慶應義塾大学学術情報リポジトリ
Keio Associated Repository of Academic resouces

Title	On concept of orthogonal functions in engineering problems : (II)
Sub Title	
Author	鬼頭, 史城(Kito, Fumiki)
Publisher	慶応義塾大学藤原記念工学部
Publication year	1969
Jtitle	Proceedings of the Fujihara Memorial Faculty of Engineering Keio University (慶応義塾大学藤原記念工学部研究報告). Vol.22, No.88 (1969.) ,p.47(1)- 64(18)
JaLC DOI	
Abstract	In this paper, general consideration about eigen-values and eigen-functions are given. Three problems have been studied here, viz:- (a) problem of free vibration of a rectangular elastic flat plate which is fitted with stiffener-ribs, (b) problem of free vibration of an elastic bar which is immersed in water region, (c) free vibration of side-wall of a rectangular water tank which is full of water. It is shown here that, for every one of them the solution may be reduced to the solution of linear homogeneous integral equation (in a generalized sense). Also, it is shown that pairs of eigen-values and eigen-functions form a set of orthogonal functions (in a generalized sense).
Notes	
Genre	Departmental Bulletin Paper
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=KO50001004-00220088-0001

慶應義塾大学学術情報リポジトリ(KOARA)に掲載されているコンテンツの著作権は、それぞれの著作者、学会または出版社/発行者に帰属し、その権利は著作権法によって保護されています。引用にあたっては、著作権法を遵守してご利用ください。

The copyrights of content available on the KeiO Associated Repository of Academic resources (KOARA) belong to the respective authors, academic societies, or publishers/issuers, and these rights are protected by the Japanese Copyright Act. When quoting the content, please follow the Japanese copyright act.

On Concept of Orthogonal Functions in Engineering Problems—(II)

(Received June 11, 1969)

Fumiki KITO*

Abstract

In this paper, general consideration about eigen-values and eigen-functions are given. Three problems have been studied here, viz:— (a) problem of free vibration of a rectangular elastic flat plate which is fitted with stiffener-ribs, (b) problem of free vibration of an elastic bar which is immersed in water region, (c) free vibration of side-wall of a rectangular water tank which is full of water. It is shown here that, for every one of them the solution may be reduced to the solution of linear homogeneous integral equation (in a generalized sense). Also, it is shown that pairs of eigen-values and eigen-functions form a set of orthogonal functions (in a generalized sense).

I. Introduction

The author has given, in the previous paper[2] of the same title as the present one, some considerations about concept of orthogonal functions which are related to engineering problems. The treatment was confined to the case of vibration of one-dimensional bodies, that is, the case of elastic bars. It is natural to extend the treatment to the case of two-dimensional elastic bodies, namely the case of elastic flat-plates. In the present paper, the author gives some account about concept of orthogonal functions in connection with the eigen-value problems of free-vibration of elastic plates. Also, the case of flat plate or bar which are vibrating in water region is considered here. The author has previously reported some actual solutions about elastic bodies which are vibrating in water regeon. It is hoped that the content of present paper will give some fundamental ground, upon which the actual calculation of previous papers by the author[8)–10)] may stand.

II. Free vibration of an elastic flat-plate fitted with stiffner ribs

Let us consider the case of an elastic flat plate of rectangular shape fitted with a number of stiffener ribs, as shown in Fig. 1. The position of the stiffener ribs are notified by their abscissa, $x_1, \ldots, x_i$. In what follows, we shall treat the case of a

*鬼 頭 史 城 Professor, Faculty of Engineering, Keio University.

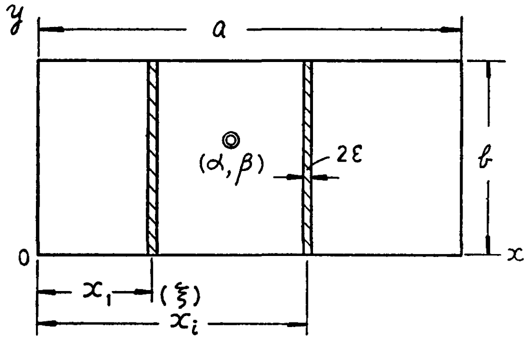

Fig. 1. Rectangular Elastic Plate fitted with Stiffener-ribs.

single stiffener-rib located at the abscissa ξ. The reasoning given below will apply to the case of several sitffener-ribs, by generalizing the expressions.

For the case of a rectangular elastic plate of length a and breadth b, and of uniform thickness h, which is vibrating with a small displacement w, the equation of motion may be written as follows,

$$D\left[\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2 \partial y^2}+\frac{\partial^4 w}{\partial y^4}\right]=$$
$$-\rho h\frac{\partial^2 w}{\partial t^2}-\left[E_1 I_1\left(\frac{\partial^4 w}{\partial y^4}\right)+\rho_1 A_1\left(\frac{\partial^2 w}{\partial t^2}\right)\right]\delta(x-\xi)$$
$$-\left[G_1 K_1\left(\frac{\partial^3 w}{\partial x \partial y^2}\right)-\rho_1 J_1\left(\frac{\partial^3 w}{\partial t^2 \partial x}\right)\right]\delta'(x-\xi)\,. \tag{1}$$

Notations used in this equation (1) are as follows:— w=transverse (small) displacement of the flat plate, D=flexural rigidity of the plate, or, $=Eh^3/[12(1-\nu^2)]$, h=thickness of the plate, E, ν=Young's modulus and Poisson's ratio of the plate material, ρ=density of the plate material, a, b=length and breadth of the rectangular flat plate, $\delta(x)$=delta function of Dirac, $\delta'(x)$=first derivative of $\delta(x)$.

As to quantities relating to the stiffener rib, we use the notation; G_1=shear modulus of elasticity, K_1=modulus of torsion, E_1=Young's modulus, I_1=secondary moment of sctional area of stiffener, ρ_1=material density, A_1=cross sectinal area, J_1=longitudinal secondary moment. This equation (1) was originally given (in more general form) by Mr. M. Higuchi(4), who obtained numerical values of eigen-values and eigen-functions by using electronic computers.

