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On the Synthesis of an Optimal Control System for Lateral-directional Motion of Airplanes

(Received March 20, 1969)

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Abstract

The dynamic system which expresses lateral-directional motion of an airplane constructs a typical multivariable system, so-called *P*-canonical structure in which each of the outputs is expressed as explicitly and wholly dependent upon all of the inputs.

In this study, an reference model was introduced to the dynamic system and the method of variational calculus was applied under condition such as the accumulated quantity of error signal between the model and the practical airplane dynamics would be restricted.

The control actions were based on the error signals and their derivatives, so that the optimal control system was synthesized. And further, it was shown that the realization of such a control would be possible under condition of comparatively small acceleration without making increment of the settling time.

At the end of this study, a numerical example was shown, in which the difference of responses was illustrated between the cases that the dynamic system would be carried out an ordinary feedback control and such as the optimal control.

I. Introduction

In such a case as the optimal control techniques are applied to a multivariable system in which the interactions arise between the internal signals, generally it will be necessary that either the non-interacted system is constructed in advance or the optimal controller is synthesized under consideration of the effect of interactions.

The synthesis of the optimal control system for lateral-directional motion of airplanes is one of the most typical example for this kind of problem. In either case as to be stated above, the dynamic characteristics of the airplane must be considered, however, in this study, it was attempted to realize the optimal control by means of rather utilizing the effect of interaction without making non-interaction in advance.

In order to linearize the equations of motion which describe the lateral-directional behaviour of the airplane in the neighbourhood of any equilibrium state of the motion, it was assumed that the flight-path angle would be comparatively small,[1] and particularly in the equations of aerodynamic forces or moments, the terms not less

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than second order were neglected[2] as to be sufficiently small.

From the practical condition, a restriction was made to the accumulated quantity of error signal between the reference model and the airplane dynamics.

Further, the synthesis of optimal controller was advanced on the supposition that the effect of rudder would be less than the effect of aileron in the side-force equation.

II. Notation

C_l rolling-moment coefficient
C_n yawing-moment coefficient
C_y side-force coefficient
I_x, I_y, I_z moment of inertia about the X, Y and Z axes
I_{xz} product of inertia
S wing area
b wing span
W weight of airplane
m mass of airplane
g acceleration due to gravity
t time
D the operator ...... d/dt
V velocity
α angle of attack
γ flight-path angle
φ angle of bank
ψ angle of yaw
β side-slip angle
p rolling velocity
r yawing velocity
δ_a aileron deflection
δ_r rudder deflection
q_0 dynamic pressure

$$C_{lp} = \frac{\partial C_l}{\partial(pb/2V)}$$

$$C_{lr} = \frac{\partial C_l}{\partial(rb/2V)}$$

$$C_{l\beta} = \frac{\partial C_l}{\partial\beta}$$

$$C_{np} = \frac{\partial C_n}{\partial(pb/2V)}$$

$$C_{nr} = \frac{\partial C_n}{\partial(rb/2V)}$$

$$C_{n\beta} = \frac{\partial C_n}{\partial\beta}$$

$$C_{y\beta} = \frac{\partial C_y}{\partial \beta}$$

$$C_{l\delta} = \frac{\partial C_l}{\partial \delta}$$

$$C_{n\delta} = \frac{\partial C_n}{\partial \delta}$$

III. Equations of lateral-directional motion for airplanes

The axes

The equations of motion generally used in airplane dynamics are described on a system of axes fixed in the airplane in which the X axis is the intersection of the plane of the symmetry and a plane perpendicular to the symmetry that contains the relative wind vector.

But, it can be regarded as the X axis coincides with the reference axis of the airplane, when the angle of attack is not so large.

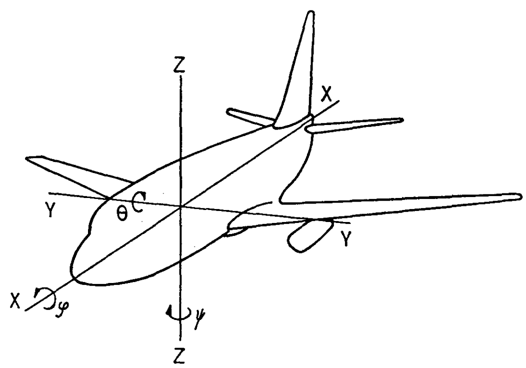

Fig. 1. Three axes fixed in the airplane and the direction of rotation.

In this study, the equations of motion are described with respect to the three axes that they are perpendicular each other to the reference axis, as to be shown in Fig. 1.

The rolling moment equation

Motions of the airplane which we are concerned to analyze are governed by the balance of three different kind of forces and moments, such as inertia forces or moments, gravity forces and aerodynamic forces or moments.

In this case, rolling moments due to inertia are represented by the following equation.

$$L_i = I_x \Delta\ddot{\varphi} + I_{xz}\Delta\ddot{\psi} \tag{3·1-1}$$

The gravity force produces no moment, because it acts through the origin of axes.

Aerodynamic rolling moments are represented by a Taylor series as

$$L_a \doteqdot \frac{\partial L}{\partial \beta}\Delta\beta + \frac{\partial L}{\partial r}\Delta r + \frac{\partial L}{\partial p}\Delta p + \frac{\partial L}{\partial \delta_a}\Delta\delta_a + \frac{\partial L}{\partial \delta_r}\Delta\delta_r \tag{3·1-2}$$

These different rolling moments are summed up and set equal to zero, then the resulting equation is reduced to the following form.

$$(D^2 - L_p D)\Delta\varphi + (-r_x D^2 - L_r D)\Delta\psi - L_\beta\Delta\beta = L_{\delta a}\Delta\delta_a + L_{\delta r}\Delta\delta_r \tag{3·1-3}$$

In the above equation, the coefficients are defined as

$$r_x=\frac{I_{xz}}{I_x} \qquad L_p=\frac{q_0Sb^2}{2VI_x}C_{lp} \qquad L_r=\frac{q_0Sb^2}{2VI_x}C_{lr}$$

$$L_\beta=\frac{q_0Sb}{I_x}C_{l\beta} \qquad L_{\delta a}=\frac{q_0Sb}{I_x}C_{l\delta a} \qquad L_{\delta r}=\frac{q_0Sb}{I_x}C_{l\delta r}$$

where C_{lp} is generally a negative number representing moments tending to resist rolling rotation and C_{lr} is a cross derivative that couples yawing and rolling motion that is sometimes (subsonic) positive and sometimes (supersonic) negative, further $C_{l\beta}$ may be either positive or negative.

