Consequently

$$\left|\frac{e'_{out}}{e'_{in}}\right|_k=\left[\frac{\alpha_a{}^2+\beta_a{}^2}{(\alpha_k+\alpha_a)^2+(\beta_k+\beta_a)^2}\right]^{\frac{1}{2}}\leqq 1, \tag{3}$$

then

$$\left|\frac{e_{out}}{e_{in}}\right|=\prod_{k=1}^{n}\left|\frac{e'_{out}}{e'_{in}}\right|_k\leqq 1. \tag{4}$$

On the other hand, from Fig. 1 (b), transfer fanction of lattice section is given by

$$\left(\frac{e_{out}}{e_{in}}\right)=\frac{Z_2Z_3-Z_1Z_4}{(Z_1+Z_3)(Z_2+Z_4)} \tag{5}$$

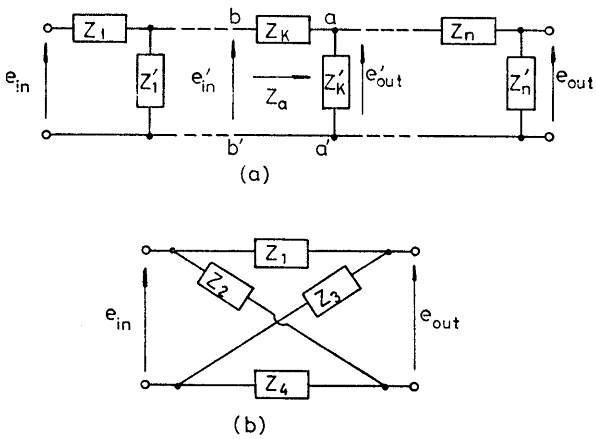

Fig. 1. (a) **Ladder notwork.** (b) **Lattice network.**

where *RC* impedances Z_1 are written by

$$Z_i(jw)=\alpha_i(w)-j\beta_i(w)$$

$$(\alpha_i(w),\ \beta_i(w) \text{ are positive})$$

by the same calculation as ladder network, we get

$$\left|\frac{e_{out}}{e_{in}}\right| \leqq 1 \tag{6}$$

that completes the proof.

Some parallel-*T* and bridged-*T* networks are capable of having higher than unity gain, for example, networks shown by Fig. 2 (*a*), and Fig. 2 (*b*) have maximum gain about 1.15 (an $n=2(1+\sqrt{2})$, $w=1$) and 1.08 ($w \simeq 0.611$), respectively. And Fig. 2 (*c*) and Fig. 2 (*d*) show their vector loci.

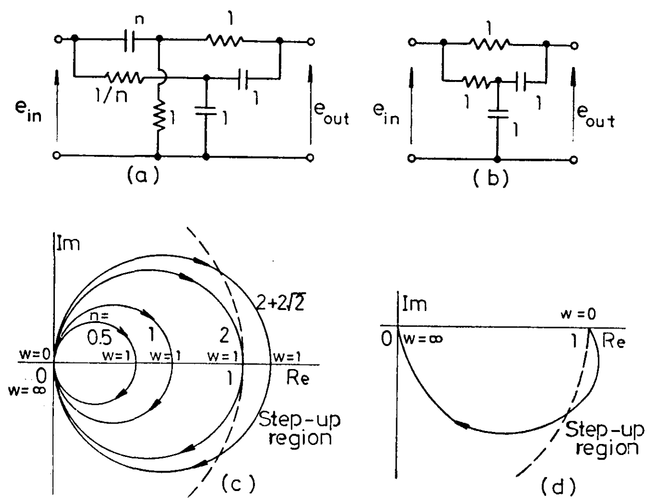

Fig. 2. Paraller-*T* and bridged-*T* networks, and their vector loci.

III. Input and output inpedances

The higher gain is obtainable from the cascade connected *RC* networks having gain greater than unity, if the each succeeding section does not load the previous section, in other words, if there is no loading effect. But the following theorem shows that the loading effect restricts the cascade connection use of the *RC* step-up networks.

(*B*): In an *RC* voltage step-up network, the absolute value of the open-circuit input impedance is less than that of the open-circuit output impedance at a certain frequency which give over-unity gain; and the absolute value of the short-circuit input admittance is more than that of the short-circuit output admittance (when $T(s)$ is current transfer function, obviously, relationship between input and output impedances is vice versa).