In the following discussions, we shall use, instead of the equation (1), more understandable form of the equation,

$$D\left[\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2 \partial y^2}+\frac{\partial^4 w}{\partial y^4}\right]$$
$$-\rho h\frac{\partial^2 w}{\partial t^2}=0, \tag{2}$$

which hold for the region of the rectangular plate, $0\leq x\leq a$, $0\leq y\leq b$, with exception of the position of stiffener rib, at which $\xi-\varepsilon<x<\xi+\varepsilon$, $0\leq y\leq b$, where ε is an infinitely small positive constant.

When the flat plate is in a state of free vibration with an angular frequency $\omega=2\pi f$, we may put

$$w = W \sin \omega t, \tag{3}$$

where W is a function of x and y which represent the amplitude of vibration at the point (x, y) on the plate.

Substituting the value of (3) into the equation of motion (2), we obtain,

$$D\left[\frac{\partial^4 W}{\partial x^4} + 2\frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4}\right] - \rho h \lambda W = 0, \tag{4}$$

where we put $\lambda = \omega^2$. This equation (4) is to hold on every point (x, y) on the rectangular plate, with exception of points on the rib, which is assumed to occupy the rectangular slender region given by,

$$\xi - \varepsilon \leq x \leq \xi + \varepsilon, \quad 0 \leq y \leq b.$$

The boundary conditions satisfied by the solution W are considered to be of usual ones. To fix our ideas, we may here confine ourselves to the case of a rectangular plate of which four edges are in state of fixed edges. In that case we have,

$$\left.\begin{array}{ll} \text{for} & x = 0 \text{ and } x = b\,[0 \leq y \leq b] \\ & W = 0, \quad \partial W/\partial x = 0, \\ \text{for} & y = 0 \text{ and } y = b\,[0 \leq x \leq a] \\ & W = 0, \quad \partial W/\partial y = 0. \end{array}\right\} \tag{5}$$

However, the similar argument may also be made for the other mode of fixation of edges, such as the case of supported edges, etc., etc.

In order to examine the nature of boundary conditions satisfied at the location of stiffener rib, we recall that the value of shearing forces and bending moments acting on cross-section of the flat plate are given by the following formula, using the customary notation such as used by S. Timoshenko;[11]

$$\left.\begin{array}{l} M_x = -D\left[\dfrac{\partial^2 w}{\partial x^2} + \nu \dfrac{\partial^2 w}{\partial y^2}\right], \\ M_y = -D\left[\dfrac{\partial^2 w}{\partial y^2} + \nu \dfrac{\partial^2 w}{\partial x^2}\right], \\ M_x + M_y = -(1+\nu)D\left[\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2}\right], \\ Q_x = -D\dfrac{\partial}{\partial x}\left[\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2}\right], \\ Q_y = -D\dfrac{\partial}{\partial y}\left[\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2}\right]. \end{array}\right\} \tag{6}$$

An approximate method for expressing the existence of stiffener-rib is to assume that Q_x and $M_x + M_y$ are connected to deflection of stiffener-rib by the following relations,

$$\left.\begin{aligned} -\left| Q_x \right|_{\xi-\varepsilon}^{\xi+\varepsilon} &= E_1 I_1 \left(\frac{\partial^4 w}{\partial y^4} \right) + \rho_1 A_1 \left(\frac{\partial^2 w}{\partial t^2} \right), \\ -\left| M_x + M_y \right|_{\xi-\varepsilon}^{\xi+\varepsilon} &= G_1 K_1 \left(\frac{\partial^3 w}{\partial x \partial y^2} \right) - \rho_1 J_1 \left(\frac{\partial^3 w}{\partial t^2 \partial x} \right), \end{aligned}\right\} \tag{7}$$

where the right hand side must be taken the values at $x=\xi$.

Now, let us take up two functions U, V of x and y which are everywhere continuous inside the rectangular region under consideration, up to fourth order derivatives, except on the location of stiffener-rib, where the discontinuity derived from the relation (7) are occurring. By the known formula

$$V[\Delta\Delta U]-U[\Delta\Delta V]=A+B, \tag{8}$$

where we write for brevity,

$$\Delta \equiv \partial^2/\partial x^2+\partial^2/\partial y^2,$$

and

$$\begin{aligned} A &= \frac{\partial}{\partial x}\left[\left\{V\frac{\partial}{\partial x}(\Delta U)-U\frac{\partial}{\partial x}(\Delta V)\right\}\right. \\ &\quad \left.-\left\{\frac{\partial V}{\partial x}(\Delta V)-\frac{\partial U}{\partial x}(\Delta V)\right\}\right], \\ B &= \frac{\partial}{\partial y}\left[\left\{V\frac{\partial}{\partial y}(\Delta U)-U\frac{\partial}{\partial y}(\Delta V)\right\}\right. \\ &\quad \left.-\left\{\frac{\partial V}{\partial y}\Delta U-\frac{\partial U}{\partial y}\Delta V\right\}\right]. \end{aligned}$$

By intergration over the rectangular area of the flat-plate, we have,

$$J_B=\int\!\!\int B\,dxdy=0, \tag{9}$$

$$J_A=\int\!\!\int A\,dxdy=\int_0^b K\,dy \tag{10}$$

Where we put,

$$\begin{aligned} K = \Big| & V\frac{\partial}{\partial x}\left(\Delta U\right)-U\frac{\partial}{\partial x}\left(\Delta V\right) \\ & -\frac{\partial V}{\partial x}\Delta U+\frac{\partial U}{\partial x}\Delta V\Big|_{\xi+\varepsilon}^{\xi-\varepsilon} \end{aligned} \tag{11}$$

When U and V plays the rôle of amplitude of vibration W, corresponding discontinuities can be derived from the equation (7). In that case, the above mentioned value K may also be written in the following form,

$$\begin{aligned} K = & -\frac{1}{D}VQ_x(U)+\frac{1}{D}VQ_x(V) \\ & +\frac{1}{(1+\nu)D}\frac{\partial V}{\partial x}\Big[M_x(U)+M_y(U)\Big] \\ & -\frac{1}{(1+\nu)D}\frac{\partial U}{\partial x}\Big[M_x(V)+M_y(V)\Big], \end{aligned} \tag{12}$$

where the notations (U) and (V) are used to indicate that they are quantities with regard to U and V, respectively, and that equation (7) is accounted for.

