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Limit Cycle of a Modified van der Pol Equation

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Abstract

A nonlinear differential equation of the form:

$$\ddot{x}-\varepsilon(1-2\beta x-x^2)\dot{x}+x=0$$

is analyzed to form a limit cycle in a phase plane by digital computer. Using Bendixson-Poincare's theorem repeatedly the limit cycle is obtained in a narrow ring-shaped domain. The amplitudes and the period of the solution x are obtained.

In analyzing a negative resistance oscillator it is very common to use the van der Pol equation.(1)(2)(3) The characteristics of the negative resistance in the actual oscillators, however, is not always symmetrical with respect to the operating point. It seems significant to analyze such an oscillator with asymmetrical negative resistance. The nonlinear differential equation for the oscillator is given by

$$\ddot{x}-\varepsilon(1-2\beta x-x^2)\dot{x}+x=0, \qquad \varepsilon>0, \tag{1}$$

where $\dot{x}=dx/dt$. Equation (1) agrees with the van der Pol equation when $\beta=0$. Therefore equation (1) is a modified van der Pol equation. Analysis is made only to form a limit cycle in the phase plane. According to the Bendixson-Poincare's theorem(4) there exists at least one limit cycle in a ring-shaped domain D if trajectories enter D through the boundary and if there are no singular points in D or on the boundary. It is easily shown that equation (1) has no singular point except at the origin. As shown in Figure 1 the ring-shaped domain D is formed in the $(x, \dot{x})$ plane by two trajectories of equation (1) (one from A_1 to A_2, the other from B_1 to B_2), and two line-segments on the x-axis ($\overline{A_1A_2}$ and $\overline{B_1B_2}$) excluding the origin. Two distinct initial points A_1 and B_1 are selected on the positive x-axis, such that two trajectories from A_1 and B_1 reach the points A_2 and B_2 on the positive x-axis respectively with the relation:

$$0<A_1<A_2<B_2<B_1.$$

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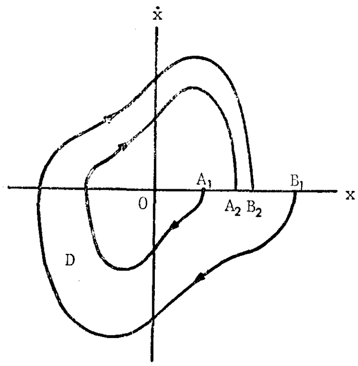

Fig. 1. Ring-shaped domain D.

The process of forming the ring-shaped domain is repeated such that

$$0<A_1<A_2<\cdots<A_{n-1}<A_n<B_n<B_{n-1}<\cdots<B_2<B_1.$$

When the difference $|A_n-B_n|$ becomes sufficiently small such that

$$|A_n-B_n|\leqq 10^{-4},$$

the final narrowest domain is obtained, and at least one limit cycle is located inside of this domain.

The limit cycles of equation (1) are shown in Fig. 2 for $\varepsilon=0.2$, 0.5, 1.0, 2.0, and $2\beta=0$, 1, 2. Numerical computation based on Runge-Kutta's method is performed by TOSBAC-3400 computer with the increment $\Delta t=0.002$. The computed values are the positive amplitude $a_1(=x_{\max})$, the negative amplitude $a_2(=x_{\min})$, the period T, and the time interval T' in which x is positive, as shown in Table 1. The negative time interval in which $x<0$ is eliminated because it can be calculated by $T-T'$.

When the parameter $|\beta|$ increases the limit cycle becomes more asymmetrical with respect to the origin. When β approaches to zero the limit cycle becomes symmetrical and agrees with the one obtained for the van der Pol equation. The time interval T' in which x is positive increases as β becomes large, while the time interval in which x is negative changes little. The positive amplitude a_1 charges little as β becomes large, while the negative amplitude $|a_2|$ becomes large as β becomes large. The estimated errors are in the order of 0.01% for both the positive and the negative amplitudes, and 0.1 % for the period. The above obtained results cover the case for $\beta<0$ if x is replaced by $-x$ in equation (1). Also equation (1) coincides with the following differential equations by simple transformations,

$$\ddot{x}-\varepsilon(1-x^2)\dot{x}+x=A, \quad |A|<1,$$

$$\ddot{x}-\varepsilon(1-\beta'\dot{x}-\dot{x}^2)\dot{x}+x=0,$$

$$\ddot{x}-\varepsilon(1-\dot{x}^2)\dot{x}+x=At+B.$$

The more detailed limit cycles of the same equation (1) are shown in Figure 3 for $\varepsilon=0.1$—2.0 with step 0.1 and $2\beta=0.1$—2.0 with step 0.1. Numerical calculation is performed by NEAC L-2 with the aid of hybrid computer system by courtesy of NEC (Nippon Electric Company). In Figure 3 the values for parameter β are not indicated, because the limit cycles change their configurations continuously, therefore the values of β are easily known by comparing Figure 3 with Figure 2.