The yawing moment equation

In this case, inertia moments are

$$N_i=-I_z\Delta\ddot{\psi}+I_{xz}\Delta\ddot{\varphi} \tag{3·2-1}$$

The moment produced by gravity is zero because the axis of moments is taken through the centre of gravity of the airplane.

Aerodynamic yawing moments are similarly represented by Taylor series expansion as previous manner.

$$N_a \doteqdot \frac{\partial N}{\partial\beta}\Delta\beta+\frac{\partial N}{\partial r}\Delta r+\frac{\partial N}{\partial p}\Delta p+\frac{\partial N}{\partial\delta_a}\partial\delta_a+\frac{\partial N}{\partial\delta_r}\Delta\delta_r \tag{3·2-2}$$

According to Newton's Laws, all kinds of yawing moments are added up and set equal to zero so that the resulting equation could be as follows.

$$(-r_zD^2-N_pD)\Delta\varphi+(D^2-N_rD)\Delta\psi-N_\beta\Delta\beta=N_{\delta a}\Delta\delta_a+N_{\delta r}\Delta\delta_r \tag{3·2-3}$$

In the above equation, the coefficients are defined as

$$r_z=\frac{I_{xz}}{I_z} \qquad N_p=\frac{q_0Sb^2}{2VI_z}C_{np} \qquad N_r=\frac{q_0Sb^2}{2VI_z}C_{nr}$$

$$N_\beta=\frac{q_0Sb}{I_z}C_{n\beta} \qquad N_{\delta a}=\frac{q_0Sb}{I_z}C_{n\delta a} \qquad N_{\delta r}=\frac{q_0Sb}{I_z}C_{n\delta r}$$

where C_{np} is a cross derivative tending to couple rolling and yawing motions that is a negative number for subsonic configurations and C_{nr} is a negative number corresponding to yawing moments against yaw rate, further $C_{n\beta}$ is a measure of the tendency of the airplane to aline itself in side slip with the relative wind.

The side force equation

Side slip motion is fully governed by the three kind of forces, that is, inertia force, gravity force and aerodynamic force.

Inertia force is given by

$$Y_i=-m(\Delta\dot{v}+V\Delta\dot{\psi}-w_0\Delta\dot{\varphi}) \tag{3·3-1}$$

Gravity force with respect to stability axis is

$$Y_g=(W\cos\gamma)\Delta\varphi+(W\sin\gamma)\Delta\psi \tag{3·3–2}$$

Aerodynamic side force is

$$Y_a\doteqdot\frac{\partial Y}{\partial\beta}\Delta\beta+\frac{\partial Y}{\partial\delta_a}\Delta\delta_a+\frac{\partial Y}{\partial\delta_r}\Delta\delta_r \tag{3·3–3}$$

These different forces are summed and equated to zero in the side force equation, which may easily be put in the following form.

$$(-W\cos\gamma)\Delta\varphi+(mVD-W\sin\gamma)+(mVD-q_0SC_{y\beta})\Delta\beta$$
$$=q_0SC_{y\delta a}\Delta\delta_a+q_0SC_{y\delta r}\Delta\delta_r \tag{3·3–4}$$

Dividing the previous equation by mV, it is reduced as

$$-K_1\Delta\varphi+(D-K_2)\Delta\psi+(D-Y_\beta)\Delta\beta=Y_{\delta a}\Delta\delta_a+Y_{\delta r}\Delta\delta_r \tag{3·3–5}$$

where

$$K_1=\frac{W\cos\gamma}{mV}\qquad K_2=\frac{W\sin\gamma}{mV}\qquad Y_\beta=\frac{q_0S}{mV}C_{y\beta}$$

In the above coefficients, $C_{y\beta}$ is a negative numer representing cross-wind force.

IV. Derivation of optimal conditions

Consider that the functional

$$J=\int_{t_1}^{t_2}f(t,x_1,x_2,\dots\dots,x_n,\dot{x}_1,\dot{x}_2,\dots\dots,\dot{x}_n)\,dt \tag{4·1}$$

, in which the functions $x_1=\phi_1(t)$, $x_2=\phi_2(t)$,……, $x_n=\phi_n(t)$ and their first order derivatives are included, could be the extreme value under conditions such as

$$x_1=x_1^1,\ x_2=x_2^1,\dots\dots,\ x_n=x_n^1\qquad\text{at}\qquad t=t_1$$
$$x_1=x_1^2,\ x_2=x_2^2,\dots\dots,\ x_n=x_n^2\qquad\text{at}\qquad t=t_2$$

are satisfied and

$$H=\int_{t_1}^{t_2}g(t,x_1,x_2,\dots\dots,x_n,\dot{x}_1,\dot{x}_2,\dots\dots,\dot{x}_n)\,dt=\text{const.} \tag{4·2}$$

Now, we introduce any functions $\xi_1(t)$, $\xi_2(t)$,……, $\xi_n(t)$ which satisfy the following conditions.

$$\xi_1(t_1)=\xi_1(t_2)=0,\qquad \xi_2(t_1)=\xi_2(t_2)=0,\dots\dots,\ \xi_n(t_1)=\xi_n(t_2)=0$$

Taking any small positive numbers ε_1, ε_2,……, ε_n and defining the variations with respect to x_1, x_2,……, x_n as

$$\Delta x_1=\varepsilon_1\xi_1(t),\qquad \Delta x_2=\varepsilon_2\xi_2(t),\dots\dots,\ \Delta x_n=\varepsilon_n\xi_n(t)$$

the total variations for J and H could be represented as

$$\Delta J=\int_{t_1}^{t_2}\Big[f(t,x_1+\varepsilon_1\xi_1(t),\ldots\ldots,\ x_n+\varepsilon_n\xi_n(t),\ \dot{x}_1+\varepsilon_1\dot{\xi}_1(t),\ldots\ldots,\ \dot{x}_n+\varepsilon_n\dot{\xi}_n(t))$$

$$-f(t,\ x_1\ldots\ldots,\ x_n,\ \dot{x}_1,\ldots\ldots,\ \dot{x}_n)\Big]dt \tag{4·3}$$

$$\Delta H=\int_{t_1}^{t_2}\Big[g(t,\ x_1+\varepsilon_1\xi_1(t),\ldots\ldots,\ x_n+\varepsilon_n\xi_n(t),\ \dot{x}_1+\varepsilon_1\dot{\xi}_1(t),\ldots\ldots\ x_n+\varepsilon_n\dot{\xi}_n(t))$$

$$-g(t,x_1\ldots\ldots,\ x_n,\ \dot{x}_1,\ldots\ldots,\ \dot{x}_n)\Big]dt\equiv 0 \tag{4·4}$$

respectively.