Based on these preliminary considerations, let us take two different solutions (w_i, λ_i) and (w_k, λ_k) of our equation (4) together with the boundary conditions (5) and also the condition (7) at the location of the rib. From two equations of the from (4), which we write for simplicity,

$$\left.\begin{aligned} L[W_i]-\rho h\lambda_i W_i=0,\\ L[W_k]-\rho h\lambda_k W_k=0, \end{aligned}\right\} \tag{13}$$

we obtain the equation

$$W_k L[W_i]-W_i L[W_k]-\rho h(\lambda_i-\lambda_k)W_i W_k=0. \tag{14}$$

We integrate both sides of this equation (14) over the rectangular area, by which we mean integration with respect to y from 0 to b, but with respect to x the integration is carried out from $x=0$ to $x=\xi-\varepsilon$, and from $x=\xi+\varepsilon$ to $x=a$. Using the above result of equation (12) for $U=W_i$ and $V=W_k$, we obtain the following result.

$$\int_0^b S\,dy-\rho h(\lambda_i-\lambda_k)\iint W_i W_k\,dx\,dy=0, \tag{15}$$

where we put

$$\begin{aligned} S=&-W_k Q_x(W_i)+W_i Q_x(W_k)\\ &+\frac{1}{1+\nu}\frac{\partial W_k}{\partial x}[M_x(W_i)+M_y(W_i)]\\ &-\frac{1}{1+\nu}\frac{\partial W_i}{\partial x}[M_x(W_k)+M_y(W_k)]\\ =&+W_k\left[E_1 I_1\frac{\partial^4 W_i}{\partial y^4}-\rho_1 A_1\lambda_i W_i\right]\\ &-W_i\left[E_1 I_1\frac{\partial^4 W_k}{\partial y^4}-\rho_1 A_1\lambda_k W_k\right]\\ &-\frac{1}{1+\nu}\frac{\partial W_k}{\partial x}\left[G_1 K_1\frac{\partial^3 W_i}{\partial x\partial y^2}+\rho_1 J_1\lambda_i\frac{\partial W_i}{\partial x}\right]\\ &+\frac{1}{1+\nu}\frac{\partial W_i}{\partial x}\left[G_1 K_1\frac{\partial^3 W_k}{\partial x\partial y^2}+\rho_1 J_1\lambda_k\frac{\partial W_k}{\partial x}\right]. \end{aligned} \tag{16}$$

Of course, at the right-hand side of this equation (16), the values must be taken on the location of the rib, that is, at $x=\xi$.

In connection with this equation (16), we notice that

$$\int_0^b\left[W_k\frac{\partial^4 W_i}{\partial y^4}-W_i\frac{\partial^4 W_k}{\partial y^4}\right]dy$$

$$=\int_0^b\frac{\partial}{\partial y}\left[W_k\frac{\partial^3 W_i}{\partial y^3}-W_i\frac{\partial^3 W_k}{\partial y^3}\right]dy$$

$$-\int_0^b \frac{\partial}{\partial y}\left[\frac{\partial W_k}{\partial y}\frac{\partial^2 W_i}{\partial y^2}-\frac{\partial W_i}{\partial y}\frac{\partial^2 W_k}{\partial y^2}\right]dy=0,$$

$$\int_0^b \left[\frac{\partial W_k}{\partial x}\frac{\partial^3 W_i}{\partial x\partial y^2}-\frac{\partial W_i}{\partial x}\frac{\partial^3 W_k}{\partial x\partial y^2}\right]dy=0,$$

on account of the boundary conditions at $y=0$ and $y=b$. Thus, the above relation (15) can be reduced into the following equation,

$$(\lambda_i-\lambda_k)\rho_1 A_1\int_0^b W_i W_k dy$$
$$+(\lambda_i-\lambda_k)\rho_1 J_1\int_0^b \frac{\partial W_i}{\partial x}\frac{\partial W_k}{\partial x}dy$$
$$+\rho h(\lambda_i-\lambda_k)\int\int W_i W_k dxdy=0.$$

Hence, we see that when $\lambda_i\neq\lambda_k$ for $i\neq k$, we have the following equation,

$$\rho h\int\int W_i W_k dxdy$$
$$+\rho_1 A_1\int W_i W_k dy+\rho_1 J_1\int \frac{\partial W_i}{\partial x}\frac{\rho W_k}{\partial x}dy=0. \tag{17}$$

In this equation (17), the double integral is to be taken over the entire area of the rectangular plate, while the simple integrals are to be taken on the location of slender stiffener-rib.

When there exists no stiffener-rib, the two eigen-functions of free vibration of the rectangular elastic plate satisfy the equation

$$\int\int W_i W_k dxdy=0 \tag{18}$$

which is called the orthogonality of two functions W_i, W_k. Thus, we are led to claim that the relation of two functions W_i and W_k which is expressed by the equation (17) is an orthogonality condition in the generalized sense. In accordance with this relation (17), we may call the "norm" of the eigen-function (in the generalized sense) to be the value N_i given by,

$$N_i=\rho h\int\int \{W_i\}^2 dxdy$$
$$+\rho_1 A_1\int \{W_i\}^2 dy+\rho_1 J_1\int \left\{\frac{\partial W_i}{\partial x}\right\}^2 dy. \tag{19}$$

There may be raised the question; how we can be sure of the existence of eigen-values λ_i and eigen-functions W_i? In order to answer to this question, we choose two solutions W and G of the equation (4), together with conditions at the boundary and at the rib. One of them, W, is taken as an actual solution of the problem of free-vibration, with an eigen-value of λ. The other one, G, is also taken to satisfy all these conditions as for W, but here we take $\lambda=0$.

