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# Green Function Theory of Ferromagnetism

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**Abstract**

Theory of ferromagnetism in Heisenberg model is investigated by Green function method. In low temperature region, the Green function appropriate to Dyson Hamiltonian and its Hermitian adjoint one is worked out by means of series expansion for the mass operator of the Green function. The behaviour of a ferromagnet in wider temperature range has been worked out using the original Hamiltonian in spin operators by means of the various decoupling approximations. They are criticized and their degrees of approximation and the merits and demerits are considered mainly from the criterion of microscopic view-ponit and are expressed in a table.

## I. Introduction

Since Tyablikov published his paper[1], many people have studied about the Heisenberg ferromagnet using the method of two time Green function. For instance, using an intuitive decoupling method, Tahir-Kheli and Ter-Haar[2] developed the theory in the low temperature region, and succeeded in getting the result which agrees with Dyson's spin wave theory.[3] In the present work we develop the successive perturbation expansion for the two time Green function, and compare the results with those obtained by others and show that the excitation energies in Dyson's ideal Hamiltonian agree with those of his Hermitian adjoint one concretely.

Now, the method of two time Green function has various merits. As one of these merits, we cite an instance, we can obtain microscopic properties i.e., elementary excitation energies[5] directly from the knowledge of the Green function, and at the same time macroscopic properties (magnetization, specific heat, etc.) can be obtained using "spectrum theorem[1]". By the way, decoupling approximations have been made taking into account of macroscopic properties. Temperature expansion of magnetization contains many terms which are due to the wave number dependence of the spin wave energy and don't reflect directly the effect of the interaction of spin waves. Thus we may decide the degree of the approximation rather directly in the temperature expansion of spin wave excitation energy. In this paper, we

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don't propose a new decoupling approximation, but we reconsider about some decoupling approximations previously done, namely, Tyablikov decoupling, Hartree-Fock type decoupling, Callen decoupling[6] and Katsura-Horiguchi decoupling[7] in the above mentioned microscopic view-point. As a result, in low temperature region, we conclude that Tyablikov decoupling does not agree, but Hartree-Fock type decoupling and Callen decoupling agree with Dyson's spin wave theory completely for any spin magnitude $S$ in the elementary excitation energies.

Getting near the Curie temperature, elementary excitation energy goes to zero by Tyablikov decoupling and Callen decoupling, but elementary excitation energy has some value according to Hartree-Fock type decoupling and Katsura-Horiguchi decoupling. Thus the improvement achieved in Katsura-Horiguchi decoupling gives us some doubts in the above view-point.

## II. Two time Green function and perturbation theory

Let $A(t)$ and $B(t')$ be any time dependent operators in the Heisenberg representation. The advanced and retarded Green functions are defined as follows,

$$\begin{aligned} &G_r(t, t') = -i\theta(t-t')\langle[A(t), B(t')]\rangle \\ &G_a(t, t') = i\theta(t'-t)\langle[A(t), B(t')]\rangle \\ &\theta(t-t') = \begin{cases} 1 & t>t' \\ 0 & t<t' \end{cases} \end{aligned} \tag{2—1}$$

The square brakets, [ ] represent the commutator or anticommutator.

$$\begin{aligned} &[A, B] = AB - \eta BA \\ &\eta = \begin{cases} 1 & \text{for commutator} \\ -1 & \text{for anticommutator} \end{cases} \end{aligned} \tag{2—2}$$

In this paper, we take the following notations, $G_{r,a}(t, t')$ or $G^{(-)}_{r,a}(t, t')$ for commutator, $G^{(+)}_{r,a}(t, t')$ for anticommutator. The angular brackets, $\langle\ \rangle$ indicate an average over a grand canonical ensemble,

$$\begin{aligned} &\langle A(t)\rangle = Z^{-1}T_r(\exp(-\beta\mathscr{H})A(t)) \\ &Z = T_r(\exp(-\beta\mathscr{H})) \\ &\beta = 1/kT \end{aligned} \tag{2-3}$$

where $\mathscr{H}$ denotes the Hamiltonian of the system and $Z$ means the grand partition function.

Let $G^{(\pm)}_{r,a}(E)$ be the Fourier component of the Green function $G^{(\pm)}_{r,a}(t, t') = G^{(\pm)}_{r,a}(t-t')$.

$$\begin{aligned} &G^{(\pm)}_{r,a}(t-t') = \int_{-\infty}^{\infty} G^{(\pm)}_{r,a}(E) e^{-iE(t-t')} dE \\ &G^{(\pm)}_{r,a}(E) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G^{(\pm)}_{r,a}(t) e^{iEt} dt \end{aligned} \tag{2-4}$$

Now, $G_r^{(\pm)}(E)$ and $G_a^{(\pm)}(E)$ are two branches of one analytical function $G^{(\pm)}(E)$.

$$G^{(\pm)}(E)=\begin{array}{ll} G_r^{(\pm)}(E) & I_m\ E>0 \\ G_a^{(\pm)}(E) & I_m\ E<0 \end{array} \tag{2-5}$$

$G^{(\pm)}(E)$ is determined from the equation of motion. This equation is expressed in energy representation as follows,

$$E\ G^{(\pm)}(E)=\frac{1}{2\pi}\langle[A,\ B]\rangle+\langle\!\langle[A(t),\ \mathscr{H}(t)]_-;\ B(t')\rangle\!\rangle_E^{(\pm)} \tag{2-6}$$

where $\langle\!\langle[A(t),\mathscr{H}(t)]_-;B(t')\rangle\!\rangle_E^{(\pm)}$ is the Fourier component of $\langle\!\langle[A(t),\mathscr{H}(t)]_-;B(t')\rangle\!\rangle^{(\pm)}$.

If we can solve the equation (2-6) using any approximation and obtain $G(E)$, we can determine $\langle B(t')A(t)\rangle$ according to "spectrum theorem[1]",

$$\langle B(t')A(t)\rangle=i\int_{-\infty}^{\infty}\frac{G^{(\pm)}(\omega+i\varepsilon)-G^{(\pm)}(\omega-i\varepsilon)}{e^{\beta\omega}-\eta}\times e^{-i\omega(t-t')}\,d\omega \quad (\varepsilon\to+0) \tag{2-7}$$

Following Pike's theorem[5] we can determine the elementary excitation energy from the pole of $G(E)$.

If we consider about the $\langle\!\langle a(t)\,;c(t')\rangle\!\rangle$ type Green function, the pole of $G(E)$ gives the elementary excitation energy,where $a$ is an annihilation operator of a particle, and $c$ is any operator.

Now we indicate the perturbation theory[4] for two time Green function briefly. We consider about the $\langle\!\langle a(t)\,;c(t')\rangle\!\rangle$ type Green function,

$$\mathscr{H}=\mathscr{H}_0+\varepsilon\,\mathscr{H}_1 \tag{2-8}$$

$$\left.\begin{array}{l}[a,\ \mathscr{H}_0]=Ka \\ [a,\ \mathscr{H}_1]=R_1a'\end{array}\right\} \tag{2-9}$$

where $\mathscr{H}_0$ is unperturbed part and $\varepsilon\mathscr{H}_1$ is perturbed part of Hamiltonian $\mathscr{H}$. The quantity $\varepsilon$ is a parameter and represents the smallness of the magnitude of the perturbation. $\mathscr{H}_0$ and $\mathscr{H}_1$ are assumed to satisfy the equations (2-8) and (2-9) respectively. $K$ and $R_1$ are some linear operators, and $a'$ is some new operator appropriate to the problem. Using the equations (2-6), (2-8) and (2-9), we construct the equation of motion for $\langle\!\langle a(t)\,;c(t')\rangle\!\rangle_E$,

$$E\langle\!\langle a(t)\,;\,c(t')\rangle\!\rangle_E=\frac{1}{2\pi}\langle[a,\ c]\rangle+K\langle\!\langle a(t)\,;\,c(t')\rangle\!\rangle_E$$
$$+\varepsilon R_1\langle\!\langle a'(t)\,;\,c(t')\rangle\!\rangle_E \tag{2-10}$$

We rewrite this equation in the symbolical form,

$$L_1G_1=I_1+\varepsilon R_1G_2$$

Then, we construct the equation of motion for $G_2$, thus we obtain the equation,

$$L_2G_2=I_2+\varepsilon R_2G_3$$

In general,we obtain the infinite chain of simultaneous equations,

$$\left.\begin{array}{l} L_1G_1=I_1+\varepsilon R_1G_2 \\ L_2G_2=I_2+\varepsilon R_2G_3 \\ \cdots\cdots\cdots \end{array}\right\} \tag{2-11}$$

The quantities $L_n$ and $I_n$ are functions of $\varepsilon$,

$$L_n=L_n^{(0)}+\varepsilon L_n^{(1)}+\ldots\ldots$$

$$I_n=I_n^{(0)}+\varepsilon I_n^{(1)}+\ldots\ldots$$

Now we denote by $G_n^{(0)}$, $G_n^{(0,0)}$ the solution of the following equations respectively,

$$L_nG_n^{(0)}=I_n$$

$$L_n^{(0)}G_n^{(0,0)}=I_n^{(0)}$$

We define the mass operator $M_1$ by the equation,

$$(L_1-M_1)G_1=I_1 \tag{2-12}$$

Comparing (2–12) with the first of equations (2–11), we see that

$$M_1=\varepsilon R_1G_2(G_1)^{-1} \tag{2-13}$$

Introducing the notation,

$$X_1=R_1G_2(I_1)^{-1} \tag{2-14}$$

and substituting $G_1$ from (2–12) into (2–13), we obtain the following equation,

$$M_1=\varepsilon X_1(L_1-M_1)$$

This can be solved as follows,

$$M_1=(1+\varepsilon X_1)^{-1}\varepsilon X_1L_1 \tag{2-15}$$

or, expanding in a series of $\varepsilon X_1$,

$$M_1=\{\varepsilon X_1-(\varepsilon X_1)^2+(\varepsilon X_1)^3+\ldots\}L_1 \tag{2-16}$$