The authors wish their thanks deeply to Mr. K. Kagiyama, research manager at Central Research Laboratory of NEC who offered willingly the enormous computer results of Figure 3.

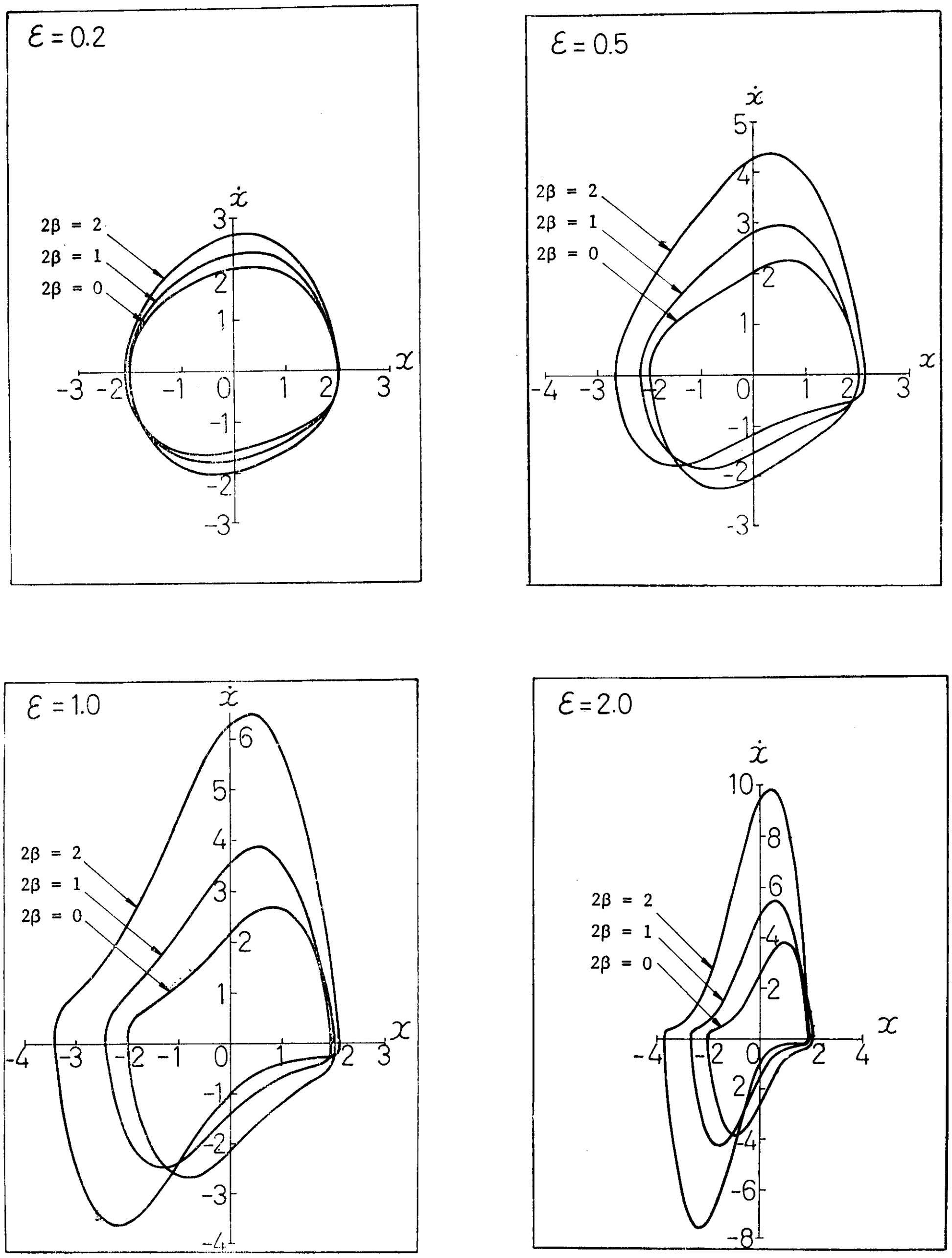

Fig. 2. Limit cycles of equation (1) for $\varepsilon=0.2,\ 0.5,\ 1.0,\ 2.0$ and $2\beta=0,\ 1,\ 2$.