Furthermore, to the function G, we impose another condition that, at a given point (α, β) it has a singularity of the order of $r^2 \log r$, where $r^2=(x-\alpha)^2+(y-\beta)^2$. Thus, we take

$$\begin{aligned} F&=G(x, y; \alpha, \beta) \\ &=-\frac{1}{8\pi} r^2 \log r+\Gamma(x, y; \xi, \eta), \end{aligned} \tag{20}$$

where $\Gamma(x, y; \xi, \eta)$ is a regular function inside the rectangular region under consideration. In the notation (13), we may write

$$\left.\begin{aligned} &L[W]-\rho h\lambda W=0 \\ &L[G]=0 \end{aligned}\right\} \tag{21}$$

Hence we have

$$GL[W]-WL[G]=\rho h\lambda WG \tag{22}$$

Next, we take the integral of both sides of this equation (22) over the entire area of the rectangular region with exception of infinitesimally small areas of following nature;

(1) Slender area on location of the rib,

(2) Circular area of infinitesimally small radius ε_1, with its center at the point (α, β).

The result of this integration gives us a linear integral equation, whose kernel is $G(x, y; \alpha, \beta)$, for unknown function $W(x, y)$. This integral equation is not a conventional form, but contains additional term pertaining to the effect of stiffener rib. Nevertheless, it may be called an integral equation in the more generalized sense.[3] From the theory of this generalized integral equation, we may conclude the existence of eigen-values λ_i, and corresponding eigen-functions $W_i(x, y)$. The integral equation mentioned above may be written as follows,

$$\begin{aligned} W(\xi, \eta)=\lambda\rho h &\iint G(x, y; \xi, \eta)W(x, y)\,dxdy \\ &+\lambda\rho_1 A_1 \int_0^b GW\,dy \ (\text{for } x=\xi) \\ &+\lambda\rho_1 J_1 \int_0^b \frac{\partial G}{\partial x}\frac{\partial W}{\partial x}\,dy \ (\text{for } x=\xi). \end{aligned} \tag{23}$$

III. Free vibration of an elastic bar in a water-region

Next, we shall take the case of an elastic bar, which is fixed in water region, as shown in Fig. 2. When the bar is vibrating, the surrounding water will also make a vibratory motion. The angular frequency of free vibration of the bar will be affected by the presence of surrounding water. Actual estimation of this effect has been made by the author, for the case of a circular bar.[8] Here, the fundamental account (theoretical basis) as to the existence of eigen-values and also the orthogonality of the eigen-functions related to this problem will be given. The equation of motion (vibraion) of the elastic bar may be written as follows

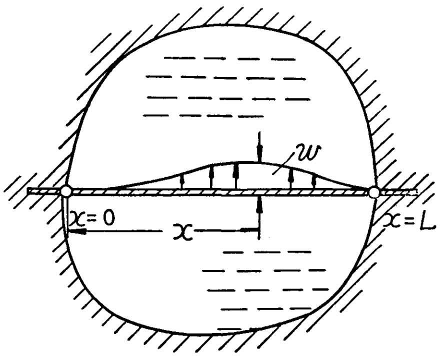

Fig. 2. Vibration of an Elastic-bar in a Water region.

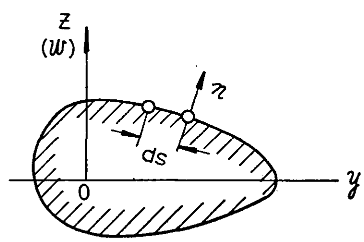

Fig. 3. Cross Section of the Elastic bar of Fig. 2.

$$\frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 w}{\partial x^2}\right]+\rho_m A\frac{\partial^2 w}{\partial t^2}+q=0 \tag{24}$$

where w is the transverse displacement (which is assumed to be of small magnitude) of the bar. EI is the flexural rigidity of cross-section of the bar, while $\rho_m A$ denote its mass per unit length of the bar. When the bar is vibrating in vacuo, $q=0$. When the bar is vibrating in water (or other liquid) region, the quantity q is added to equation (24), representing the resultant effect of fluid pressure, which act on the surface of the bar.

When the elastic bar is vibrating, surrounding water also vibrates. This vibratory motion of water can be expressed by a velocity potential ϕ, which is a function of x, y, z and t, satisfying the Laplace equation $\Delta\phi=0$ or

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0. \tag{25}$$

Here, we assume that the water (or, other fluid) to be an incompressible, non viscous fluid, and that the fluid motion is the one which is called potential flow in hydrodynamics. The velocity potential ϕ must satisfy, together with the Laplace equation (25), the boundary conditions at boundary surfaces of fluid region. The boundary walls are considered to consist of rigid wall and surface of the vibrating bar. The water region may be made of finite space or it may be extended to infinity. So that the rigid wall may be of finite extent (as shown in Fig. 2), or it may be extended to infinity.

(*a*) On the surface of the rigid wall, we must have

$$\frac{\partial \phi}{\partial n}=0 \tag{26}$$

where $\partial\phi/\partial n$ denotes the derivative in direction normal to the rigid wall.

(*b*) On the surface of the vibrating bar, we must have (see Fig. 3)

$$\frac{\partial \phi}{\partial n}=\frac{\partial w}{\partial t}\cos(n,\ z), \tag{27}$$

where $\cos(n, z)$ is cosine of the angle subtended between the normal drawn to the surface of the bar and direction of z-axis. It is assumed that direction of transverse displacement w of the bar is taking place in the direction of z-axis.