But we must not forget that $X_1$ is a function of $\varepsilon$. From the system of equations (2–11) we obtain

$$\begin{aligned} G_2&=G_2^{(0)}+\varepsilon\frac{1}{L_2}R_2G_3 \\ &=G_2^{(0)}+\varepsilon\frac{1}{L_2}R_2G_3^{(0)}+\varepsilon^2\frac{1}{L_2}R_2\frac{1}{L_3}R_3G_4^{(0)}+\ldots \end{aligned} \tag{2-17}$$

Using (2–14), (2–16) and (2–17), we can obtain the mass operator $M_1$ in a series of the parameter $\varepsilon$ formally.

$$\left.\begin{array}{l} M_1=\varepsilon M_1'+\varepsilon^2M_1''+\ldots \\ M_1'=R_1G_2^{(0)}\{G_1^{(0)}\}^{-1} \\ M_1''=R_1\frac{1}{L_2}R_2G_3^{(0)}\{G_1^{(0)}\}^{-1}-R_1G_2^{(0)}\{G_1^{(0)}\}^{-1}\frac{1}{L_1}R_1G_2^{(0)}\{G_1^{(0)}\}^{-1} \end{array}\right\} \tag{2-18}$$

Then we can obtain $G_1$ approximately in a series expansion of the parameter $\varepsilon$. If $\varepsilon\mathscr{H}_1$ is really smaller than $\mathscr{H}_0$, the representation of the equation (2–18) is valid.

## III. Dyson's spin wave; application of the previous perturbation expansion

The Heisenberg Hamiltonian is given by

$$\mathscr{H}=-g\mu_B H\sum_l S_l^z-\sum_{l,m}J(l-m)S_l\cdot S_m \tag{3–1}$$

where $\mu_B$ is the Bohr magneton, $g$ the Landé $g$-factor, and $H$ is the applied magnetic field which is assumed to be along the $+z$-direction. $S_l$ is the spin operator (in units of $h/2\pi$) at site $l$, and $J(l-m)$ is the exchange integral between ions at site $l$ and $m$. Now, we introduce the following transformation (Maleev transformation[8]),

$$\begin{aligned}S_l^+&=(2S)^{1/2}(a_l-a_l\dagger a_l a_l/2S)\\S_l^-&=(2S)^{1/2}a_l\dagger\\S_l^z&=S-a_l\dagger a_l\end{aligned} \tag{3–2}$$

where $a_l\dagger$ and $a_l$ are the boson creation and annihilation operators referring to the spin deviation at the lattice site $l$. These operators satisfy the following commutation relations,

$$[a_l,\ a_m\dagger]_-=\delta_{l,m}\quad,\quad[a_l\dagger,\ a_m\dagger]_-=[a_l,\ a_m]_-=0 \tag{3–3}$$

The original Heisenberg Hamiltonian is expressed after the transformation (3–2) into the form of Dyson's ideal Hamiltonian $\mathscr{H}_D$,

$$\begin{aligned}\mathscr{H}_D=&-g\mu_B HSN-N\mathscr{J}(0)S^2\\&+(g\mu_B H+2S\mathscr{J}(0))\sum_l a_l\dagger a_l-2S\sum_{l,m}J(l-m)a_l\dagger a_m\\&+\sum_{l,m}J(l-m)(a_l\dagger a_m\dagger a_m a_m-a_l\dagger a_l a_m\dagger a_m)\end{aligned} \tag{3–4}$$

where $N$ is the total number of spins in the lattice, and $\mathscr{J}(\nu)$ is defined by

$$\mathscr{J}(\nu)=\sum_f J(f)e^{if\cdot\nu} \tag{3–5}$$

The last term in $\mathscr{H}_D$ is not Hermitian. But we shall show later that this difficulty is out of the question.

Now, we introduce the spin wave creation and annihilation operators in wave number representation,

$$\begin{aligned}a_l\dagger&=\frac{1}{\sqrt{N}}\sum_\nu e^{-i(l\cdot\nu)}a_\nu\dagger\quad,\quad a=\frac{1}{\sqrt{N}}\sum_\nu e^{i(l\cdot\nu)}a_\nu\\a_\nu\dagger&=\frac{1}{\sqrt{N}}\sum_l e^{i(l\cdot\nu)}a_l\dagger\quad,\quad a_\nu=\frac{1}{\sqrt{N}}\sum_l e^{-i(l\cdot\nu)}a_l\end{aligned} \tag{3–6}$$

where the sum over $\nu$ are carried out within the first Brillouin zone. We can easily show that operators $a_\nu\dagger$ and $a_\nu$ satisfy the boson type commutation relations.

$$[a_\lambda, a_\mu\dagger]_- = \delta_{\lambda,\mu}. \quad [a_\lambda\dagger, a_\mu\dagger]_- = [a_\lambda. a_\mu]_- = 0 \tag{3-7}$$

Now we get

$$\mathscr{H}_D = \mathscr{H}_0 + \mathscr{H}_1 \tag{3-8}$$

where

$$\mathscr{H}_0 = -g\mu_B HSN - N\mathscr{F}(0)S^2 + \sum_\nu [g\mu_B H + 2S(\mathscr{F}(0) - \mathscr{F}(\nu))]a_\nu\dagger a_\nu \tag{3-9}$$

$$\mathscr{H}_1 = \frac{1}{N}\sum_{\lambda,\mu,\nu}[\mathscr{F}(\lambda) - \mathscr{F}(\lambda-\nu)]\, a_\lambda\dagger a_\mu\dagger a_\nu a_{\lambda+\mu-\nu} \tag{3-10}$$

and further we remark,

$$\mathscr{H}_0\dagger = \mathscr{H}_0 \tag{3-9'}$$

$$\mathscr{H}_1\dagger = \frac{1}{N}\sum_{\lambda,\mu,\nu}[\mathscr{F}(\lambda+\mu-\nu) - \mathscr{F}(\lambda-\nu)]\, a_\lambda\dagger a_\mu\dagger a_\nu a_{\lambda+\mu-\nu} \tag{3-10'}$$

In the low temperature, we can assume that the term $\mathscr{H}_1$ is the small perturbation. So, we introduce the parameter $\varepsilon$ and rewrite the equation (3-8) as,

$$\mathscr{H}_D = \mathscr{H}_0 + \varepsilon\mathscr{H}_1 \tag{3-8'}$$

For our discussions we consider the following Green function in wave number space,

$$G_1(t, t') = \langle\langle a_\kappa(t)\,;\, a_\kappa\dagger(t')\rangle\rangle \tag{3-11}$$

In order to construct the equation of motion for this Green function, we take the commutator $a_\kappa$ with $\mathscr{H}_0$ and $\mathscr{H}_1$.

$$[a_\kappa, \mathscr{H}_0]_- = \{g\mu_B H + 2S[\mathscr{F}(0) - \mathscr{F}(\kappa)]\}a_\kappa \tag{3-12}$$

$$\varepsilon_\kappa^{(0)} = g\mu_B H + 2S[\mathscr{F}(0) - \mathscr{F}(\kappa)] \tag{3-13}$$

$$[a_\kappa, \mathscr{H}_1] = \frac{1}{N}\sum_{\lambda,\mu}[\mathscr{F}(\lambda) + \mathscr{F}(\kappa) - \mathscr{F}(\lambda-\mu) - \mathscr{F}(\kappa-\mu)]a_\lambda\dagger a_\mu a_{\kappa+\lambda-\mu} \tag{3-14}$$

$$D(\alpha, \beta, \gamma, \delta) = \mathscr{F}(\alpha) + \mathscr{F}(\beta) - \mathscr{F}(\gamma) - \mathscr{F}(\delta) \tag{3-15}$$

Comparing the equations (3-8'), (3-12) and (3-14) with (2-8) and (2-9) we see that we can use our perturbation theory previously described in low temperature region.

1) The zeroth order approximation

We neglect the pertubation term $\varepsilon\mathscr{H}_1$ in this approximation. We construct the equation of motion for $\langle\langle a_\kappa\,;\, a_\kappa\dagger\rangle\rangle_E$ from the equations (2-6) and (3-12). Then we obtain,

$$E\langle\langle a_\kappa\,;\, a_\kappa\dagger\rangle\rangle_E^{(0)} = \frac{1}{2\pi} + \varepsilon_\kappa^{(0)}\langle\langle a_\kappa\,;\, a_\kappa\dagger\rangle\rangle_E^{(0)}$$

which can be solved as,

$$\langle\langle a_\kappa\,;\, a_\kappa\dagger\rangle\rangle_E^{(0)} = \frac{1}{2\pi}\frac{1}{E - \varepsilon_\kappa^{(0)}} \tag{3-16}$$

From Pike's theorem, $\varepsilon_\kappa^{(0)}$ is the elementary excitation energy of a non-interacting spin wave with wave vector $\kappa$.