Proof: First we show the property of open-circuit impedances. Suppose the two terminal-pair network, together with two ideal transformers, is connected in the manner indicated in Fig. 3.. The impedance $Z_{(s)}$ is easily shown to be given by

$$Z(s)=m^2Z_{11}(s)+2mnZ_{12}(s)+n^2Z_{22}(s), \qquad (7)$$

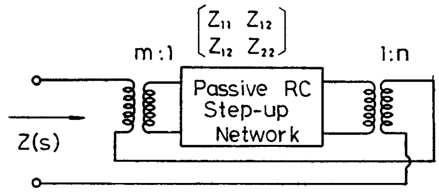

Fig. 3. Connection yielding $Z(s)=m^2Z_{11}+2mnZ_{12}+n^2Z_{22}$.

where the turn ratios m, n are arbitrary. The impedance function $Z_{ij}(jw)$ can be expressed in the form

$$Z_{ij}(jw)=\gamma_{ij}e^{-j\varphi_{ij}},$$

$$\text{where} \quad \gamma_{ij}\geqq 0 \qquad (8)$$

Since Z, Z_{11}, Z_{22} are passive RC driving point impedances

$$0\leqq\varphi,\ \varphi_{11},\ \varphi_{22},\ \leqq\frac{\pi}{2}. \qquad (9)$$

Substituting (8) and (9) in (7), and comparing with real and imaginary parts, respectively.

$$\frac{n^2}{\gamma_{11}\cos\varphi_{11}}\left[\left(\frac{m}{n}\gamma_{11}\cos\varphi_{11}+\gamma_{12}\cos\varphi_{12}\right)^2-\left\{\left(\gamma_{12}\cos\varphi_{12}\right)^2\right.\right.$$
$$\left.\left.-\gamma_{11}\gamma_{22}\cos\varphi_{11}\cos\varphi_{22}\right\}\right]=\gamma\cos\varphi \qquad (10)$$

$$\frac{n^2}{\gamma_{11}\sin\varphi_{11}}\left[\left(\frac{m}{n}\gamma_{11}\sin\varphi_{11}+\gamma_{12}\sin\varphi_{11}\right)^2-\left\{\left(\gamma_{12}\sin\varphi_{12}\right)^2\right.\right.$$
$$\left.\left.-\gamma_{11}\gamma_{22}\sin\varphi_{11}\sin\varphi_{22}\right\}\right]=\gamma\sin\varphi \qquad (11)$$

Since $\gamma\cos\varphi$, $\gamma\sin\varphi\geqq 0$, for any real number m, n, (10) and (11) must be written as:

$$\gamma_{11}\gamma_{22}\cos\varphi_{11}\cos\varphi_{22}-(\gamma_{12}\cos\varphi_{12})^2\geqq 0 \qquad (12)$$

$$\gamma_{11}\gamma_{22}\sin\varphi_{11}\sin\varphi_{22}-(\gamma_{12}\sin\varphi_{12})^2\geqq 0. \qquad (13)$$

Consequently

$$\frac{\gamma_{22}}{\gamma_{11}}\cos(\varphi_{11}\sim\varphi_{22})\geqq\left(\frac{\gamma_{12}}{\gamma_{11}}\right)^2 \qquad (14)$$

Obviously

$$0 \leqq \cos\ (\varphi_{11} \sim \varphi_{22}) \leqq 1 \tag{15}$$

$$|T(jw_0)| = \left|\frac{Z_{12}(jw_0)}{Z_{11}(jw_0)}\right| = \frac{\gamma_{12}}{\gamma_{11}} > 1 \tag{16}$$

from (14), (15) and (16), the following result is obtained dirictly

$$\gamma_{22} > \gamma_{11} \tag{17}$$

This completes the proof of the first part, and remainder part of the theorem can be shown same as above.

Here, appending the above theorem, we get the following

(*C*): Symmetrical quadripoles cannot be *RC* step-up network.

IV. Maximum gain

The voltage or current gain which can be obtained from typical passive *RC* step-up network is greater than unity. but actualy it is soldom to exceed two.[2] So, it is ieteresting to examine the theoretical maximum gain of the general passive *RC* network.[3]

(*D*): The theoretical maximum gain obtainable from a passive *RC* step-up network can be infinite.

To demonstrate this property, let $T_1(s)$, $T_2(s)$ be the transfer functions of the realizable passive three terminal *RC* step-up networks which belong to two terminal-pair networks, and can be written in the forms

$$T_1(s) = \sum_{i=0}^{n} a_i s^i \Big/ \sum_{i=0}^{n} b_i s^i \tag{18}$$

$$T_2(s) = \sum_{j=0}^{m} a_j' s^j \Big/ \sum_{j=0}^{m} b_j' s^j \tag{19}$$

and

$$|T_1(jw_0)| > 1, \qquad |T_2(jw_0)| > 1. \tag{20}$$

It is well known that the necessary and sufficient conditions that a real rational function $T(s)$ given by (18) or (19) is the transfer function of an *RC* three-termimal network are

(1) $T(s)$ must have simple pole lying on the negative real axis.