The velocity potential ϕ may be considered to consist of two parts namely

$$\phi=\phi_1+\phi_2. \tag{28}$$

ϕ_2 is the part corresponding to steady motion (if any) of the fluid, while ϕ_1 is the part corresponding to vibratory motion of the fluid which is assumed to be of infinitely small amplitude. However, at least in the present stage of the study, we take the case of no steady flow, for simplicity of the discussion. The fluid pressure is given by

$$\rho=\rho_w\frac{\partial \phi}{\partial t}-\frac{1}{2}\rho_w\left\{\left(\frac{\partial \phi}{\partial x}\right)^2+\left(\frac{\partial \phi}{\partial y}\right)^2+\left(\frac{\partial \phi}{\partial z}\right)^2\right\}$$

But taking only the non-steady part, we have

$$p=\rho_w\frac{\partial \phi_1}{\partial t}-\rho_w\left[\frac{\partial \phi_1}{\partial x}\frac{\partial \phi_2}{\partial x}+\frac{\partial \phi_1}{\partial y}\frac{\partial \phi_2}{\partial y}+\frac{\partial \phi_1}{\partial z}\frac{\partial \phi_2}{\partial z}\right]$$

Moreover, restricting ourselves to the case of no steady flow, we may put approximately

$$p=\rho_w\frac{\partial \phi}{\partial t}\ . \tag{29}$$

For the case of free vibration with an angular frequency $\omega=2\pi f$, we may put

$$\left.\begin{aligned}\phi&=\omega\Phi\cos\omega t\\ w&=W\sin\omega t\\ q&=Q\sin\omega t\end{aligned}\right\} \tag{30}$$

where Φ is a function of x, y, z which satisfy the Laplace equation $\Delta\Phi=0$. W is a function of x which represents the amplitude of vibration of the bar. Q is a function of x, which represents the amplitude of fluid force acting on the bar.

The equation of free vibration, then becomes as follows,

$$\frac{d^2}{dx^2}\left[EI\frac{d^2W}{dx^2}\right]-\rho_m A\lambda W+Q=0 \tag{31}$$

where we put $\lambda=\omega^2$.

The function Φ must satisfy together with the Laplace equation, the following boundary conditions;

(a) On the rigid wall

$$\frac{\partial \Phi}{\partial n}=0, \tag{32}$$

(b) On the surface of the bar

$$\frac{\partial \Phi}{\partial n}=W \cos (n,\ z). \tag{33}$$

Moreover we have

$$p=-\lambda \rho_w \Phi \sin \omega t$$

$$q=-\lambda \rho_w \sin \omega t \int \Phi \cos (n,\ z)\, ds,$$

where the last integral means an integration over the surface of the bar at the position x. Thus, ds means a line-element of the cross-sectional curve of the bar, as shown in Fig. 3. The above expresson for q may be written in the form

$$\left.\begin{aligned} &q=Q \sin \omega t, \qquad Q=-\lambda \rho_w R \\ &R=\int \Phi \cos (n,\ z)\, ds \end{aligned}\right\} \tag{34}$$

Now, let us take two pairs of solutions $(W_i,\ \lambda_i)$, $(W_k,\ \lambda_k)$ of this problem of free vibration. Corresponding values of Q and R will be denoted respectively by Q_i, R_i and Q_k, R_k.

From the equation (31) we have

$$\frac{d^2}{dx^2}\left[EI \frac{d^2 W_i}{dx^2}\right]-\rho_m A \lambda_i W_i+Q_i=0, \tag{35}$$

$$\frac{d^2}{dx^2}\left[EI \frac{d^2 W_k}{dx^2}\right]-\rho_m A \lambda_k W_k+Q_k=0 \tag{36}$$

Whence we obtain, by making the equation

$$W_k \cdot [\text{eq. } (35)]-W_i \cdot [\text{eq. } (36)]=0,$$

and integrating both sides of this equation, and taking into account the end-conditions of the bar (for example, case of fixed ends), we arrive at the following result,

$$\begin{aligned} &-\rho_m(\lambda_i-\lambda_k) \int W_i W_k\, dx \\ &\quad+\int [Q_i W_k-Q_k W_i]\, dx=0, \end{aligned} \tag{37}$$

the integration being made for the entire length of the bar.

From the equation (34), we remark that

$$Q_i=-\rho_w \lambda_i R_i, \qquad Q_k=-\rho_w \lambda_k R_k.$$

Next, if we put

$$I_{ik}=\int R_i W_k\, dx, \qquad I_{ki}=\int R_k W_i\, dx$$

we notice (from the discussion given in the next section) that

$$I_{ki}=I_{ki}$$

Hence, if $\lambda_i \neq \lambda_k$ for $i \neq k$, we obtain the following equation from eq. (37)

$$\rho_m \int W_i W_k \, dx - \rho_w \int R_i W_k \, dx = 0. \tag{38}$$

the integration being made for the whole length of the bar.

As we see in the next section, R_i is a linear function of W_i. If water does not exist, we have $\rho_w=0$, and the equation (38) reduces into usual orthogonality relation. If $\rho_w \neq 0$, the equation (38) is not an orthogonality relation in the conventional sense. However, we may call this relation (38) an orthogonality relation in the generalized sense. [see ref. (3)]

IV. Existence of the solution of problem in the previous section

The potential function Φ mentioned in the previous section must satisfy the Laplace equation $\Delta\phi=0$, inside the water region, and must satisfy the boundary conditions (32) and (33). Moreover, if the fluid extend to infinity, it must be such that Φ tend to zero of order of $1/r^2$, as the distance from the origin, r tend to infinity. When the function W in the equation (33) is regarded to be a known function, the problem to find the function Φ becomes the Neumann's problm in potential theory.(7)

According to the theory of Neumann's problem,(7) we introduce the Green's function of the second kind, by the formula,

$$G(Q,\ P)=\frac{1}{r}+V(Q,\ P) \tag{39}$$

where Q, P are two points in the region under consideration. $N(Q, P)$ and $V(Q, P)$, regarded as functions of point Q in the region, must satisfy the Laplace equation, namely

$$\Delta_Q G(Q,\ P)=0, \qquad \Delta_Q V(Q,\ P)=0 \tag{40}$$

In equation (39) r is the distance of two points P and Q. The point P is the singular point of the Green's function $G(Q,\ P)$. In order to use the Neumann's problem, $V(Q,\ P)$ regarded as a function of Q must be regular inside the region, and must be constructed in such way that the normal derivative of V is different from that of $-1/r$ only by a constant. This means that the Green's function $G(Q.\ P)$ is such that its normal derivatives on the boundary surface shall be null (or a constant).