Now we use the relation (2-7), "spectrum theorem". In the following we confine to the nearest neighbor approximation, that is,

$$\begin{cases} J(l-m)=J & \text{if } l \text{ and } m \text{ are nearest neighbors} \\ J(l-m)=0 & \text{otherwise,} \end{cases}$$

and calculate only about the simple cubic lattice. Then we get,

$$\langle a_\kappa{}^\dagger a_\kappa\rangle = \frac{1}{e^{\beta\varepsilon\kappa^{(0)}}-1} \tag{3-17}$$

$$\begin{aligned}\langle n\rangle &= \frac{1}{N}\sum_{\kappa}\langle a_\kappa{}^\dagger a_\kappa\rangle \\ &= Z(3/2)\tau^{3/2}+\frac{3\pi}{4}Z(5/2)\tau^{5/2}+\frac{33}{32}\pi^2 Z(7/2)\tau^{7/2}+\ldots \end{aligned} \tag{3-18}$$

and $\quad \langle S^z\rangle = S-\langle n\rangle$

where $\tau = kT/8\pi JS$ and $Z(p)=\sum_{n=1}^{\infty}\frac{1}{n^p}e^{-ng\mu_B H\beta}$.

From the equation (3-18), we understand that $n$ represents the deviation of magnetization from the saturated value $S$. In low temperature region, the average of this deviation $\langle n\rangle$ is so small that it is adequate to neglect the fourth order term in the Hamiltonian $\mathscr{H}_D$. Equation (3-17) represents a boson-distribution for the spin wave occupation numbers.

2) The first order approximation

We construct the equation of motion for $\langle\langle a_\kappa\,;\,a_\kappa{}^\dagger\rangle\rangle_E$ , accurately from the equations (2-6), (3-12), (3-13), (3-14) and (3-15),

$$\begin{aligned} E\langle\langle a_\kappa\,;\,a_\kappa{}^\dagger\rangle\rangle_E &= \frac{1}{2\pi}+\varepsilon_\kappa{}^{(0)}\langle\langle a_\kappa\,;\,a_\kappa{}^\dagger\rangle\rangle_E \\ &+\frac{\varepsilon}{N}\sum_{\lambda,\mu}D(\lambda,\,\kappa,\,\lambda-\mu,\,\kappa-\mu)\langle\langle a_\lambda{}^\dagger a_\mu a_{\kappa+\lambda-\mu}\,;\,a_\kappa{}^\dagger\rangle\rangle_E \end{aligned} \tag{3-19}$$

As we want to apply the perturbation theory described previously to this problem, we proceed to construct the equation of motion for $\langle\langle a_\lambda{}^\dagger a_\mu a_{\kappa+\lambda-\mu}\,;\,a_\kappa{}^\dagger\rangle\rangle_E$.

Using the following relations,

$$\langle[a_\lambda{}^\dagger a_\mu a_{\kappa+\lambda-\mu},\,a_\kappa{}^\dagger]\rangle = \langle a_\lambda{}^\dagger a_\lambda\rangle(\delta_{\kappa,\,\mu}+\delta_{\lambda,\,\mu}) \tag{3-20}$$

and

$$[a_\lambda{}^\dagger a_\mu a_{\kappa+\lambda-\mu},\,\mathscr{H}_0] = (-\varepsilon_\lambda{}^{(0)}+\varepsilon_\mu{}^{(0)}+\varepsilon_{\kappa+\lambda-\mu}^{(0)})a_\lambda{}^\dagger a_\mu a_{\kappa+\lambda-\mu} \tag{3-21}$$

we obtain the equation required within the first order approximation,

$$\begin{aligned} E\langle\langle a_\lambda{}^\dagger a_\mu a_{\kappa+\lambda-\mu}\,;\,a_\kappa{}^\dagger\rangle\rangle_E{}^{(0,0)} &= \frac{1}{2\pi}\langle n_\lambda\rangle(\delta_{\kappa,\mu}+\delta_{\lambda,\mu}) \\ &+(-\varepsilon_\lambda{}^{(0)}+\varepsilon_\mu{}^{(0)}+\varepsilon_{\kappa+\lambda-u}^{(0)})\langle\langle a_\lambda{}^\dagger a_\mu a_{\kappa+\lambda-\mu}\,;\,a_\kappa{}^\dagger\rangle\rangle_E{}^{(0,0)} \end{aligned} \tag{3-22}$$

where $n_\lambda = a_\lambda{}^\dagger a_\lambda$. Then we solve $\langle\langle a_\lambda{}^\dagger a_\mu a_{\kappa+\lambda-\mu}\,;\,a_\kappa{}^\dagger\rangle\rangle_E{}^{(0,0)}$ to get,

$$\langle\langle a_\lambda{}^\dagger a_\mu a_{\kappa+\lambda-\mu}\,;\,a_\kappa{}^\dagger\rangle\rangle_E{}^{(0,0)} = \frac{\langle n_\lambda\rangle}{2\pi}\frac{1}{E-\varepsilon_\kappa{}^{(0)}}(\delta_{\kappa,\mu}+\delta_{\lambda,\mu}) \tag{3-23}$$

We combine this with the equation (3-16).

$$\langle\langle a_\lambda{}^\dagger a_\mu a_{\kappa+\lambda-\mu}\,;\,a_\kappa{}^\dagger\rangle\rangle_E{}^{(0,0)} = \langle n_\lambda\rangle(\delta_{\kappa,\mu}+\delta_{\lambda,\mu})\langle\langle a_\kappa\,;\,a_\kappa{}^\dagger\rangle\rangle_E{}^{(0)} \tag{3-24}$$

Equation (3–24) corresponds exactly to the Hartree-Fock approximation and agrees with the Tahir-Kheli and Ter-Haar decoupling.

We rewrite the equations (3–19) and (3–22) using the notations defined previously,

$$L_1\langle\langle a_\kappa\,;\,a_\kappa{}^\dagger\rangle\rangle_E=I_1+\varepsilon R_1\langle\langle a_\lambda{}^\dagger a_\mu a_{\kappa+\lambda-\mu}\,;\,a_\kappa{}^\dagger\rangle\rangle_E$$

$$L_2{}^{(0)}\langle\langle a_\lambda{}^\dagger a_\mu a_{\kappa+\lambda-\mu}\,;\,a_\kappa{}^\dagger\rangle\rangle_E{}^{(0,0)}=I_2$$

$$L_1=E-\varepsilon_\kappa{}^{(0)},\qquad L_2{}^{(0)}=E+\varepsilon_\lambda{}^{(0)}-\varepsilon_\mu{}^{(0)}-\varepsilon_{\kappa+\lambda-\mu}^{(0)}$$

$$I_1=\frac{1}{2\pi},\qquad I_2=\frac{1}{2\pi}\langle n_\lambda\rangle\,(\delta_{\kappa,\mu}+\delta_{\lambda,\mu})$$

and
$$R_1=\frac{1}{N}\sum_{\lambda,\mu}D(\lambda,\,\kappa,\,\lambda-\mu,\,\kappa-\mu)$$

Using the equation (2–18), we obtain the mass operator in this approximation.

$$\begin{aligned}M_1{}'=\Delta\varepsilon_\kappa&=R_1G_2{}^{(0,0)}\{G_1{}^{(0)}\}^{-1}\\&=\frac{2}{N}\sum_\lambda D(\lambda,\,\kappa,\,\kappa-\lambda,\,0)\,\langle n_\lambda\rangle\end{aligned}\tag{3–25}$$

$$\langle\langle a_\kappa\,;\,a_\kappa{}^\dagger\rangle\rangle_E{}^{(1)}=\frac{1}{2\pi}\frac{1}{E-\varepsilon_\kappa{}^{(0)}-\Delta\varepsilon_\kappa}\tag{3–26}$$

We calculate the temperature dependence of $\Delta\varepsilon_\kappa$ in low temperature region.

$$\Delta\varepsilon_\kappa=-[\mathscr{J}(0)-\mathscr{J}(\kappa)]\,\frac{2}{N}\sum_\lambda\left(1-\frac{\mathscr{J}(\lambda)}{\mathscr{J}(0)}\right)\langle n_\lambda\rangle\tag{3–27}$$

$$\frac{1}{N}\sum_\lambda\left(1-\frac{\mathscr{J}(\lambda)}{\mathscr{J}(0)}\right)\langle n_\lambda\rangle=\pi Z(5/2)\tau^{5/2}+\frac{5}{4}\pi^2Z(7/2)\tau^{7/2}+\dots\tag{3–28}$$

Then, the elementary excitation energy in the first order approximation is

$$\begin{aligned}\varepsilon_\kappa{}^{(1)}&=\varepsilon_\kappa{}^{(0)}+\Delta\varepsilon_\kappa\\&=g\mu_BH+2S\,[\mathscr{J}(0)-\mathscr{J}(\kappa)]\left(1-\frac{\pi}{S}Z(5/2)\tau^{5/2}+\dots\right)\end{aligned}\tag{3–29}$$

The elementary excitation energy in the zero applied field depends on the temperature with the leading term proportional to $T^{5/2}$ through the interaction between the spin waves. This temperature dependence agrees with the result of Keffer and London theory.

Next, we calculate the magnetization $\langle S^z\rangle$.

$$\langle S^z\rangle=S-\langle n\rangle\tag{3–18}$$

$$\begin{aligned}\langle n\rangle=&Z(3/2)\tau^{3/2}+\frac{3}{4}\pi Z(5/2)\tau^{5/2}+\frac{33}{32}\pi^2Z(7/2)\tau^{7/2}\\&+\frac{3}{2S}\pi Z(3/2)Z(5/2)\tau^4+\dots\end{aligned}\tag{3–18′}$$

This agrees with Dyson's spin wave theory without imperceptible differences.