In the equations (4·3) and (4·4), if all $x_1,\ x_2,\ldots\ldots,\ x_n,\ \xi_1(t),\ \xi_2(t),\ldots\ldots,\ \xi_n(t)$ could be determined, since they are the functions with respect to t, either ΔJ and ΔH would be the functions with respect to only $\varepsilon_1,\ \varepsilon_2,\ldots\ldots\ \varepsilon_n$, that is,

$$\Delta J=F(\varepsilon_1,\ \varepsilon_2,\ldots\ldots,\ \varepsilon_n),\qquad \Delta H=G(\varepsilon_1,\ \varepsilon_2,\ldots\ldots,\ \varepsilon_n)\equiv 0$$

Therefore, putting

$$\Phi(\varepsilon_1,\ \varepsilon_2,\ldots\ldots,\ \varepsilon_n,\ \lambda)=F(\varepsilon_1,\ \varepsilon_2\ldots\ldots.\ \varepsilon_n)+\lambda G(\varepsilon_1,\ \varepsilon_2\ldots\ldots,\ \varepsilon_n) \tag{4·5}$$

$$\frac{\partial\Phi}{\partial\varepsilon_1}=0,\qquad \frac{\partial\Phi}{\partial\varepsilon_2}=0,\ldots\ldots,\ \frac{\partial\Phi}{\partial\varepsilon_n}=0,\qquad \frac{\partial\Phi}{\partial\lambda}=0 \tag{4·6}$$

when $\varepsilon_1=0,\ \varepsilon_2=0,\ldots\ldots,\ \varepsilon_n=0,\ \lambda=\lambda_0$

On the other hand,deriving $\partial F/\partial\varepsilon_1,\ \partial F/\partial\varepsilon_2,\ldots\ldots:\ \partial F/\partial\varepsilon_n$ and $\partial G/\partial\varepsilon_1,\ \partial G/\partial\varepsilon_2,\ldots$ $\ldots,\ \partial G/\partial\varepsilon_n$ from the equations (4·3). (4·4) and substituting them into equation (4·6), we may obtain the set of equations

$$\int_{t_1}^{t_2}\Big[\xi_1(t)\frac{\partial f}{\partial x_1}+\dot{\xi}_1(t)\frac{\partial f}{\partial\dot{x}_1}+\lambda_0\Big\{\xi_1(t)\frac{\partial g}{\partial x_1}+\dot{\xi}_1(t)\frac{\partial g}{\partial\dot{x}_1}\Big\}\Big]dt=0 \tag{4·7}$$

$$\int_{t_1}^{t_2}\Big[\xi_n(t)\frac{\partial f}{\partial x_n}+\dot{\xi}_n(t)\frac{\partial f}{\partial\dot{x}_n}+\lambda_0\Big\{\xi_n(t)\frac{\partial g}{\partial x_n}+\dot{\xi}_n(t)\frac{\partial g}{\partial\dot{x}_n}\Big\}\Big]dt=0 \tag{4·8}$$

since all the equations of (4·6) except $\partial\Phi/\partial\lambda$ could be

$$\frac{\partial\Phi}{\partial\varepsilon_1}=\frac{\partial F}{\partial\varepsilon_1}+\lambda_0\frac{\partial G}{\partial\varepsilon_1}=0 \tag{4·9}$$

$$\vdots$$

$$\frac{\partial\Phi}{\partial\varepsilon_n}=\frac{\partial F}{\partial\varepsilon_n}+\lambda_0\frac{\partial G}{\partial\varepsilon_n}=0 \tag{4·10}$$

at $\varepsilon_1=0,\ \varepsilon_2=0,\ldots\ldots,\ \varepsilon_n=0$ respectively, while

$$\frac{\partial\Phi}{\partial\lambda}=G(\varepsilon_1,\ ,\varepsilon_2,\ldots\ldots,\ \varepsilon_n)=0$$

Since,

$$\int_{t_1}^{t_2} \xi_1(t) \frac{\partial(f+\lambda_0 g)}{\partial \dot{x}_1} dt = \xi_1(t) \frac{\partial(f+\lambda_0 g)}{\partial \dot{x}_1} \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \xi_1(t) \frac{d}{dt}\left\{\frac{\partial(f+\lambda_0 g)}{\partial \dot{x}_1}\right\} dt \tag{4·11}$$

from the equation (4·9)

$$\begin{aligned}
&\int_{t_1}^{t_2} \left\{\xi_1(t) \frac{\partial(f+\lambda_0 g)}{\partial x_1} + \dot{\xi}_1(t) \frac{\partial(f+\lambda_0 g)}{\partial \dot{x}_1}\right\} dt \\
&= \int_{t_1}^{t_2} \xi_1(t) \frac{\partial(f+\lambda_0 g)}{\partial x_1} dt + \int_{t_1}^{t_2} \dot{\xi}_1(t) \frac{\partial(f+\lambda_0 g)}{\partial \dot{x}_1} dt \\
&= \int_{t_1}^{t_2} \xi_1(t) \left\{\frac{\partial(f+\lambda_0 g)}{\partial x_1} - \frac{d}{dt}\left(\frac{\partial(f+\lambda_0 g)}{\partial \dot{x}_1}\right)\right\} dt
\end{aligned} \tag{4·12}$$

Consequently, we obtain the following equation as the Euler's Equation.

$$\frac{\partial(f+\lambda_0 g)}{\partial x_1} - \frac{d}{dt}\left[\frac{\partial(f+\lambda_0 g)}{\partial \dot{x}_1}\right] = 0 \tag{4·13}$$

Similarly, for $x_2 \ldots\ldots x_n$, we may represent them as follows.

$$\frac{\partial(f+\lambda_0 g)}{\partial x_n} - \frac{d}{dt}\left[\frac{\partial(f+\lambda_0 g)}{\partial \dot{x}_n}\right] = 0 \tag{4·14}$$

The optimal conditions for the original dynamic system would be satisfied by the solution of them.

V. Optimal conditions of the control system

For the sake of convenience, let's substitute φ, ψ and β for $\Delta\varphi$, $\Delta\psi$ and $\Delta\beta$.
Now, we define such error signals as

$$e_\varphi = \dot{\varphi}_m - \dot{\varphi} \tag{5·1}$$

$$e_\psi = \dot{\psi}_m - \dot{\psi} \tag{5·2}$$

They would establish the exciting functions for which a integral error would be minimized as $\delta_a = x_1$ and $\delta_r = x_2$. In the above equations, φ_m and ψ_m are the outputs of the desired models so that they are optimum responses for the system.