Using this Green's function of the second kind, we have

$$U(P)=\frac{1}{4\pi}\iint \frac{\partial U}{\partial \nu} G(Q,\ P)\, dS$$
$$+ \text{ additive constant,} \tag{41}$$

the surface integral being taken over the boundary surface. This formula enables us to obtain the value $U(P)$ of potential function, when the value of $\partial U/\partial \nu$ along the boundary surface is given. When, as in the case of previous section, the function sought for (U) is velocity potential (Φ) of fluid motion, the additive constant in the formula (41) will not affect in any way actual results of our problem.

Applying this formula (41) to our problem in the previous section for vibration of elastic bar, we have

$$\Phi = \frac{1}{4\pi} \iint [W \cos(n, z)] G(Q, P)\, dS \tag{42}$$

where the surface integral is extended to whole surface of the bar which is immersed in fluid region (Fig. 2).

Corresponding value of the factor R, defined by (34), is given by

$$\begin{aligned} R &= \int_s \Phi \cos(n, z)\, ds \\ &= \int_s \cos(n, z)\, ds \int_0^L dx \int_s [W \cos(n, z)] \\ &\quad \times G(Q, P)\, ds \end{aligned} \tag{43}$$

Hence, we have

$$\begin{aligned} R_i &= \frac{1}{4\pi} \int_s \cos(n, z)\, ds \int_0^L dx \int_s G(Q, P) \\ &\quad \times [W_i \cos(n, z)]\, ds \end{aligned} \tag{44}$$

where the integration ds is carried out for the surface area of the bar, while the integration dx is carried out over the whole length of the bar. Since $\cos(n, z)$ and $G(Q, P)$ are known functions, so long as the configuration of our problem is given, this equation (43) tells us that R is a linear homogeneous function of W. So also, by (44), R_i is a linear homogeneous function of W_i. Moreover, the value of factor I_{ik} defined in the previous section is given by the following formula,

$$\begin{aligned} I_{ik} &= \int_0^L R_i W_k\, dx \\ &= \frac{1}{4\pi} \int_0^L W_k \int_s \cos(n, z)\, ds \int_0^L dx \\ &\quad \times \int_s G(Q, P) [W_i \cos nz]\, ds \end{aligned} \tag{45}$$

It may be remarked that this formula (45) gives only formal expression, and not the full expression of an individual case.

The expression (43), being a homogeneous linear function for W, can be written in the form,

$$R=\int_0^L H(x,\xi)\,W(\xi)\,d\xi \tag{46}$$

Using this form for R, together with the relation (34), the equation (31) of free vibration of our present problem can be rewritten into the following form,

$$\frac{d^2}{dx^2}\left[EI\frac{d^2W(x)}{dx^2}\right]-\rho_m A\lambda W(x)$$
$$-\rho_w\lambda\int_0^L H(x,\xi)W(\xi)\,d\xi=0 \tag{47}$$

This is an integro-differential equation with respect to the unknown function $W(x)$.

In order to examine the qeustion of existence of eigen-value λ and eigen-function $W(x)$ about this equation (47), let us take another Green function defined in the following manner;

(a) It satisfies the differential equation

$$\frac{d^2}{dx^2}\left[EI\frac{d^2g(x,\alpha)}{dx^2}\right]=0 \tag{48}$$

(b) It satisfies the same end-conditions as for the solution $W(x)$, for example those of fixed ends.

(c) At the point $x=\alpha$ on the bar, $g(x,\alpha)$ has a discontinuity defined by

$$\left|\frac{d}{dx}\left[EI\frac{d^2}{dx^2}g(x,\alpha)\right]\right|_{\alpha-0}^{\alpha+0}=1 \tag{49}$$

Then, from equations (47) and (48), we obtain

$$g(x,\alpha)\frac{d^2}{dx^2}\left[EI\frac{d^2W(x)}{dx^2}\right]$$
$$-W(x)\frac{d^2}{dx^2}\left[EI\frac{d^2g(x,\alpha)}{dx^2}\right]$$
$$-\rho_m A\lambda W(x)g(x,\alpha)$$
$$-\rho_w\lambda g(x,\alpha)\int_0^L H(x\ \ \xi)W(\xi)d\xi=0 \tag{50}$$

Next, we integrate both sides of this equation from $x=0$ to $x=\alpha-\varepsilon$, and also from $z=\alpha+\varepsilon$ to $x=L$, and make $\varepsilon\to 0$. The result is as follows;

$$W(\alpha)+\rho_m\lambda A\int_0^L W(x)g(x,\ \alpha)\,dx$$

$$+\lambda\rho_w\int_0^L g(x,\ \alpha)\,dx\int_0^L H(x,\ \xi)W(\xi)\,d\xi=0,$$

or

$$W(\alpha)+\lambda\int_0^L W(x)K(x,\ \alpha)\,dx=0 \tag{51}$$

where we put

$$K(x,\ \alpha)=\rho_m Ag(x,\ \alpha)+\rho_w\int_0^L d\eta\int_0^L d\xi g(\eta,\ \alpha)H(\eta,\ \xi) \tag{52}$$

This equation (51) is an integral equation of conventional form. We may claim, from what is known about the theory of linear integral equations, the existence of eigen-function $W(x)$ and eigen-value λ.