(2) $0 \leqq a_i \leqq b_i \qquad b_n \neq 0 \neq b_0$

Now, consider the function $T_{12}(s)$ defined by

$$T_{12}(s) = T_1(s) \cdot T_2(s) = \frac{\sum_{k=0}^{m+n} \left(\sum_{h=0}^{k} a_h a'_{k-h} \right) S^k}{\sum_{k=0}^{m+n} \left(\sum_{h=0}^{k} b_h b'_{k-h} \right) S^k}. \tag{21}$$

From the conditions mentioned above, the $T_{12}(s)$ must satisfy the following conditions

(1) $T_{12}(s)$ has poles lying on the negative real axis.

(2) $0 \leqq \sum_{h=0}^{k} a_h a'_{k-h} \leqq \sum_{h=0}^{k} b_h b'_{k-h}$, $b_n b_m' \neq 0 \neq b_0 b_0'$

then, it has been shown that this conditions are equivalent to realizable conditions indicated above and $T_{12}(s)$ must be three terminal passive RC realizable function only if the poles of $T_{12}(s)$ are simple. When $T_{12}(s)$ has the multiple poles $P = (s+\sigma)^l$ (where l is an integer and $\sigma > 0$). By continuity considerations we may form $P' = \prod_{i=1}^{l} (S+\delta_i)$ with the δ_i sufficiently close to σ and distinct. Therefore, the modified transfer function $T_{12}(s)$ must be three terminal passive RC step-up realizable function which has gain greater than $|T_1(jw_0)|$ and $|T_2(jw_0)|$.

Next, consider the function $T_{12\ldots q}(s)$ defined by

$$T_{12\ldots q}(s) = \prod_{i=1}^{q} T_i(s), \tag{22}$$

where $T_1(s), \ldots, T_q(s)$ are the realizable tranfer functions of the three terminal RC step-up networks. From the same treatment mentioned above, $T_{12\ldots q}(s)$ must be the realizable function of the passive RC step-up networks. And larger $\prod_{i=1}^{q} |T_i(jw_0)|$ could be obtained by going to higher values of q. Then, it may be shown that this implies the statement as given in the property (D). Thus, theoretical maximum gain is infiinite, but practically it is very difficult to construct the RC network which has gain much greater than unity because of wide spread of dispersion in RC element values of the network. Where w_0 is some finite frequency, and the next is evident.

(E): The gain at zero and infinite frequecies is never greater than unity.

V. *RC* network function of degree two

In this section we are concerned with the investigation of the most simple and practical three terminal passive RC step-up networks with degree two. And the following properties are agreeable without the proof.

(F): The simplest realizable passive RC step-up network function is of degree two.

The normalized transfer function of degree two can be written as

$$T(s) = \frac{a_2 s^2 + a_1 s + a_0}{s^2 + bs + 1}, \tag{23}$$

where, from RC realizable conditions, $b > 2$, $0 \leqq a_2 \leqq 1$, $0 \leqq a_1 < b$, $0 \leqq a_0 \leqq 1$.
And the following is evident.

(G): The necessary and sufficient condition that a function $T(s)$ given by (23) is the transfer function of an RC step-up nework is

$$a_1^2 - b^2 2a_0 a_2 + 2 > 2\sqrt{(1-a_2^2)(1-a_0^2)} \tag{24}$$

and $T(s)$ has gain greater than unity over a band of frequencies only.
Last, we give the following

(H): The transfer function of degree two with maximum gain is given by

$$\frac{2S+1}{S^2+(2+\varepsilon)S+1}, \tag{25}$$

where the $\varepsilon > 0$ sufficiently close to zero and maximum gain is about 1.15 at $w=0.707$.

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Reference

(1) A. Fialkow and I. Gerst, "The maximum Gain of an *RC* Network", Proc. I. R. E. 1953, Vol. 41, pp. 392-395.
(2) J. G. Holbrook, "Laplace transforms for Electronic Engineers", Chap. 6, Pergamon Press, 1959.
(3) H. Epstin, "Synthesis of passive *RC* networks witn gain greater than unity", Proc. I. R. E., pp. 833-835, July, 1951.