Suppose that f, the variational integrand in equation (4·1). is chosen to be some function of the error signals with respect to φ, ψ and their time derivatives. From equations (5·1) and (5·2), we may obtain the following relations as the effect of aileron and rudder deflections.

$$\frac{\partial e_\varphi}{\partial \delta_a} = -\frac{\partial \dot{\varphi}}{\partial \delta_a}, \qquad \frac{\partial e_\varphi}{\partial \delta_r} = -\frac{\partial \dot{\varphi}}{\partial \delta_r} \tag{5·3}$$

$$\frac{\partial e_\psi}{\partial \delta_a} = -\frac{\partial \dot{\psi}}{\partial \delta_a}, \qquad \frac{\partial e_\psi}{\partial \delta_r} = -\frac{\partial \dot{\psi}}{\partial \delta_r} \tag{5·4}$$

On the other hand, from the original equations of motion. the effect of φ, $\dot{\varphi}$ and $\ddot{\varphi}$ due to the aileron and the rudder deflections could be derived as follows.

$$\frac{\partial\ddot{\varphi}}{\partial\delta_a}=\frac{A_{\delta a}}{1-r_x r_z} \tag{5·5}$$

$$\frac{\partial\dot{\varphi}}{\partial\delta_a}=-\frac{A_{\delta a}}{L_p+r_x N_p} \tag{5·6}$$

$$\frac{\partial\varphi}{\partial\delta_a}=-\frac{A_{\delta a}}{K_1(L_r+r_x N_r)} \tag{5·7}$$

$$\frac{\partial\ddot{\varphi}}{\partial\delta_r}=\frac{A_{\delta r}}{1-r_x r_z} \tag{5·8}$$

$$\frac{\partial\dot{\varphi}}{\partial\delta_r}=-\frac{A_{\delta r}}{L_p+r_x N_p} \tag{5·9}$$

$$\frac{\partial\varphi}{\partial\delta_r}=-\frac{A_{\delta r}}{K_1(L_r+r_x N_r)} \tag{5·10}$$

In the equations from (5·5) to (5·10), $A_{\delta a}$ and $A_{\delta r}$ are

$$A_{\delta a}=L_{\delta a}+r_x N_{\delta a}+Y_{\delta a}(L_r+r_x N_r)$$
$$A_{\delta r}=L_{\delta r}+r_x N_{\delta r}+Y_{\delta r}(L_r+r_x N_r)$$

Similarly, the effect of ψ, $\dot{\psi}$ and $\ddot{\psi}$ due to the aileron and the rudder deflections are represented by the following equations, that is,

$$\frac{\partial\ddot{\psi}}{\partial\delta_a}=\frac{B_{\delta a}}{K_1(1-r_x r_z)-(N_p+r_z L_p)} \tag{5·11}$$

$$\frac{\partial\dot{\psi}}{\partial\delta_a}=-\frac{B_{\delta a}}{K_1(N_r+r_z L_r)-K_2(N_p+r_z L_p)} \tag{5·12}$$

$$\frac{\partial\psi}{\partial\delta_a}=-\frac{A_{\delta a}}{K_2(L_r+r_x N_r)} \tag{5·13}$$

$$\frac{\partial\ddot{\psi}}{\partial\delta_r}=\frac{B_{\delta r}}{K_1(1-r_x r_z)-(N_p+r_z L_p)} \tag{5·14}$$

$$\frac{\partial\dot{\psi}}{\partial\delta_r}=-\frac{B_{\delta r}}{K_1(N_r+r_z L_r)-K_2(N_p+r_z L_p)} \tag{5·15}$$

$$\frac{\partial\psi}{\partial\delta_r}=-\frac{A_{\delta r}}{K_2(L_r+r_x N_r)} \tag{5·16}$$

where,

$$B_{\delta a}=K_1(N_{\delta a}+r_z L_{\delta a})$$
$$B_{\delta r}=K_1(N_{\delta r}+r_z L_{\delta r})$$

Further, the effect of time derivatives of error signals due to aileron and rudder deflections would be calculated as follows.

$$\frac{\partial\dot{e}_\varphi}{\partial\delta_a}=-\frac{\partial\ddot{\varphi}}{\partial\delta_a}, \qquad \frac{\partial\dot{e}_\varphi}{\partial\delta_r}=-\frac{\partial\ddot{\varphi}}{\partial\delta_r} \tag{5·17}$$

$$\frac{\partial\dot{e}_\psi}{\partial\delta_a}=-\frac{\partial\ddot{\psi}}{\partial\delta_a}, \qquad \frac{\partial\dot{e}_\psi}{\partial\delta_r}=-\frac{\partial\ddot{\psi}}{\partial\delta_r} \tag{5·18}$$

Then, in order to synthesize the control system based on the error signals and their derivatives, let's choose a function described by the following form as a variational integrand.

$$f=(\alpha_1 \varDelta\delta_a+\alpha_2 \varDelta\delta_r)^2+(\beta_1 \varDelta\delta_a+\beta_2 \varDelta\delta_r)^2 \tag{5·19}$$

where,

$$\alpha_1=-\frac{A_{\delta a}}{1-r_x r_z}+\frac{A_{\delta a}}{L_p+r_x N_p}$$

$$\alpha_2=-\frac{A_{\delta r}}{1-r_x r_z}+\frac{A_{\delta r}}{L_p+r_x N_p}$$

$$\beta_1=-\frac{B_{\delta a}}{K_1(1-r_x r_z)-(N_p+r_z L_p)}+\frac{B_{\delta a}}{K_1(N_r+r_z L_r)-K_2(N_p+r_z K_p)}$$

$$\beta_2=-\frac{B_{\delta r}}{K_1(1-r_x r_z)-(N_p+r_z L_p)}+\frac{B_{\delta r}}{K_1(N_r+r_z L_r)-K_2(N_p+r_z L_p)}$$

and $\varDelta\delta_a$, $\varDelta\delta_r$ are the displacements of aileron and rudder which must be shifted at the time t.