3) The second order approximation

We shall improve the approximation further. So, we need the equation of motion for $\langle\langle a_\lambda{}^\dagger a_\mu a_{\kappa+\lambda-\mu}\,;\,a_\kappa{}^\dagger\rangle\rangle_E$ completely. Thus the following commutator is required.

$$[a_{\lambda 1}{}^\dagger a_{\mu 1} a_{\kappa+\lambda 1-\mu 1},\ \mathscr{H}_1]$$
$$=\frac{1}{N}\sum_{\lambda,\nu}\{-D(\lambda,\lambda,\lambda-\lambda_1,\lambda-\lambda_1)\, a_\lambda{}^\dagger a_\nu{}^\dagger a_{\lambda+\nu-\lambda 1} a_{\mu 1} a_{\kappa+\lambda 1-\mu 1}$$
$$+D(\lambda,\mu_1,\lambda-\nu,\mu_1-\nu) a_{\lambda 1}{}^\dagger a_\lambda{}^\dagger a_\nu a_{\lambda+\mu 1-\nu} a_{\lambda+\lambda 1-\mu 1}$$
$$+D(\lambda,\kappa+\lambda_1-\mu_1,\lambda-\nu,\kappa+\lambda_1-\mu_1-\nu) a_{\lambda 1}{}^\dagger a_{\mu 1} a_\lambda{}^\dagger a_\nu a_{\kappa+\lambda+\lambda 1-\mu 1-\nu}\} \tag{3-30}$$

We adjust the last term in the above equation into the form $a^\dagger a^\dagger aaa$. Then the last term becomes,

$$\frac{1}{N}\sum_{\nu} D(\mu_1,\kappa+\lambda_1-\mu_1,\mu_1-\nu,\kappa+\lambda_1-\mu_1-\nu) a_{\lambda 1}{}^\dagger a_\nu a_{\kappa+\lambda 1-\nu}$$
$$+\frac{1}{N}\sum_{\lambda,\nu} D(\lambda,\kappa+\lambda_1-\mu_1,\lambda-\nu,\kappa+\lambda_1-\mu_1-\nu) a_{\lambda 1}{}^\dagger a_\lambda{}^\dagger a_{\mu 1} a_\nu a_{\kappa+\lambda 1+\lambda-\mu 1-\nu} \tag{3-31}$$

From the equations (3–20), (3–21), (3–30) and (3–31), we obtain the equation of motion for $\langle\langle a_{\lambda 1}{}^\dagger a_{\mu 1} a_{\kappa+\lambda 1-\mu 1}\,;\,a_\kappa{}^\dagger\rangle\rangle_E$.

$$[E+\varepsilon_{\lambda 1}{}^{(0)}-\varepsilon_{\mu 1}{}^{(0)}-\varepsilon_{\kappa+\lambda 1-\mu 1}{}^{(0)}]\,\langle\langle a_{\lambda 1}{}^\dagger a_{\mu 1} a_{\kappa+\lambda 1-\mu 1}\,;\,a_\kappa{}^\dagger\rangle\rangle_E$$
$$-\frac{\varepsilon}{N}\sum_{\nu} D(\mu_1,\kappa+\lambda_1-\mu_1,\mu_1-\nu,\kappa+\lambda_1-\mu_1-\nu)\,\langle\langle a_{\lambda 1}{}^\dagger a_\nu a_{\kappa+\lambda 1-\nu}\,;\,a_\kappa{}^\dagger\rangle\rangle_E$$
$$=\frac{1}{2\pi}\langle n_{\lambda 1}\rangle\,(\delta_{\kappa,\mu 1}+\delta_{\lambda 1,\mu 1})$$
$$+\frac{\varepsilon}{N}\sum_{\lambda,\nu} D(\lambda,\kappa+\lambda_1-\mu_1,\lambda-\nu,\kappa+\lambda_1-\mu_1-\nu)\langle\langle a_{\lambda 1}{}^\dagger a_\lambda{}^\dagger a_{\mu 1} a_\nu a_{\kappa+\lambda 1+\lambda-\mu 1-\nu}; a_\kappa{}^\dagger\rangle\rangle_E$$
$$+\frac{\varepsilon}{N}\sum_{\lambda,\nu} D(\lambda,\mu_1,\lambda-\nu,\mu_1-\nu)\,\langle\langle a_{\lambda 1}{}^\dagger a_\lambda{}^\dagger a_\nu a_{\lambda+\mu 1-\nu} a_{\kappa+\lambda 1-\mu 1}\,;\,a_\kappa{}^\dagger\rangle\rangle_E$$
$$-\frac{\varepsilon}{N}\sum_{\lambda,\nu} D(\lambda,\lambda,\lambda-\lambda_1,\lambda-\lambda_1)\,\langle\langle a_\lambda{}^\dagger a_\nu{}^\dagger a_{\lambda+\nu-\lambda 1} a_{\mu 1} a_{\kappa+\lambda 1-\mu 1}\,;\,a_\kappa{}^\dagger\rangle\rangle_E \tag{3-32}$$

We must obtain the equation of motion for $\langle\langle a_{\kappa 1}{}^\dagger a_{\kappa 2}{}^\dagger a_{\kappa 3} a_{\kappa 4} a_{\kappa 5}\,;\,a_\kappa{}^\dagger\rangle\rangle_E$ required within the second order approximation. Using the equations,

$$[a_{\kappa 1}{}^\dagger a_{\kappa 2}{}^\dagger a_{\kappa 3} a_{\kappa 4} a_{\kappa 5}\,;\,\mathscr{H}_0]_-=(-\varepsilon_{\kappa 1}{}^{(0)}-\varepsilon_{\kappa 2}{}^{(0)}+\varepsilon_{\kappa 3}{}^{(0}+\varepsilon_{\kappa 4}{}^{(0)}+\varepsilon_{\kappa 5}{}^{(0})a_{\kappa 1}{}^\dagger a_{\kappa 2}{}^\dagger a_{\kappa 3} a_{\kappa 4} a_{\kappa 5} \tag{3-33}$$

and

$$\mathcal{I}=\langle[a_{\kappa 1}{}^\dagger a_{\kappa 2}{}^\dagger a_{\kappa 3} a_{\kappa 4} a_{\kappa 5},\ a_\kappa{}^\dagger]_-\rangle$$
$$=(\delta_{\kappa 1,\kappa 5}\delta_{\kappa 2,\kappa 3}\delta_{\kappa 4,\kappa}+\delta_{\kappa 1,\kappa 3}\delta_{\kappa 2,\kappa 5}\delta_{\kappa 4.\kappa}+\delta_{\kappa 1,\kappa 4}\delta_{\kappa 2,\kappa 3}\delta_{\kappa 5,\kappa}$$
$$+\delta_{\kappa 1,\kappa 3}\delta_{\kappa 2,\kappa 4}\delta_{\kappa 5,\kappa}+\delta_{\kappa 1,\kappa 4}\delta_{\kappa 2,\kappa 5}\delta_{\kappa 3,\kappa}$$
$$+\delta_{\kappa 1,\kappa 5}\delta_{\kappa 2,\kappa 4}\delta_{\kappa 3,\kappa})\,\langle n_{\kappa 1}\rangle\,\langle n_{\kappa 2}\rangle \tag{3-34}$$

we obtain a series of equations from the equations (3–19), (3–32), (3–33) and (3–34).

$$L_1\langle\langle a_\kappa\,;\,a_\kappa{}^\dagger\rangle\rangle_E=I_1+\varepsilon R_1\langle\langle a_{\lambda 1}{}^\dagger a_{\mu 1} a_{\kappa+\lambda 1-\mu}\,;\,a_\kappa{}^\dagger\rangle\rangle_E$$
$$L_2\langle\langle a_{\lambda 1}{}^\dagger a_{\mu 1} a_{\kappa+\lambda 1-\mu 1}\,;\,a_\lambda{}^\dagger\rangle\rangle_E=I_2+\varepsilon R_2\langle\langle a_{\kappa 1}{}^\dagger a_{\kappa 2}{}^\dagger a_{\kappa 3} a_{\kappa 4} a_{\kappa 5}\,;\,a_\kappa{}^\dagger\rangle\rangle_E$$
$$L_3{}^{(0)}\langle\langle a_{\kappa 1}{}^\dagger a_{\kappa 2}{}^\dagger a_{\kappa 3} a_{\kappa 4} a_{\kappa 5}\,;\,a_\kappa{}^\dagger\rangle\rangle_E{}^{(0,0)}=I_3$$
$$L_2=E+\varepsilon_{\lambda 1}{}^{(0)}-\varepsilon_{\kappa+\lambda 1-\mu 1}{}^{(0)}-\varepsilon_{\mu 1}{}^{(0)}+\varepsilon L_2{}^{(1)}$$

$$L_3^{(0)}=E+\varepsilon_{\kappa1}{}^{(0)}+\varepsilon_{\kappa2}{}^{(0)}-\varepsilon_{\kappa3}{}^{(0)}-\varepsilon_{\kappa4}{}^{(0)}-\varepsilon_{\kappa5}{}^{(0)}$$

$$R_2=\frac{1}{N}\sum_{\kappa1}{}_{\kappa5}\{D(\kappa_2,\kappa+\lambda_1-\mu_1,\kappa_2-\kappa_4,\kappa+\lambda_1-\mu_1-\kappa_4)\delta_{\kappa1,\lambda1}\delta_{\kappa3,\mu1}\delta_{\kappa5,\kappa+\lambda1+\kappa2-\mu1-\kappa4}$$
$$+D(\kappa_2,\mu_1,\kappa_2-\kappa_3,\mu_1-\kappa_3)\delta_{\kappa1,\lambda1}\delta_{\kappa4,\kappa+\lambda1-\mu1}\delta_{\kappa5,\kappa2+\mu1-\kappa3}$$
$$-D(\kappa_1,\kappa_1,\kappa_1-\lambda_1,\kappa_1-\lambda_1)\delta_{\kappa4,\mu1}\delta_{\kappa3,\kappa1+\kappa2-\mu1}\delta_{\kappa5,\kappa+\lambda1-\mu1}\}$$

$$I_3=\mathcal{J}/2\pi$$

where $L_2^{(1)}$ represents the second term in the equation (3–32) (l.h.s.). At first, we obtain the expression of $\langle\!\langle a_{\lambda1}{}^\dagger a_{\mu1}a_{\kappa+\lambda1-\mu1}\,;\,a_\kappa{}^\dagger\rangle\!\rangle_E{}^{(0)}$ within the second order approximation. From the equation (3–32), we obtain the equation for $G_2$ correct up to the first order in $\varepsilon$.