V. Free vibration of side-wall of a water-tank, full of water

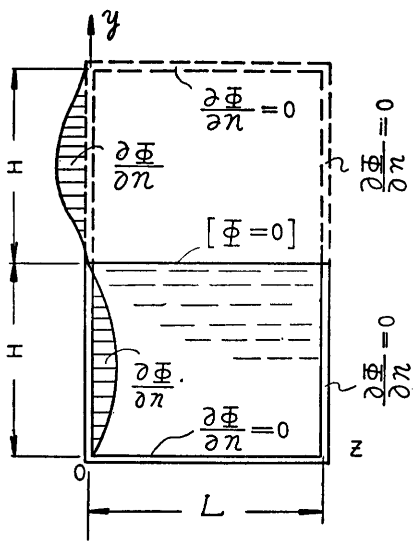

Fig. 4. Vibration of Side-wall of a Water tank.

We consider a rectangular water tank of height H, length L, and breadth B, which is full of water. Taking rectangular axis as shown in Fig. 4, this tank occupies the space

$$0\leq x\leq B,\ 0\leq y\leq H,\ 0\leq z\leq L.$$

In the present paper we take up only one case in which a panel

$$0\leq x\leq B,\ 0\leq y\leq H,\ z=0$$

of side wall is vibrating, the other four panels are here assumed to stand still. The top surface at

$$0\leq x\leq B,\ y=H,\ 0\leq z\leq L$$

is assumed to be in state of free surface. This being so, the arguments similar to that given below may be made for other cases of vibration of side-walls, such as, for example, (*a*) both side-walls are vibrating simultaneously, (*b*) only the bottom panel is vibrating, etc., etc.

Actual values of eigen-values and eigen-functions have already been sought by the author.[9] Here we shall make studies about the general nature and existence of eigen-values and eigen-functions. The displacement of rectangular elastic plate at $x=0$ ($0\leq x\leq B$, $0\leq y\leq H$) will be denoted by w which is a function of x and y; of infinitesimally small magnitude. Here also, we shall put $w=W\sin\omega t$ as given in equation (3).

The equation of free (transverse) vibration of this plate may also be written as our equation (4),

$$D\left[\frac{\partial^4 W}{\partial x^4}+2\frac{\partial^4 W}{\partial x^2\partial y^2}+\frac{\partial^4 W}{\partial y^4}\right]-\rho h\lambda W+Q=0 \tag{4a}$$

Here, the term Q is added in order to represent the effect of hydraulic pressure acting on the plate, putting that

$$q=Q\sin\omega t.$$

The boundary conditions for the rectangular plate, such as given in eq. (5) must also be imposed here.

When the side-wall is vibrating, the water (or other fluid) filled up in the tank will also make a vibratory motion. This motion of water will be expressed by a velocity potential ϕ. In general, this velocity potential ϕ will consist of two parts as shown by equation (28). But, here, for shortness, we take up the case of fluid vibration of infinitesmally small amplitude, omitting the effect. if any, of steady flow. The velocity potential ϕ must satisfy the Laplace equation (25) together with the boundary condition required for the present problem. This boundary condition may be written as

(*a*) at vibrating side wall, for which we have $z=0$. $0\leq x\leq B$, $0\leq y\leq H$.

$$\frac{\partial\phi}{\partial z}=w(x,\ y\ ;\ t)=W\sin\omega t,$$

(*b*) at side walls which stand still,

$$\frac{\partial\phi}{\partial n}=0,$$

(*c*) at the bottom wall plate, at which we have $y=0$, $0\leq x\leq B$. $0\leq z\leq L$,

$$\frac{\partial\phi}{\partial y}=0,$$

(*d*) at the top surface, at which we have $y=H$, $0\leq x\leq B$, $0\leq z\leq L$,

$$p=\rho_w\frac{\partial\phi}{\partial t}=0,$$

this equation being deduced from equation (29). Thus the top surface $y=H$ is considered to be in state called "free-surface" in hydrodynamical theory.

The transverse load $q=Q\sin\omega t$ which act on the rectangular plate, may at the present problem, be taken as hydraulic pressure p itself. So that we have,

$$\phi=\omega\,\Phi\cos\omega t,\ p=-\rho_w\lambda\,\Phi\sin\omega t,\ Q=-\rho_w\lambda\Phi.$$

Now, we must show how to find out the potential function Φ, which satisfies the Laplace equation $\Delta\Phi=0$ inside the water region together with the boundary condition as follows;

(a) at the vibrating wall. for which $z=0$,

$$\frac{\partial\Phi}{\partial z}=W(x,\ y),$$

(b) at side walls which stand still

$$\frac{\partial\Phi}{\partial n}=0,$$

(c) at the bottom surface, at which $y=0$,

$$\frac{\partial\Phi}{\partial y}=0,$$

(d) at the top surface at which $y=H$,

$$\Phi=0.$$

This potential problem may be termed mixed-type boundary value problem, for which we have not known explicit formula as Green's or Neumann's. Nevertheless, since the top surface is always a horizontal plane, we may use the method of images, as shown in Fig. 5. Thus, by adding an imaginary boundary region, as shown in Fig. 5, we may omit the condition at a horizontal plane surface, and reduce our problem to classical case of Neumann's problem. For the case of rectangular water tank with a free surface, as shown in Fig. 4, we imagine that an inverted fictitous water tank is added to the actual tank, and reduce our boundary value problem to that of water tank with height of $2H$.

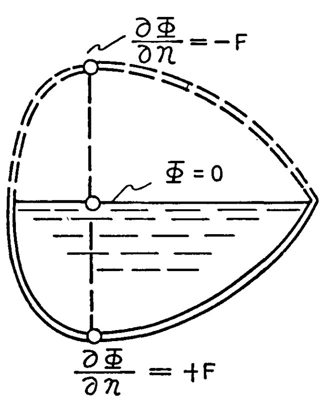

Fig. 5. Image-boundary Region introduced for Solving the problem with free-Surface.