Consequently, if the function g is specified, the Euler's equations for this kind of variational problem subject to the integral constraint would be represented by the following equations.

$$\frac{\partial(f+\lambda_0 g)}{\partial \varDelta\delta_a}-\frac{d}{dt}\frac{\partial(f+\lambda_0 g)}{\partial \varDelta\dot{\delta}_a}=0 \tag{5·20}$$

$$\frac{\partial(f+\lambda_0 g)}{\partial \varDelta\delta_r}-\frac{d}{dt}\frac{\partial(f+\lambda_0 g)}{\partial \varDelta\dot{\delta}_r}=0 \tag{5·21}$$

and

$$\int_0^t g(t, e_\varphi, e_\psi, \dot{e}_\varphi, \dot{e}_\psi)\, dt=\text{const.} \tag{5·22}$$

From the equation (5·19).

$$\frac{\partial f}{\partial \varDelta\dot{\delta}_a}=0 \quad \text{and} \quad \frac{\partial f}{\partial \varDelta\dot{\delta}_r}=0$$

Further, the partial derivative of g with respect to $\varDelta\dot{\delta}_a$ is represented as

$$\frac{\partial g}{\partial \varDelta\dot{\delta}_a}=\frac{\partial g}{\partial e_\varphi}\frac{\partial e_\varphi}{\partial \varDelta\dot{\delta}_a}+\frac{\partial g}{\partial \dot{e}_\varphi}\frac{\partial \dot{e}_\varphi}{\partial \varDelta\dot{\delta}_a}=-\frac{\partial \dot{\varphi}}{\partial \varDelta\dot{\delta}_a}\frac{\partial g}{\partial e_\varphi}-\frac{\partial \ddot{\varphi}}{\partial \varDelta\dot{\delta}_a}\frac{\partial g}{\partial \dot{e}_\varphi}$$

in which

$$\frac{\partial \dot{\varphi}}{\partial \varDelta\dot{\delta}_a}=0 \quad \text{and} \quad \frac{\partial \ddot{\varphi}}{\partial \varDelta\dot{\delta}_a}=0$$

therefore,

$$\frac{\partial g}{\partial \varDelta\dot{\delta}_a}=0 \tag{5·23}$$

But the other side, the partial derivative of g with respect to $\varDelta\dot{\delta}_r$ is to be

$$\frac{\partial g}{\partial \Delta \dot{\delta}_r}=\frac{\partial g}{\partial e_\psi}\frac{\partial e_\psi}{\partial \Delta \dot{\delta}_r}+\frac{\partial g}{\partial \dot{e}_\psi}\frac{\partial \dot{e}_\psi}{\partial \Delta \dot{\delta}_r}=-\frac{\partial \dot{\psi}}{\partial \Delta \dot{\delta}_r}\frac{\partial g}{\partial e_\psi}-\frac{\partial \ddot{\psi}}{\partial \Delta \dot{\delta}_r}\frac{\partial g}{\partial \dot{e}_\psi}$$

in which

$$\frac{\partial \dot{\psi}}{\partial \Delta \dot{\delta}_r}=\frac{(N_p+r_zL_p)Y_{\delta r}}{K_1(N_r+r_zL_r)-K_2(N_p+r_zL_p)}$$

$$\frac{\partial \ddot{\psi}}{\partial \Delta \dot{\delta}_r}=-\frac{(N_p+r_zL_p)Y_{\delta r}}{K_1(1-r_xr_z)-(N_p+r_zL_p)}$$

and if $\qquad Y_{\delta r}\doteqdot 0$

then,

$$\frac{\partial g}{\partial \Delta \dot{\delta}_r}\doteqdot 0 \tag{5·24}$$

Similarly, since tne partial derivatives for f and g with respect to $\Delta\delta_a$ and $\Delta\delta_r$ are to be

$$\frac{\partial f}{\partial \Delta\delta_a}=2\left[\alpha_1(\alpha_1\Delta\delta_a+\alpha_2\Delta\delta_r)+\beta_1(\beta_1\Delta\delta_a+\beta_2\Delta\delta_r)\right] \tag{5·25}$$

$$\frac{\partial f}{\partial \Delta\delta_r}=2\left[\alpha_2(\alpha_1\Delta\delta_a+\alpha_2\Delta\delta_r)+\beta_2(\beta_1\Delta\delta_a+\beta_2\Delta\delta_r)\right] \tag{5·26}$$

$$\begin{aligned}\frac{\partial g}{\partial \Delta\delta_a}&=\frac{\partial g}{\partial e_\varphi}\frac{\partial e_\varphi}{\partial \Delta\delta_a}+\frac{\partial g}{\partial \dot{e}_\varphi}\frac{\partial \dot{e}_\varphi}{\partial \Delta\delta_a}=-\frac{\partial \dot{\varphi}}{\partial \Delta\delta_a}\frac{\partial g}{\partial e_\varphi}-\frac{\partial \ddot{\varphi}}{\partial \Delta\delta_a}\frac{\partial g}{\partial \dot{e}_\varphi}\\&=A_{\delta a}\left[\frac{1}{(L_p+r_xN_p)}\frac{\partial g}{\partial e_\varphi}-\frac{1}{(1-r_xr_z)}\frac{\partial g}{\partial \dot{e}_\varphi}\right]\end{aligned} \tag{5·27}$$

$$\begin{aligned}\frac{\partial g}{\partial \Delta\delta_r}&=\frac{\partial g}{\partial e_\psi}\frac{\partial e_\psi}{\partial \Delta\delta_r}+\frac{\partial g}{\partial \dot{e}_\psi}\frac{\partial \dot{e}_\psi}{\partial \Delta\delta_r}=-\frac{\partial \dot{\psi}}{\partial \Delta\delta_r}\frac{\partial g}{\partial e_\psi}-\frac{\partial \ddot{\psi}}{\partial \Delta\delta_r}\frac{\partial g}{\partial \dot{e}_\psi}\\&=B_{\delta r}\left[\frac{1}{K_1(N_r+r_zL_r)-K_2(N_p+r_zL_p)}\frac{\partial g}{\partial e_\psi}\right.\\&\qquad\left.-\frac{1}{K_1(1-r_xr_z)-K_2(N_p+r_zL_p)}\frac{\partial g}{\partial \dot{e}_\psi}\right]\end{aligned} \tag{5·28}$$

substituting the equations from (5·25) to (5·28) into the equations (5·20) and (5·21), we may obtain the following relations.

$$\begin{aligned}&2\left[\alpha_1(\alpha_1\Delta\delta_a+\alpha_2\Delta\delta_r)+\beta_1(\beta_1\Delta\delta_a+\beta_2\Delta\delta_r)\right]\\&\quad+\lambda_0A_{\delta a}\left[\frac{1}{(L_p+r_xN_p)}\frac{\partial g}{\partial e_\varphi}-\frac{1}{(1-r_xr_z)}\frac{\partial g}{\partial \dot{e}_\varphi}\right]=0\end{aligned} \tag{5·29}$$

$$\begin{aligned}&2\left[\alpha_2(\alpha_1\Delta\delta_a+\alpha_2\Delta\delta_r)+\beta_2(\beta_1\Delta\delta_a+\beta_2\Delta\delta_r)\right]\\&\quad+\lambda_0B_{\delta a}\left[\frac{1}{K_1(N_r+r_zL_r)-K_2(N_p+r_zL_p)}\frac{\partial g}{\partial e_\psi}\right.\\&\quad\left.-\frac{1}{K_1(1-r_xr_z)-(N_p+r_zL_p)}\frac{\partial g}{\partial \dot{e}_\psi}\right]=0\end{aligned} \tag{5·30}$$