$$[E+\varepsilon_{\lambda1}{}^{(0)}-\varepsilon_{\mu1}{}^{(0)}-\varepsilon_{\kappa+\lambda1-\mu1}{}^{(0)}]\langle\!\langle a_{\lambda1}{}^\dagger a_{\mu1}a_{\kappa+\lambda1-\mu1}\,;\,a_\kappa{}^\dagger\rangle\!\rangle_E{}^{(0)}$$
$$=\frac{1}{2\pi}\langle n_{\lambda1}\rangle(\delta_{\kappa,\mu1}+\delta_{\lambda1,\mu1}) \tag{3-35}$$
$$+\frac{\varepsilon}{N}\sum_{\nu}D(\mu_1,\kappa+\lambda_1-\mu_1.\mu_1-\nu,\kappa+\lambda_1-\mu_1-\nu)\langle\!\langle a_{\lambda1}{}^\dagger a_\nu a_{\kappa+\lambda1-\nu}\,;\,a_\kappa{}^\dagger\rangle\!\rangle_E{}^{(0)}$$

We solve this using the successive approximation, that is, the second term in the right hand side is expressed with the use of zero order one (3–23).

$$\langle\!\langle a_{\lambda1}{}^\dagger a_{\mu1}a_{\kappa+\lambda1-\mu1}\,;\,a_\kappa{}^\dagger\rangle\!\rangle_E{}^{(0)}=\frac{\langle n_{\lambda1}\rangle}{2\pi}\frac{1}{E-\varepsilon_\kappa{}^{(0)}}(\delta_{\kappa,\mu1}+\delta_{\lambda1,\mu1})$$
$$+\varepsilon\frac{2}{N}\frac{\langle n_{\lambda1}\rangle}{2\pi}\frac{1}{E-\varepsilon_\kappa{}^{(0)}}\frac{D(\mu_1,\,\kappa+\lambda_1-\mu_1,\,\kappa-\mu_1,\,\lambda_1-\mu_1)}{E+\varepsilon_{\lambda1}{}^{(0)}-\varepsilon_{\mu1}{}^{(0)}-\varepsilon_{\kappa+\lambda1-\mu1}{}^{(0)}} \tag{3-36}$$

According to the equation (2–18), $M_1'$ is obtained as,

$$M_1'=R_1G_2{}^{(0)}\{G_1{}^{(0)}\}^{-1}$$
$$=\frac{2}{N}\sum_{\lambda1}D(\lambda_1,\,\kappa,\,\kappa-\lambda_1,\,0)\langle n_{\lambda1}\rangle$$
$$+\frac{\varepsilon}{N^2}\sum_{\lambda1,\mu1}\frac{D(\lambda_1,\,\kappa,\,\lambda_1-\mu_1,\kappa-\mu_1)D(\mu_1,\,\kappa+\lambda_1-\mu_1,\,\kappa-\mu_1,\,\lambda_1-\mu_1)}{E+\varepsilon_{\lambda1}{}^{(0)}-\varepsilon_{\mu1}{}^{(0)}-\varepsilon_{\kappa+\lambda1-\mu1}{}^{(0)}}\langle n_{\lambda1}\rangle \tag{3-37}$$

Next we have to obtain $\langle\!\langle a_{\kappa1}{}^\dagger a_{\kappa2}{}^\dagger a_{\kappa3}a_{\kappa4}a_{\kappa5}\,;\,a_\kappa{}^\dagger\rangle\!\rangle_E{}^{(0,0)}$ to get the second order mass operator $M_1''$.

$$\langle\!\langle a_{\kappa1}{}^\dagger a_{\kappa2}{}^\dagger a_{\kappa3}a_{\kappa4}a_{\kappa5}\,;\,a_\kappa{}^\dagger\rangle\!\rangle_E{}^{(0,0)}$$
$$=\frac{1}{2\pi}\frac{\langle n_{\kappa1}\rangle\langle n_{\kappa2}\rangle}{E-\varepsilon_\kappa{}^{(0)}}\{\delta_{\kappa1,\kappa4}\delta_{\kappa2,\kappa3}\delta_{\kappa,\kappa5}+\delta_{\kappa1,\kappa3}\delta_{\kappa2,\kappa4}\delta_{\kappa,\kappa5}+\delta_{\kappa1,\kappa5}\delta_{\kappa2,\kappa3}\delta_{\kappa,\kappa4}+\delta_{\kappa1,\kappa3}\delta_{\kappa2,\kappa5}\delta_{\kappa,\kappa4}$$
$$+\delta_{\kappa1,\kappa5}\delta_{\kappa2,\kappa4}\delta_{\kappa,\kappa3}+\delta_{\kappa1,\kappa4}\delta_{\kappa2,\kappa5}\delta_{\kappa,\kappa3}\}$$

After a series of tedious calculation, we get $M_1''$ explicitely.

$$M_1''=R_1L_2{}^{-1}R_2G_3{}^{(0,0)}\{G_1{}^{(0)}\}^{-1}-R_1G_2{}^{(0)}\{G_1{}^{(0)}\}^{-1}L_1{}^{-1}R_1G_2{}^{(0)}\{G_1{}^{(0)}\}^{-1}$$
$$=\frac{2}{N^2}\sum_{\lambda1,\mu1}\frac{D(\lambda_1,\,\kappa,\,\lambda_1-\mu_1,\,\kappa-\mu_1)\,D(\mu_1,\,\kappa+\lambda_1-\mu_1,\,\kappa-\mu_1,\,\lambda_1-\mu_1)}{E+\varepsilon_{\lambda1}{}^{(0)}-\varepsilon_{\mu1}{}^{(0)}-\varepsilon_{\kappa+\lambda1-\mu1}{}^{(0)}}$$
$$\times(\langle n_{\lambda1}\rangle\langle n_{\mu1}\rangle+\langle n_{\lambda1}\rangle\langle n_{\kappa+\lambda1-\mu1}\rangle-\langle n_{\mu1}\rangle\langle n_{\kappa+\lambda1-\mu1}\rangle) \tag{3-38}$$

Then the mass operator $M_1$ accurate within the second order approximation is expressed as,

$$M_1=\varepsilon\frac{2}{N}\sum_{\lambda} D(\lambda,\kappa,\kappa-\lambda,0)\langle n_\lambda\rangle$$
$$+\varepsilon^2\frac{2}{N^2}\sum_{\lambda,\mu}\frac{D(\lambda,\kappa,\lambda-\mu,\kappa-\mu)\,D(\mu,\kappa+\lambda-\mu,\kappa-\mu,\lambda-\mu)}{E+\varepsilon_\lambda^{(0)}-\varepsilon_\mu^{(0)}-\varepsilon_{\kappa+\lambda-\mu}^{(0)}}$$
$$\times(\langle n_\lambda\rangle+\langle n_\lambda\rangle\langle n_\mu\rangle+\langle n_\lambda\rangle\langle n_{\kappa+\lambda-\mu}\rangle-\langle n_\mu\rangle\langle n_{\kappa+\lambda-\mu}\rangle) \tag{3-39}$$

This result agrees with that Tahir-Kheli and Ter-Haar have obtained in their higher order decoupling theory. For the Green function in this accuracy, we have,

$$\langle\langle a_\kappa\,;a_\kappa{}^\dagger\rangle\rangle_E{}^{r,a}=\frac{1}{2\pi}\frac{1}{E-\varepsilon_\kappa{}^{(0)}-\Delta\varepsilon_\kappa-R_\kappa(E)\pm i\gamma_\kappa(E)} \tag{3-40}$$

where

$$R_\kappa(E)=\frac{2}{N^2}P\sum_{\lambda,\mu}\frac{D(\lambda,\kappa,\lambda-\mu,\kappa-\mu)\,D(\mu,\kappa+\lambda-\mu,\kappa-\mu,\lambda-\mu)}{E+\varepsilon_\lambda^{(0)}-\varepsilon_\mu^{(0)}-\varepsilon_{\lambda+\lambda-\mu}^{(0)}}$$
$$\times\{\langle n_\lambda\rangle+\langle n_\lambda\rangle\langle n_\mu\rangle+\langle n_\lambda\rangle\langle n_{\kappa+\lambda-\mu}\rangle-\langle n_\mu\rangle\langle n_{\kappa+\lambda-\mu}\rangle\} \tag{3-41}$$

and

$$\gamma_\kappa(E)=\frac{2\pi}{N^2}\sum D(\lambda,\kappa,\lambda-\mu,\kappa-\mu)\,D(\mu,\kappa+\lambda-\mu,\kappa-\mu,\lambda-\mu)$$
$$\times\{\langle n_\lambda\rangle+\langle n_\lambda\rangle\langle n_\mu\rangle+\langle n_\lambda\rangle\langle n_{\kappa+\lambda-\mu}\rangle-\langle n_\mu\rangle\langle n_{\kappa+\lambda-\mu}\rangle\}$$
$$\times\delta(E-\varepsilon_\mu{}^{(0)}-\varepsilon_{\kappa+\lambda-\mu}^{(0)}+\varepsilon_\lambda{}^{(0)}) \tag{3-42}$$

$\gamma_\kappa$ means the damping coefficient of spin waves, and appears firstly in this approximation. We can expect that $\gamma_\kappa$ is much smaller than $\Delta\varepsilon_\kappa+R_\kappa$. So we can assume that the spin wave in low temperature region is the well defined quasi boson with the elementary excitation energy $\varepsilon_\kappa{}^{(2)}$.

$$\varepsilon_\kappa{}^{(2)}=\varepsilon_\kappa{}^{(0)}+\Delta\varepsilon_\kappa+R_\kappa(\varepsilon_\kappa{}^{(0)}) \tag{3-43}$$

The main effect of $R_\kappa$ in low temperature proves to be minor modification of the coefficient of $T^{5/2}$ term and the appearance of higher terms in temperature.