The formal solution of this problem may be written by equation (41), where the surface integral is to extend to two (one actual, one imaginary) panels at $z=0$. Thus we have

$$\Phi=\frac{1}{4\pi}\iint WF(Q,\ P)\,dS \tag{53}$$

where we put

$$F(Q.\ P)=G(Q,\ P)-G(Q,\ P')$$

P' being a point which is the image of actual point P. The integration in this formula (53) is carried out over a panel of actual rectangular plate, at which we have $z=0$, $0\leq y\leq H$, $0\leq x\leq B$.

The relation (53) shows us that Φ is a linear function of W. The equation (4*a*) of free vibration may therefore, be written in the following form,

$$D\left[\frac{\partial^4 W}{\partial x^4}+2\frac{\partial^4 W}{\partial x^2\partial y^2}+\frac{\partial^4 W}{\partial y^4}\right] -\rho h\lambda_m W-\rho_w\lambda\frac{1}{4\pi}\iint WF(Q,\ P)\,dS=0 \tag{54}$$

Let us assume then that there exist pairs of $(W_i,\ \lambda_i)$ and $(W_k,\ \lambda_k)$ of solutions. Then we shall have two pairs of equations of the form of (54), wherein $(w,\ \lambda)$ are replaced by $(W_i,\ \lambda_i)$ and $(W_k,\ \lambda_k)$ respectively. Making the equation

$$W_k\times[\text{eq. (54) for } (W_i,\ \lambda_i) -W_i\times[\text{eq. (54) for } (W_k,\ \lambda_k)]=0 \tag{55}$$

and integrating both sides of this equation (55) over the rectangular area, we obtain, taking into account the boundary conditions similar to (5)

$$\rho_m H\iint W_iW_k\,dS -\left(\frac{1}{4\pi}\right)\rho_w\iint dS_1\iint W_i(P_1)W_k(P_2)F(P_1,\ P_2)dS_2=0. \tag{56}$$

which hold so long as $\lambda_i\neq\lambda_k(i\neq k)$. This relation (56) shows us that eigen-functions W_i and W_k are orthogonal each other. But here the orthogonality is understood to be meant in the generalized sense.

Furthermore, let us take the solution of the equation

$$D\left[\frac{\partial^4 J}{\partial x^4}+2\frac{\partial^4 J}{\partial x^2\partial y^2}+\frac{\partial^4 J}{\partial y^4}\right]=0. \tag{57}$$

J is taken to be a function of x, y which satisfies this equation togther with the boundary condition as for W, but that it has a singularity at a point on the rectangular plate $(\alpha,\ \beta)$ in such manner that

$$J(x,\ y;\ \alpha,\ \beta)=-\frac{1}{8\pi}r^2\log r+\Gamma(x,\ y;\ \alpha,\ \beta) \tag{58}$$

where we put

$$r^2=(x-\alpha)^2+(y-\beta)^2$$

Here again, we obtain from equations (54) and (58), after making integration over the whole area of the rectangular region in which a circular area of small radius (at center α, β) is omitted beforehand,

$$DW(\alpha,\ \beta)=\rho_m h\lambda\iint W(x,\ y)J(x,\ y;\ \alpha,\ \beta)dS +\rho_w\lambda\frac{1}{4\pi}\iint J(x,\ y;\ \alpha,\ \beta)\,dx\,dy\iint W(x,\ y)F(Q,\ P)dS \tag{59}$$

Thus we see that, once again, the solution of our problem must satisfy a linear homogeneous integral equation (59). Therefore, the existence of eigen-values and eigen-functions may also be deduced from theory of generalized integral equations, such as given in reference (3).

VI. Concluding remarks

Heretofore, much has been discussed about existence of solution of boundary value problems in hydrodynamics. [see ref. (5) and (6)] In the present paper the author has made some general considerations about the existence of solution of eigen-value problems in hydro-elasticity. Some actual results of numerical calculation of individual problems, have already been reported by the author. [see ref. (8) and (9), (10)]

References

(1) A. Kneser, Die Integralgleichungen, 1922 (Vieweg & Sohn) Section 31, Belastete Integralgleichugen.

(2) F. Kito, On Concept of Orthogonal Functions in Engineering Problems—I, This Proceedings, Vol. 21, No. 4, 1968.

(3) N. Gunther, Sur les Integrales de Stieltjes, (Chelsea Publ. Co) 1949, Chap. 3 (Section 15); Chap. 4.

(4) M. Higuchi, Calculation of Characteristic Value of Reinforced Rectangular Plate in Vibration, by Means of Electric Computer (1st report), Trans. Inst. Nav. Architecture Japan, No. 125, Nov., 1966.

(5) L. Lichtenstein, Grundlagen der Hydromechanik, 1929 (Julius Springer).

(6) C. W. Oseen, Neuere Methoden und Ergebnisse in der Hydrodynamik, 1927 (Akademishe Verlags Gesellschaft).

(7) O. D. Kellog, Foundations of Potential Theory, 1929 (Julius Springer).

(8) F. Kito, On Vibration of an Elastic Bar immersed in the Water region lying between the two parallel plane walls, This Proceedings, I (Vol. 10, No. 39, 1957), II (Vol. 16, No. 61, 1963).

(9) F. Kito, On Virtual mass of Water contained in a Rectangular Tank whose side Walls are vibrating, This Proceedings, I (Vol. 11, No. 40, 1958), II (Vol. 12, No. 46, 1959), III (Vol. 12, No. 47, 1959), IV (Vol. 13, No. 49, 1960), VI (Vol. 15, No. 56, 1962), VII (Vol. 15, No. 56, 1962), VIII (Vol. 15, No. 57, 1962).

(10) F. Kito, On Vibration of a Circular Elastic Plate which is in Contact with Water, This Proceedings, Vol. 19, No. 1, 1966.

(11) S. Timoshenko, Theory of Plates and Shells, 1940 (McGraw-Hill Co.).