Therefore, $\Delta\delta_a$ and $\Delta\delta_r$ could be derived as

$$\Delta\delta_a = -\frac{1}{2}\frac{\lambda_0}{(\alpha_1\beta_2-\alpha_2\beta_1)^2}\Big[(\alpha_2^2+\beta_2^2)P-Q\Big] \tag{5·31}$$

$$\Delta\delta_r = \frac{1}{2}\frac{\lambda_0}{(\alpha_1\beta_2-\alpha_2\beta_1)^2}\Big[(\alpha_1\alpha_2+\beta_1\beta_2)P-(\alpha_1^2+\beta_1^2)Q\Big] \tag{5·32}$$

where P and Q are represented by equations (5·27) and (5·28) respectively.

If the function g is given to be

$$g=(\dot{e}_\varphi+e_\varphi)^2+(\dot{e}_\psi+e_\psi)^2 \tag{5·33}$$

the partial derivatives of g with respect to e_φ, $\dot{e}_\varphi$, e_ψ and $\dot{e}_\psi$ are represented as follows.

$$\frac{\partial g}{\partial e_\varphi}=2(\dot{e}_\varphi+e_\varphi) \quad , \quad \frac{\partial e}{\partial \dot{e}_\varphi}=2(\dot{e}_\varphi+e_\varphi)$$

$$\frac{\partial g}{\partial e_\psi}=2(\dot{e}_\psi+e_\psi) \quad , \quad \frac{\partial g}{\partial \dot{e}_\psi}=2(\dot{e}_\psi+e_\psi)$$

Substituting previous four relations into equations (5·31) and (5·32), then we may obtain the following two equations as $\Delta\delta_a$ and $\Delta\delta_r$.

$$\Delta\delta_a = -\frac{\lambda_0}{(\alpha_1\beta_2-\alpha_2\beta_1)}\,{}'A_{\delta a}(\alpha_2^2+\beta_2^2)(\dot{e}_\varphi+e_\varphi)\left[\frac{1}{(L_p+r_xN_p)}\right.$$
$$\left.-\frac{1}{(1-r_xr_z)}\right]-B_{\delta r}(\dot{e}_\psi+e_\psi)\left[\frac{1}{K_1(N_r+r_xL_r)-K_2(N_p+r_zL_p)}\right.$$
$$\left.\left.-\frac{1}{K_1(1-r_xr_z)-(N_p+r_xL_p)}\right]\right\} \tag{5·34}$$

$$\Delta\delta_r = \frac{\lambda_0}{(\alpha_1\beta_2-\alpha_2\beta_1)}\Big\{A_{\delta a}(\alpha_1\alpha_2+\beta_1\beta_2)(\dot{e}_\varphi+e_\varphi)\left[\frac{1}{(L_p+r_xN_p)}\right.$$
$$\left.-\frac{1}{(1-r_xr_z)}\right]-B_{\delta r}(\alpha_1^2+\beta_1^2)(\dot{e}_\psi+e_\psi)\left[\frac{1}{K_1(N_r+r_xL_r)-K_2(N_p+r_zL_p)}\right.$$
$$\left.\left.-\frac{1}{K_1(1-r_xr_z)-(N_p+r_zL_p)}\right]\right\} \tag{5·35}$$

VI. Transfer functions and matrix for lateral-directional motion

As to be derived in previous section, the fundamental equations of lateral-directional motion for an airplane may be represented by the equations (3·1–3), (3·2–3) and (3·3–5).

For convenience, let's describe the three equations of motion by the following form.

$$a_{11}\varphi(s)+a_{12}\psi(s)-a_{13}\beta(s)=d_{11}\delta_a(s)+d_{12}\delta_r(s) \tag{6·1}$$

$$a_{21}\varphi(s)+a_{22}\psi(s)-a_{23}\beta(s)=d_{21}\delta_a(s)+d_{22}\delta_r(s) \tag{6·2}$$

$$-a_{31}\varphi(s)+a_{32}\psi(s)+a_{33}\beta(s)=d_{31}\delta_a(s)+d_{32}\delta_r(s) \tag{6·3}$$

Applying the Cramer's rule for the above equations, the characteristic equation Δ may be formed by expanding the major determinant of them as

$$\Delta = a_{11}(a_{22}a_{33}+a_{23}a_{32}) - a_{21}(a_{12}a_{33}+a_{13}a_{32}) - a_{31}(a_{13}a_{22}-a_{12}a_{23})$$
$$= S(C_4S^4+C_3S^3+C_2S^2+C_1S+C_0) \qquad (6\cdot4)$$

where
$$C_4 = 1-r_xr_z$$
$$C_3 = L_p'-N_r'-Y_\beta(1-r_xr_z)$$
$$C_2 = (N_r'-L_p')Y_\beta+(L_pN_r-N_pL_r)+N_\beta'$$
$$C_1 = -(L_pN_r-N_pL_r)Y_\beta+(L_\beta N_p-N_\beta L_p)-K_1L_\beta'-K_2N_\beta'$$
$$C_0 = -K_1(L_rN_\beta-N_rL_\beta)-K_2(L_\beta N_p-N_\beta L_p)$$

In the above equations which represent the coefficients of the characteristic equation, primed notations are expressed as follows.

$$L_p' = L_p+r_xN_p \qquad N_r' = N_r+r_zL_r$$
$$L_\beta' = L_\beta+r_xN_\beta \qquad N_\beta' = N_\beta+r_zL_\beta$$

Further, the equations formed numerators of each transfer functions may be readily calculated as follows.