At last we investigate the difficulty of the non-Hermitic character in $\mathscr{H}_D$. So we calculate the Green function using $\mathscr{H}_D{}^\dagger$ instead of $\mathscr{H}_D$, we obtain the results.

$$\Delta\varepsilon_\kappa=\frac{2}{N}\sum_{\lambda} D(\lambda,\kappa,\lambda-\kappa,0)\langle n_\lambda\rangle \tag{3-25$'$}$$

$$R_\kappa=\frac{2}{N^2}P\sum_{\lambda,\mu}\frac{D(\kappa,\lambda,\lambda-\mu,\kappa-\mu)\,D(\kappa+\lambda-\mu,\kappa+\lambda-\mu,\kappa-\mu,\lambda-\mu)}{E+\varepsilon_\lambda{}^{(0)}-\varepsilon_\mu{}^{(0)}-\varepsilon_{\kappa+\lambda-\mu}^{(0)}}$$
$$\times(\langle n_\lambda\rangle+\langle n_\lambda\rangle\langle n_\mu\rangle+\langle n_\lambda\rangle\langle n_{\kappa+\lambda-\mu}\rangle-\langle n_{\kappa+\lambda-\mu}\rangle\langle n_\mu\rangle) \tag{3-41$'$}$$

$$\gamma_\kappa=\frac{2\pi}{N^2}\sum_{\lambda,\mu} D(\kappa,\lambda,\lambda-\mu,\kappa-\mu)\,D(\kappa+\lambda-\mu,\kappa+\lambda-\mu,\kappa-\mu,\lambda-\mu)$$
$$\times(\langle n_\lambda\rangle+\langle n_\lambda\rangle\langle n_\mu\rangle+\langle n_\lambda\rangle\langle n_{\kappa+\lambda-\mu}\rangle-\langle n_\mu\rangle\langle n_{\kappa+\lambda-\mu}\rangle)$$
$$\times\delta(E-\varepsilon_\mu{}^{(0)}-\varepsilon_{\kappa+\lambda-\mu}^{(0)}+\varepsilon_\lambda{}^{(0)}) \tag{3-42$'$}$$

Equation (3-25′) agrees with equation (3-25). But equations (3-41′) and (3-42′) are different from equations (3-41) and (3-42) respectively. The difference is

$$\Delta R_\kappa=\frac{2}{N^2}P\sum_{\lambda,\mu}\frac{D(\kappa,\lambda,\lambda-\mu,\kappa-\mu)(\mathscr{F}(\mu)-\mathscr{F}(\kappa+\lambda-\mu))}{E+\varepsilon_\lambda^{(0)}-\varepsilon_\mu^{(0)}-\varepsilon_{\kappa+\lambda-\mu}^{(0)}}$$
$$\times(\langle n_\lambda\rangle+\langle n_\lambda\rangle\langle n_\mu\rangle+\langle n_\lambda\rangle\langle n_{\kappa+\lambda-\mu}\rangle-\langle n_{\kappa+\lambda-\mu}\rangle\langle n_\mu\rangle)$$

We transform the sum over $\mu$ into the sum over $\kappa+\lambda-\mu$. The sum over $\mu$ is invariant for this transformation. Then,

$$\Delta R_\kappa=\frac{2}{N^2}P\sum_{\lambda,\mu}\frac{D(\kappa,\lambda,-\kappa+\mu,-\lambda+\mu)(\mathscr{F}(\kappa+\lambda-\mu)-\mathscr{F}(\mu))}{E+\varepsilon_\lambda^{(0)}-\varepsilon_{\kappa+\lambda-\mu}^{(0)}-\varepsilon_\mu^{(0)}}$$
$$\times(\langle n_\lambda\rangle+\langle n_\lambda\rangle\langle n_{\kappa+\lambda-\mu}\rangle+\langle n_\lambda\rangle\langle n_\mu\rangle-\langle n_{\kappa+\lambda-\mu}\rangle\langle n_\mu\rangle)=-\Delta R_\kappa$$

Thus we have $\Delta R_\kappa=0$. Similarly, $\Delta\gamma_\kappa=0$.
This property seems to conserve in higher order terms. So, $\mathscr{H}_D$ and $\mathscr{H}_D\dagger$ have the same effect for the elementary excitation energy.

## IV. Tyablikov, Callen and Hartree-Fock type decoupling

So far our main interest is concerned only with the behaviour in low temperature region, where the boson treatment is a good approximation. When the temperature is raised towards the Curie point, however, interaction between spin waves dynamical as well as kinematical will predominate and the effects of unphysical states may not be ignored. Thus we return to the original Heisenberg Hamiltonian (3–1) and think about the problem in spin operators $S^+$, $S^-$ and $S^z$. Using the equations,

$$\mathscr{H}=-g\mu_BH\sum_l S_l{}^z-\sum_{l,m}J(l-m)(\tfrac{1}{2}S_l^+S_m{}^-+\tfrac{1}{2}S_l^-S_m^++S_l{}^zS_m{}^z)\tag{4—1}$$

$$[S_f{}^+,\mathscr{H}]=g\mu_BHS_f{}^+-2\sum_{f'}J(f-f')(S_f{}^zS_{f'}{}^+-S_f{}^+S_{f'}{}^z)\tag{4—2}$$

and introducing the quantity,

$$\langle[S_f{}^+,\ e^{\alpha S_f^z}S_f{}^-]\rangle=\Theta(\alpha)\tag{4-3}$$

we construct the equation of motion for the Green function,

$$G_{f^\alpha,g}(t,t')=\langle\langle S_f{}^+(t)\,;e^{\alpha S_g^z}S_g{}^-(t')\rangle\rangle\tag{4-4}$$

We write down the equation with the Fourier component in time,

$$EG_{f,g}^\alpha(E)=\frac{\Theta(\alpha)}{2\pi}\delta_{f,g}+g\mu_BHG_{f,g}^\alpha(E)$$
$$-2\sum_{f'}J(f-f')\langle\langle(S_f{}^zS_{f'}{}^+-S_f{}^+S_{f'}{}^z)\,;e^{\alpha S_g^z}S_g{}^-\rangle\rangle_E\tag{4-5}$$

where $G_{f,g}^\alpha(E)$ is the Fourier transform of $G_{f,g}^\alpha(t,t')$,

$$G_{f,g}^\alpha(E)=\frac{1}{2\pi}\int_{-\infty}^{\infty}G_{f,g}^\alpha(t-t')e^{iE(t-t')}dt$$

The last term in equation (4–5) is a new higher order Green function. So, we must use some approximation to solve the equation of motion (4–5). The new Green function in the equation is expressed by means of the original one and the statistical parameters, namely the approximation of decoupling is introduced, thus we may obtain the good properties in the wider range of temperature, including some higher order effects properly.

In this paragraphs we consider the following three decoupling types.

Tyablikov; $\langle\langle S_f^z S_{f'}^+ ; e^{\alpha S_g^z} S_g^- \rangle\rangle \to \langle S^z \rangle \langle\langle S_{f'}^+ ; e^{\alpha S_g^z} S_g^- \rangle\rangle$ (4–6)

Hartree-Fock; $\langle\langle S_f^z S_{f'}^+ ; e^{\alpha S_g^z} S_g^- \rangle\rangle \to \langle S^z \rangle \langle\langle S_{f'}^+ ; e^{\alpha S_g^z} S_g^- \rangle\rangle$

$$-\frac{1}{2S}\langle S_f^- S_{f'}^+ \rangle \langle\langle S_f^+ ; e^{\alpha S_g^z} S_g^- \rangle\rangle \tag{4–6′}$$

Callen; $\langle\langle S_f^z S_{f'}^+ ; e^{\alpha S_g^z} S_g^- \rangle\rangle \to \langle S^z \rangle \langle\langle S_{f'}^+ ; e^{\alpha S_g^z} S_g^- \rangle\rangle$

$$-\frac{1}{2S}\frac{\langle S^z \rangle}{S}\langle S_f^- S_{f'}^+ \rangle \langle\langle S_f^+ ; e^{\alpha S_g^z} S_g^- \rangle\rangle \tag{4–6″}$$

Taking into account that $G_{f,g}^{\alpha}(E)$ depends only on the difference of the lattice vectors $f$ and $g$, we can change to the wave number representation.

$$G_{f,g}^{\alpha}(E) = \frac{1}{N}\sum_{\nu} e^{i[(f-g)\cdot\nu]} G_{\nu}^{\alpha}(E) \tag{4–7}$$