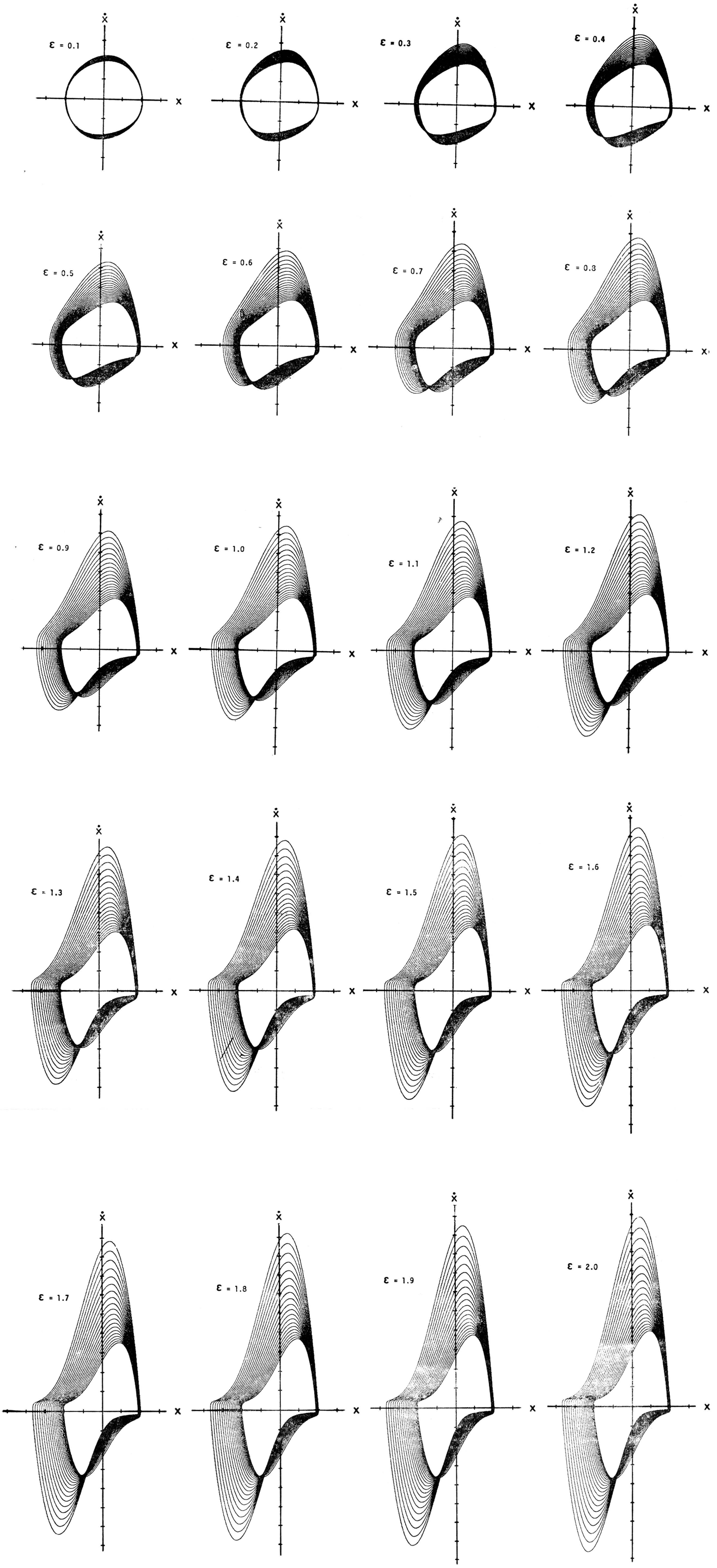

Fig. 3. Limit cycles of equation (1) for $\varepsilon=0.2$—2.0 with step 0.1 and $2\beta=0.1$—2.0 with step 0.1.
The distance ⊢—⊣ is 1 for $\dot{x}$-and x-axes.

Table 1. Positive amplitude a_1, negative amplitude a_2, period T, and positive interval T'.

ε	2β	a_1	a_2	T	T'
0.2	0	2.0004	−2.0003	6.30	3.15
	1	2.0003	−2.0287	6.34	3.19
	2	2.0299	−2.6457	6.47	3.29
0.5	0	2.0025	−2.0024	6.38	3.19
	1	1.9944	−2.1635	6.61	3.43
	2	2.0818	−2.6457	7.45	4.08
1.0	0	2.0086	−2.0086	6.66	3.33
	1	1.9416	−2.4253	7.25	4.01
	2	2.0818	−3.4172	9.11	5.62
2.0	0	2.0199	−2.0199	7.63	3.82
	1	1.8540	−2.6336	8.79	5.21
	2	1.9589	−3.7136	12.45	8.37

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