The Rolling Motion;

$$p_{\varphi\delta} = D_3S^3+D_2S^2+D_1S+D_0 \qquad (6\cdot5)$$

where
$$D_3 = L_\delta+r_xN_\delta$$
$$D_2 = -L_\delta(Y_\beta+N_r)+N_\delta(L_r-r_xY_\beta)+Y_\delta L_\beta'$$
$$D_1 = L_\delta(N_rY_\beta+N_\beta)-N_\delta(L_\beta+L_rY_\beta)+Y_\delta(L_rN_\beta-N_rL_\beta)$$
$$D_0 = K_2(N_\delta L_\beta-N_\beta L_\delta)$$

The Yawing Motion;

$$p_{\psi\delta} = E_3S^3+E_2S^2+E_1S+E_0 \qquad (6\cdot6)$$

where
$$E_3 = N_\delta+r_zL_\delta$$
$$E_2 = -N_\delta(L_p+Y_\beta)+L_\delta(N_p-r_zY_\beta)+Y_\delta N_\beta'$$
$$E_1 = N_\delta L_pY_\beta-L_\delta N_pY_\beta+Y_\delta(L_\beta N_p-N_\beta L_p)$$
$$E_0 = K_1(L_\delta N_\beta-N_\delta L_\beta)$$

The Side Slip;

$$p_{\beta\delta} = S(F_3S^3+F_2S^2+F_1S+F_0) \qquad (6\cdot7)$$

where
$$F_3 = Y_\delta(1-r_xr_z)$$
$$F_2 = -Y_\delta(N_r'+L_p')-L_\delta r_z-N_\delta$$
$$F_1 = Y_\delta(L_pN_r-N_pL_r)-L_\delta(N_p-r_zK_2-K_1)+N_\delta(r_xK_1+L_p+K_2)$$
$$F_0 = K_1(N_\delta L_r-L_\delta N_r)+K_2(L_\delta N_p-N_\delta L_p)$$

Utilizing the equations from (6·4) to (6·7), the transfer functions which express the lateral-directional motion of an airplane may be written as following form.

$$P_{\varphi\delta a}(S)=\frac{p_{\varphi\delta a}}{\Delta} \tag{6·8}$$

$$P_{\varphi\delta r}(S)=\frac{p_{\varphi\delta r}}{\Delta} \tag{6·9}$$

$$P_{\phi\delta a}(S)=\frac{p_{\phi\delta a}}{\Delta} \tag{6·10}$$

$$P_{\phi\delta r}(S)=\frac{p_{\phi\delta r}}{\Delta} \tag{6·11}$$

$$P_{\beta\delta a}(S)=\frac{p_{\beta\delta a}}{\Delta} \tag{6·12}$$

$$P_{\beta\delta r}(S)=\frac{p_{\beta\delta r}}{\Delta} \tag{6·13}$$

where P_{ij} generally expresses the transfer function for the i-th response due to the j-th excitation.

Therefore, the transfer matrix for this kind of system may be represented as follows.

$$\boldsymbol{P}=\begin{vmatrix} P_{\varphi\delta a} & P_{\varphi\delta r} \\ P_{\phi\delta a} & P_{\phi\delta r} \\ P_{\beta\delta a} & P_{\beta\delta r} \end{vmatrix} \tag{6·14}$$

As to be obvious from the above matrix (6·14), this system constructs a multivariable system which consists of two-inputs and three-outputs, and according to the Mesarović's theory,[3] it is the multivariable system which is the P-canonical structure.

VII. Construction of the optimal control system

Let's construct the optimal control system for lateral-directional motion of an airplane, as to be based on the manner discussed previously.

As to be shown in Fig. 2, the control system may be consisted of a reference model, an optimal controller, servo-actuators and some detectors.

The reference inputs are imposed on the model which yields the optimal responses subject to the specification determined in advance.

The angular velocities and accelerations for the rolling and yawing rotations are picked up as the optimum responses of the model. The model outputs are compared with the actual airplane outputs, and they are subsequently supplied into the controller in which the signals to execute the optimal control will be decided utilizing the relations of equations (5·34) and (5·35).

In this case, as to be shown in Fig. 3, the optimal controller consists of the combination of four proportional gain constants.

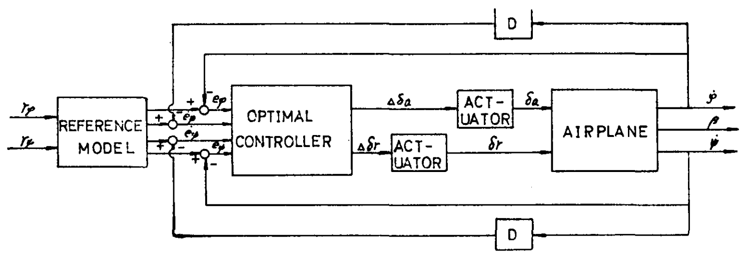

Fig. 2. Optimal control system.

The correction signals which the controller yields, are supplied to the servo-actuators, and further the outputs of those actuators operate the aileron and the rudder respectively.

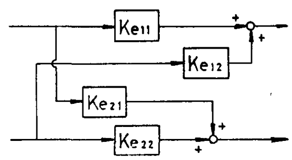

Fig. 3. Optimal controller.

VIII. Numerical example

In this section, we illustrate a numerical example based on the result of this study.

As to be apparent previously, the fundamental task to synthesize the control system is to be decided the four gain constants of the optimal controller.

The parameters used in this example are shown in Table 1 and they were originated from the literature[1] refered on the end of this paper.

Table 1. Parameters used in this example.

V	ft/sec	778	C_{np}	par rad	−0.0120
S	sq ft	287.9	C_{lr}	par rad	0.108
b	ft	37.1	C_{nr}	par rad	−0.197
q_0	lb/sq ft	222.5	$C_{l\delta r}$	par rad	0.0155
I_{xz}	slug-ft	−83	$C_{n\delta r}$	par rad	−0.0742
$C_{l\beta}$	per rad	−0.741	$C_{y\delta r}$	par rad	0.160
$C_{n\beta}$	per rad	0.1273	$C_{l\delta a}$	par rad	0.111
$C_{y\beta}$	per rad	−0.733	$C_{n\delta a}$	par rad	0.0081
C_{lp}	per rad	−0.385	$C_{y\delta a}$	par rad	0.004

Calculating the equations (5·34) and (5·35) with these parameters, $\Delta\delta_a$ and $\Delta\delta_r$ may be written as equations (8·1) and (8·2) respectively,

$$\Delta\delta_a=4.317(\dot{e}_\varphi+e_\varphi)-2.754(\dot{e}_\psi+e_\psi) \tag{8·1}$$

$$\Delta\delta_r=0.197(\dot{e}_\varphi+e_\varphi)+3.106(\dot{e}_\psi+e_\psi) \tag{8·2}$$

where $K_{e11}=4.317$. $K_{e12}=-2.754$, $K_{e21}=0.197$ and $K_{e22}=3.106$.

Consequently, the optimal control system may be performed through the estimation of such values as e_φ, $\dot{e}_\varphi$, e_ψ and $\dot{e}_\psi$.