Then we obtain the results,

$$G_{\nu}^{\alpha}(E) = \frac{\Theta(\alpha)}{2\pi}\frac{1}{E-E_{\nu}} \tag{4–8}$$

$$\overline{N}_{\nu} = (e^{\beta E_{\nu}}-1)^{-1} \tag{4–9}$$

$$P_s = \frac{1}{N}\sum_{\nu}\overline{N}_{\nu} \tag{4–10}$$

Tyablikov; $E_{\nu} = g\mu_B H + 2\langle S^z \rangle [\mathscr{F}(0) - \mathscr{F}(\nu)]$ (4–11)

Hartree-Fock; $E_{\nu} = g\mu_B H + 2\langle S^z \rangle (1+f) [\mathscr{F}(0) - \mathscr{F}(\nu)]$ (4–11′)

Callen; $E_{\nu} = g\mu_B H + 2\langle S^z \rangle \left(1+\frac{\langle S^z \rangle}{S} f\right) [\mathscr{F}(0) - \mathscr{F}(\nu)]$ (4–11″)

where

$$f = \frac{1}{SN}\sum_{\lambda}\frac{\mathscr{F}(\lambda)}{\mathscr{F}(0)}\overline{N}_{\lambda} \tag{4–12}$$

Callen[6] and Tahir-Kheli et al[9] showed the following relation holds good self-consistently.

$$\langle S^z \rangle = \frac{(S-P_S)(1+P_S)^{2S+1} + (1+S+P_S)P_S^{2S+1}}{(1+P_S)^{2S+1} - P_S^{2S+1}} \tag{4–13}$$

In the low temperature, we obtain the same results for three approximation up to the $T^{5/2}$—term. But in the higher order these results contain different terms.

Tyablikov;

$$\langle S^z \rangle = S - Z(3/2)\tau^{3/2} - \frac{3\pi}{4}Z(5/2)\tau^{5/2} - \frac{33}{32}\pi^2 Z(7/2)\tau^{7/2}$$
$$-\frac{3}{2S}Z^2(3/2)\tau^3 - \frac{3\pi}{S}Z(3/2)Z(5/2)\tau^4 + (2S+1)Z^{2S+1}(3/2)\tau^{3S+\frac{3}{2}}\ldots \tag{4–14}$$

Hartree-Fock and Callen;

$$\langle S^z\rangle = S - Z(3/2)\tau^{3/2} - \frac{3\pi}{4} Z(5/2)\tau^{5/2} - \frac{33}{32}\pi^2 Z(7/2)\tau^{7/2}$$
$$-\frac{3}{2S}\pi Z(3/2)Z(5/2)\tau^4 + (2S+1)Z^{2S+1}(3/2)\tau^{3S+\frac{3}{2}} \quad (4\text{–}14')$$

$$f = \frac{1}{S} Z(3/2)\tau^{3/2} - \frac{\pi}{4S} Z(5/2)\tau^{5/2} + \dots \quad (4\text{–}15)$$

Comparing the equations (3–17′), (3–18′) with (4–14) and (4–14′), we see that the results of Hartree-Fock and Callen decoupling correspond to Dyson's spin wave result except for the case $S=\frac{1}{2}$. For the case $S=\frac{1}{2}$, the term $\tau^{3S+\frac{3}{2}}$ in the equation (4–14′) gives incorrect contribution to the $\tau^3$ term. The higher order term $\tau^{3S+\frac{5}{2}}$ also gives incorrect contribution to the $\tau^4$ term. But the result of Tyablikov decoupling does not agree with Dyson's spin wave theory for all $S$. Combining the equations (4–14), (4–14′), (4–15) and (4–11, 11′, 11″) respectively we can determine the temperature dependence of the elementary excitation energy $E_\nu$ in low temperature region.

Tyablikov;

$$E_\nu = g\mu_B H + 2S[\mathscr{J}(0) - \mathscr{J}(\nu)]\left(1 - \frac{1}{S} Z(3/2)\tau^{3/2} - \frac{3\pi}{4S} Z(5/2)\tau^{5/2} + \dots\right) \quad (4\text{–}16)$$

Hartree-Fock and Callen;

$$E_\nu = g\mu_B H + 2S[\mathscr{J}(0) - \mathscr{J}(\nu)]\left(1 - \frac{\pi}{S} Z(5/2)\tau^{5/2} + \dots\right) \quad (4\text{–}16')$$

The equation (4–16′) corresponds to the equation (3–29) and Dyson's result for all $S$. The microscopic property is accurate in these two approximations for all $S$. This means that the error of Hartree-Fock or Callen decoupling arises from taking the statistics. The equation (4–16) is different from Dyson's spin wave theory. So, Tyablikov decoupling has the error in microscopic and macroscopic natures in the low temperature.

Next, we consider at Curie temperature. For simplicity, we consider only about the zero field case, $H=0$. At Curie temperature, we assume $\langle S^z\rangle$ is vanishingly small. Then we can prove by Tyablikov and Callen decoupling approximation,

$$E_\nu = 0 \quad (4\text{–}17)$$

but from Hartree-Fock approximation, we may conclude

$$E_\nu \neq 0 \quad (4\text{–}17')$$

at Curie temperature. We, however, don't get Curie point by Hartree-Fock approximation directly, so we take extrapolatic Curie point in this approximation. This circumstance also corresponds to Boson approximation in III.

$E_\nu \to 0$ means that numbers of elementary excitation may be very large, so we may predict that the life time of the elementary excitation is very short. This state of affairs seems to be one expected at Curie temperature. Although, we don't have any information about the life time by these decoupling approximations, we may conclude that Callen and Tyablikov decoupling can be used at near Curie temperature too, but Hartree-Fock type decoupling fails at near Curie temperature.

## V. Katsura and Horiguchi decoupling

This decoupling was proposed to improve the magnetization dependence on temperature in wider range for the case $S=\frac{1}{2}$. But it is difficult to extend the method for arbitrary $S$ values. This seems to be one of the defects of this decoupling.

In the case $S=\frac{1}{2}$, the spin operators can be expressed in terms of the Pauli operators.

$$\begin{aligned} S_f^+ &= b_f \\ S_f^- &= b_f^+ \\ S_f^z &= \tfrac{1}{2} - b_f^+ b_f \end{aligned} \tag{5-1}$$

Pauli operators satisfy the following commutation or anticommutation relations.

$$\begin{aligned} &[b_f, b_f^+]_+ = 1 \\ &[b_f, b_f]_+ = [b_f^+, b_f^+]_+ = 0 \\ &[b_f, b_g^+]_- = (1-2b_f^+ b_f)\,\delta_{f,g} \end{aligned} \tag{5-2}$$

Introducing the expression (5-1) into the Hamiltonian (3-1), we obtain

$$\begin{aligned} \mathscr{H} = &-\frac{1}{4} N\mathscr{J}(0) - \frac{1}{2} Ng\mu_B H + \{\mathscr{J}(0) + g\mu_B H\} \sum_l b_l^+ b_l \\ &- \sum_{l,m} J(l-m) b_l^+ b_m - \sum_{l,m} J(l-m)\, b_l^+ b_l b_m^+ b_m \end{aligned} \tag{5-3}$$

We consider about the following Green functions,

$$G_{f,g}^{(\pm)}(t, t') = \langle\langle b_f(t)\,,\, b_g^+(t') \rangle\rangle^{(\pm)} \tag{5-4}$$

and obtain the equations of motion in energy representation as follows.

$$\begin{aligned} [E - g\mu_B H - \mathscr{J}(0)]\, G_{f,g}^{(+)}(E) = &\frac{1}{2\pi} \langle 2b_g^+ b_f + (1-2n_f)\,\delta_{f,g} \rangle \\ &- \sum_{f'} J(f-f')\, G_{f',g}^{(+)}(E) + 2\sum_{f'} J(f-f')\, \langle\langle b_f^+ b_f b_{f'}\,;\, b_g^+ \rangle\rangle_E^{(+)} \\ &-2\sum_{f'} J(f-f')\, \langle\langle b_{f'}^+ b_{f'} b_f\,;\, b_g^+ \rangle\rangle_E^{(+)} \end{aligned} \tag{5-5}$$

$$\begin{aligned} [E - g\mu_B H - \mathscr{J}(0)]\, G_{f,g}^{(-)}(E) = &\frac{1}{2\pi} \langle (1-2n_f)\,\delta_{f,g} \rangle \\ &- \sum_{f'} J(f-f')\, G_{f',g}^{(-)}(E) + 2\sum_{f'} J(f-f')\, \langle\langle b_f^+ b_f b_{f'}\,;\, b_g^+ \rangle\rangle_E^{(-)} \\ &-2\sum_{f'} J(f-f') \langle\langle b_{f'}^+ b_{f'} b_f\,;\, b_g^+ \rangle\rangle_E^{(-)} \end{aligned} \tag{5-6}$$

Katsura and Horiguchi proposed the following decoupling approximation,

$$\begin{aligned} \langle\langle b_f^+ b_f b_{f'}\,;\, b_g^+ \rangle\rangle^{(\pm)} \to &\langle b_f^+ b_f \rangle\, \langle\langle b_{f'}\,;\, b_g^+ \rangle\rangle^{(\pm)} \\ &+ \langle b_f^+ b_{f'} \rangle\, \langle\langle b_f\,;\, b_g^+ \rangle\rangle^{(\mp)} \end{aligned} \tag{5-7}$$

and could solve the resulting simultaneous equations,

$$G_\nu^{(+)}(E)=\frac{1}{2\pi}\left[\frac{n_\nu}{E-E_\nu+Q_\nu}+\frac{n_\nu+\sigma}{E-E_\nu-Q_\nu}\right]$$

$$G_\nu^{(-)}(E)=\frac{1}{2\pi}\left[\frac{-n_\nu}{E-E_\nu+Q_\nu}+\frac{n_\nu+\sigma}{E-E_\nu-Q_\nu}\right] \tag{5-8}$$

where

$$G_{f,g}^{(\pm)}(E)=\frac{1}{N}\sum_\nu e^{i\{(f-g)\cdot\nu\}}G_\nu^{(\pm)}(E)$$

$$\langle b_g^+b_f\rangle=\frac{1}{N}\sum_\nu e^{i\{(f-g)\cdot\nu\}}n_\nu$$

$$\sigma=1-2\langle b_f^+b_f\rangle \tag{5-9}$$

$$E_\nu=g\mu_BH+\sigma[\mathscr{F}(0)-\mathscr{F}(\nu)]$$

$$Q_\nu=[\mathscr{F}(0)-\mathscr{F}(\nu)]p$$

$$p=\frac{2}{N}\sum_\lambda\frac{\mathscr{F}(\lambda)}{\mathscr{F}(0)}n_\lambda$$

In the equations (5–8), we remark, there are two kinds of elementary excitation with the energies $\tilde{E}_\nu$,

$$\tilde{E}_\nu=g\mu_BH+[\mathscr{F}(0)-\mathscr{F}(\nu)](\sigma\pm p) \tag{5-10}$$

In the low temperature, we get $\sigma$ and $p$ as the functions of $\tau$, then we obtain the temperature dependence of elementary excitation energies explicitly.

$$\tilde{E}_\nu=g\mu_BH+[\mathscr{F}(0)-\mathscr{F}(\nu)](1\mp2\pi Z(5/2)\tau^{5/2}+\dots) \tag{5-10'}$$

Comparing the equations (5–10′) with (3–29), one of the elementary excitations corresponds to the Dyson's spin wave theory. But the other does not correspond to Dyson's one. We interpret these excitations as follows. The reason why two kinds of excitations arise is that this decoupling approximation does not close in one kind of approximations. So, two excitation energies correspond to $\langle b_f^+b_f\rangle^{(-)}$ system obtained from $G^{(-)}(E)$ and $\langle b_f^+b_f\rangle^{(+)}$ system obtained from $G^{(+)}(E)$ respectively. But we have to conclude $\langle b_f^+b_f\rangle^{(-)}=\langle b_f^+b_f\rangle^{(+)}$, if the approximation is accurate. Because of these discussions, we can't assent to this decoupling approximation.

Next, we consider at Curie temperature in case of no applied field. At Curie temperature, we suppose $\sigma$ to vanish, but the elementary excitation energy may not vanish from the equations (5–10).

## VI. Conclusion and summary

We have studied Heisenberg ferromagnet in preceding two paragraphs using the method of Green function by some decoupling approximations from the microscopic viewpoint. We obtain some defects and meanings of these decoupling approximation that is not explicit in the macroscopic viewpoint.

We construct the table to understand the merits and demerits of the decoupling approximations at the same time. The table consists of the grand columns and the sub-columns. The sub-columns represent the order of the approximations in accuracy. The mark $\times$ means that this approximation can not be used. Two columns of the left side represent the elementary excitation energy (microscopic property). The next two columns represent the macroscopic properties, and we indicate the reason of the error. The underlines mean that these terms are wrong only in the numerical values of the coefficients. The last column represents Curie point, $kT_c/J$. The upper lines are for the case $S=\frac{1}{2}$ and the lower lines for the case $S=1$. To get Curie point in the perturbation expansion approximation we have to use the extrapolation method, and we cite in the table the result of Harada.[10]

At last, we present one problem. We solve Hamiltonian (3–8) using the method of temperature Green function by the diagram technique[11] used in quantum statistical mechanics. We can correspond this result to the results (3–13), (3–26) and (3–27) within the first order approximation. But we proceed into the second order appraximation to get the following result.

**Table 1.** Degrees of the various approximations

| Types of approximation | elementary excitation energy | | | | magnetization $T\ll T_c$ | | | | Curie point $kT_c/J$ |
|---|---|---|---|---|---|---|---|---|---|
| | $T\ll T_c$ | | $T\lesssim T_c,\ H=0$ | | $S=\frac{1}{2}$ | | $S\geqq 1$ | | |
| Boson (Perturbation theory) | 1 | $g\mu_B H + 2S[\mathscr{F}(0)-\mathscr{F}(\nu)] \times\left(1-\frac{\pi}{S}Z\left(\frac{5}{2}\right)\tau^{\frac{5}{2}}+\ldots\right)$ | $\times$ | | 1 | standard | 1 | standard | 2.84<br>5.68 |
| Tyablikov | 3 | $g\mu_B H + 2S[\mathscr{F}(0)-\mathscr{F}(\nu)] \times\left(1-\frac{1}{S}Z\left(\frac{3}{2}\right)\tau^{\frac{3}{2}}-\frac{3\pi}{4S}Z\left(\frac{5}{2}\right)\tau^{\frac{5}{2}}+\ldots\right)$ | 1 | $2S[\mathscr{F}(0)-\mathscr{F}(\nu)] \times\sqrt{\Gamma_s\tau_0(1-\tau/\tau_0)}$ | 4 | incorrect $\tau^3$, $\underline{\tau^4}$ terms | 3 | incorrect $\tau^3$, $\underline{\tau^4}$ terms | 1.98<br>5.28 |
| Hartree —Fock | 2 | $g\mu_B H + 2S[\mathscr{F}(0)-\mathscr{F}(\nu)] \times\left(1-\frac{\pi}{S}Z\left(\frac{2}{5}\right)\tau^{\frac{5}{2}}+\ldots\right)$ | $\times$ | | 3 | incorrect $\tau^3$, $\underline{\tau^4}$ terms | 2 | correct | ? |
| Callen | 2 | $g\mu_B H + 2S[\mathscr{F}(0)-\mathscr{F}(\nu)] \times\left(1-\frac{\pi}{S}Z\left(\frac{5}{2}\right)\tau^{\frac{5}{2}}+\ldots\right)$ | 1 | $2S[\mathscr{F}(0)-\mathscr{F}(\nu)] \times\sqrt{A\tau_0(1-\tau/\tau_c)}$ | 3 | incorrect $\tau^3$, $\underline{\tau^4}$ terms | 2 | correct | 2.65<br>6.48 |
| Katsura and Horiguchi | 4 | $g\mu_B H + [\mathscr{F}(0)-\mathscr{F}(\nu)] \times\left(1\mp 2\pi Z\left(\frac{5}{2}\right)\tau^{\frac{5}{2}}+\ldots\right)$ | $\times$ | | 2<br>? | incorrect $\tau^3$, $\tau^4$ terms | | ? | 2.51<br>? |

$$\Gamma_s=\frac{10C(S+1)}{4S^2+4S+5C-3},\quad A=\Gamma_s\left\{1-\frac{1}{2z}\Gamma_s\frac{C}{(5S+2)C-2(S+1)}\right\}^{-1}$$

$$\tau_0=2(S+1)/3C,\quad \tau_c=2(S+1)\{(4S+1)C-(S+1)\}/9SC^2$$

$$C=\frac{v}{(2\pi)^3}\int\frac{\mathscr{F}(0)}{\mathscr{F}(0)-\mathscr{F}(\nu)}d^3\nu$$

$z$ is the number of nearest neighbours in a crystal.

$$2\pi\langle\langle a_\kappa\,;\,a_\kappa{}^\dagger\rangle\rangle_E=\frac{1}{E-\varepsilon_\kappa{}^{(0)}}+\frac{1}{(E-\varepsilon_\kappa{}^{(0)})^2}\frac{2}{N}\sum_\mu D(\kappa,\,\mu,\,\kappa-\mu,\,0)\,\langle n_\mu\rangle$$

$$+\frac{1}{(E-\varepsilon_\kappa{}^{(0)})^2}\frac{2}{N^2}\sum_{\lambda,\mu}\frac{D(\lambda,\,\kappa,\,\lambda-\mu,\,\kappa-\mu)\,D(\mu,\,\kappa+\lambda-\mu,\,\mu-\lambda,\,\kappa-\mu)}{E+\varepsilon_\lambda{}^{(0)}-\varepsilon_\mu{}^{(0)}-\varepsilon^{(0)}_{\kappa+\lambda-\mu}}$$

$$\times\{\langle n_\lambda\rangle+\langle n_\lambda\rangle\,\langle n_\mu\rangle+\langle n_\lambda\rangle\,\langle n_{\kappa+\lambda-\mu}\rangle-\langle n_\mu\rangle\,\langle n_{\kappa+\lambda-\mu}\rangle\}$$

$$+\frac{1}{(E-\varepsilon_\kappa{}^{(0)})^3}\left\{\frac{2}{N}\sum_\mu D(\kappa,\,\mu,\,\kappa-\mu,\,0)\,\langle n_\mu\rangle\right\}^2$$

$$+\frac{1}{(E-\varepsilon_\kappa{}^{(0)})^2}\frac{4}{N^2}\,\beta\sum_{\lambda,\mu}D(\kappa,\,\lambda,\,\kappa-\lambda,\,0)\,D(\lambda,\,\mu,\,\lambda-\mu,\,0)$$

$$\times\{\langle n_\mu\rangle\,\langle n_\lambda\rangle+\langle n_\mu\rangle\,\langle n_\lambda\rangle^2\}$$

The last term does not appear in the equation (3-40). We expect that the result obtained in use of the perturbation expansion for the mass operator of the two time Green function and the result obtained in use of the perturbation theory in the temperature Green function must coincide with each other because both methods use the expansion procedure in the parameter of the same perturbation. We don't still solve this problem.

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