Now, let's illustrate the result of simulation by the analogue computer. As the control inputs, triangular signals were used instead of pure impulse signals as to be shown in Figs. 4 and 5. It will be recognized that the responses of non-controlled system be rather stable as to be shown in Figs. 6 and 7.

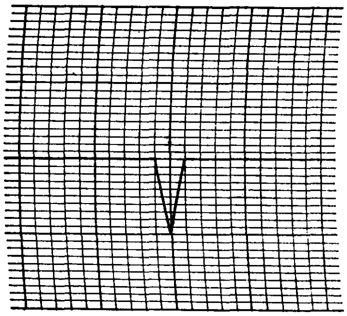

Fig. 4. Triangular input used for the aileron deflection.

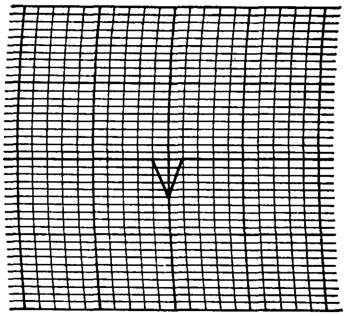

Fig. 5. Triangular input used for the rudder deflection.

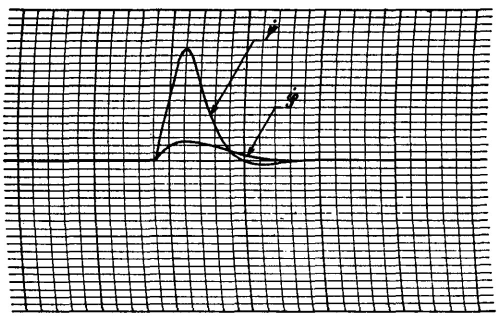

Fig. 6. The responses of non-controlled system, due to the aileron deflection.

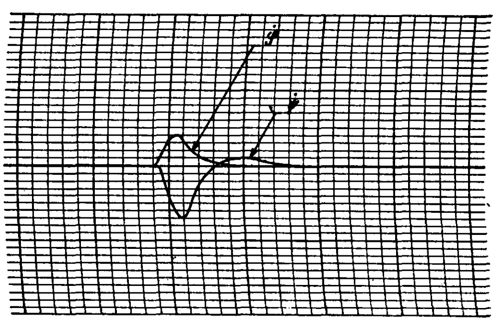

Fig. 7. The responses of non-controlled system, due to the rudder deflection.

Further. the responses in the case that the controlled object would be controlled through the optimal control system were compared with the responses in the case that the controlled object would be controlled through the simple feedback control system.

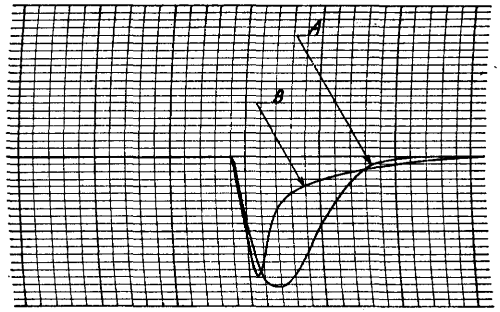

Fig. 8. The rolling rate.

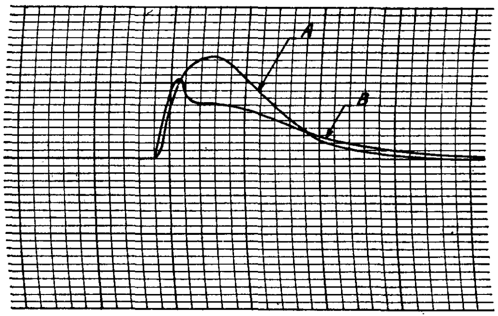

Fig. 9. The yawing rate.

In Figs. 8, 9 and 10, the behaviours of the output signals for such cases as to be stated above are illustrated.

In these figures, curves indicated by the symbol "A" are the responses in the case that the optimal control be applied, while curves indicated by the symbol "B" are the responses in the case that the ordinary simple feedback control be applied.

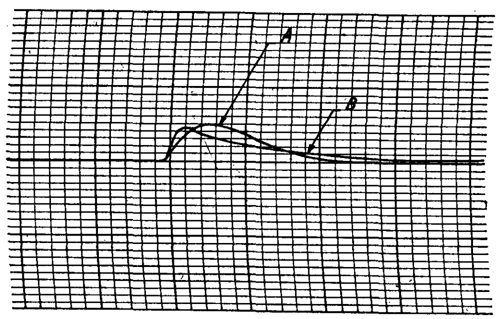

Fig. 10. The side-slip angle.

IX. Conclusion

In a case that the controlled object is the multivariable dynamic system as to be treated in this study, if it is tried to carry out the optimal control for the system, there is such a possibility as one of the loops included in the control system is optimized, the other loops escape from the optimal condition. For this reason, when the optimal control techniques are applied to the dynamic system of this kind, it is necessary either that the dynamic system is made a non-interacted system in advance and the optimal control techniques are applied to each of the loops, or that the control system is synthesized under consideration of the effect caused by the interaction between the signals produced in the dynamic system.

In this study, as to be stated at the begining of this paper, the latter method was preferred, and it was tried to synthesize the control system in order to perform the optimal control for lateral-directional motion of an airplane which is the dynamic system described by P-canonical structure.

In this method, it was shown that if the side-force coefficient is nearly constant to the rudder deflection, the optimal control may be performed through only the four proportional gain constants of the optimal controller, and the synthesis of the optimal control system will be comparatively easy.

From the restlt of simulation for this problem, since all the responses of the system with the optimal controller obviously approach to the optimum behaviours for the triangular inputs, the acceleration and the deceleration in the rolling and yawing rotations are performed more smoothly comared with the case of simple feedback control,[4] especially in the early period of the responses, the rapid ascent of the rolling and yawing rates is not observed.

Further, it is also recognized that the settling time of the responses is shortened compared with the case of simple feedback control, and the follow-up characteristics of the control system has been improved.

On the other hand, the maximum magnitudes of the rolling and yawing rates are rather large compared with the case of simple feedback control and also the side-slip angle become larger than the case of simple feedback control.

But, since the difference between the response of the optimal control system and the response of the ordinary simple feedback control system is not so remarkable, they seem to be more advatageous to lighten the rapid ascent of the rotational rates and to shorten the settling time of the responses.

Consequently, from the matters discussed in this study, it would be concluded that the optimal control system could be synthesized by the introduction of a reference model without making a non-interacted system.

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