慶應義塾大学学術情報リポジトリ
Keio Associated Repository of Academic resouces

Title	A generalized method to evaluate the Lipschitz constant associated with a class of simultaneous operator systems
Sub Title	
Author	柳井, 浩(Yanai, Hiroshi)
Publisher	慶応義塾大学藤原記念工学部
Publication year	1967
Jtitle	Proceedings of the Fujihara Memorial Faculty of Engineering Keio University (慶応義塾大学藤原記念工学部研究報告). Vol.20, No.79 (1967.) ,p.93(1)- 143(51)
JaLC DOI	
Abstract	
Notes	
Genre	Departmental Bulletin Paper
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=KO50001004-00200079-0001

慶應義塾大学学術情報リポジトリ(KOARA)に掲載されているコンテンツの著作権は、それぞれの著作者、学会または出版社/発行者に帰属し、その権利は著作権法によって保護されています。引用にあたっては、著作権法を遵守してご利用ください。

The copyrights of content available on the KeiO Associated Repository of Academic resources (KOARA) belong to the respective authors, academic societies, or publishers/issuers, and these rights are protected by the Japanese Copyright Act. When quoting the content, please follow the Japanese copyright act.

A Generalized Method to Evaluate the Lipschitz Constant Associated with a Class of Simultaneous Operator Systems

(Received December 6, 1967)

Hiroshi YANAI*

Abstract

As is well known, a sequence

$$x_0, x_1, \dots, x_n, \dots$$

on a complete metric space converges to a limit, if the sequence is defined by

$$x_{n+1}=\varphi(x_n),$$

where φ maps the space into itself and is associated with a Lipschitz constant less than unity; moreover the limit point is the unique fixed point of the operator.

This article provides a unified method to evaluate the Lipschitz constant of a class of simultaneous operator systems, which include the system of the form

$$\boldsymbol{\varphi}(\boldsymbol{x})=\begin{cases}\varphi^1(x^1, x^2, \dots, x^M)\\ \dots\dots\\ \varphi^M(x^1, x^2, \dots, x^M)\end{cases}$$

defined on a complete metric space $X^1\times X^2\times\dots\times X^M$, $(x^i\in X^i)$ and of the form

$$\psi(\boldsymbol{x})=\begin{cases}\varphi^1(x^1, x^2, \dots, x^M) & y^1=\varphi^1(x^1, x^2, \dots, x^M)\\ \varphi^2(y^1, x^2, \dots, x^M) & y^2=\varphi^2(y^1, x^2, \dots, x^M)\\ \dots\dots & \dots\dots\\ \varphi^i(y^1, \dots, y^{i-1}, x^i, \dots, x^M) & y^i=\varphi^i(y^1, \dots, y^{i-1}, x^i, \dots, x^M)\\ \dots\dots & \dots\dots\\ \varphi^M(y^1, \dots, y^{M-1}, x^M) & y^M=\varphi^M(y^1, \dots, y^{M-1}, x^M)\end{cases}$$

defined on a complete metric space $X^1\times X^2\times\dots\times X^M$; this form represents the functional structure of Gauss-Seidel process solving system of simultaneous linear equations, relaxation method solving ordinary and partial differential equations, etc.

Precise characteristics may be known as to some of the individual operators and only rough upper bounds are known as to the others. For example, the formers may be given in the form of partial derivatives, the latter, in the form of the Lipschitz constant of the individual operators. The algorithm to be presented here joins those characteristics to obtain the Lipschitz constant of the system as a whole.

The method of joining those characteristics depends on the metric defined on the product space on which the operator system is defined. Introducing 'B-operator', the algorithms, based on the 'B-operator' algebra, enable as to treat most of the important metrics in a unified manner.

The method is applied to various forms of the operator system of the class, obtaining some new convergence criteria.

---

*柳 井 浩 Lecturer, Faculty of Engineering, Keio University.

I. Introduction

As it is seen in our daily activities, partwise modification processes are carried out in the hope to achieve some objective. It seems, unfortunately, to be not rare that we reach at a point far from the aimed goal and in some miserable cases, we are led to chaos, however it is also often the case that this is the only way we can follow, so far as our knowledge and ability are limited. And even in the case some other way can be followed, we are tempted to follow this attractively simple way. The partwise ordering process, sometimes employed in ordering a set of numbered cards, may be an example of this kind.

As one of the typical methods based on this idea, we can mention one-variable-at-a-time method of seeking the optimum (minimum or maximum) of a function of several variables. More precisely, the simplest idea of seeking the minimum (or the maximum)* of a function $F(\boldsymbol{x})=F(x^1, x^2,\dots, x^M)$ defined on a product set $D^1\times D^2 \times\dots\times D^M$,

$$F(\check{\boldsymbol{x}})=\min_{\boldsymbol{x}\in D^1\times D^2\times\dots\times D^M} F(\boldsymbol{x}) \tag{1}$$

is to use the successive approximation technique giving the n-th approximation

$$\boldsymbol{x}_n=(x_n^1,\ x_n^2,\dots,\ x_n^M) \tag{2}$$

by the recurrence relation

$$F(x_n^1,\ x_n^2,\dots,\ x_n^i,\ x_{n-1}^{i+1},\dots,\ x_{n-1}^M)=\min_{\substack{x^i\in D^i \\ i=1,\ 2,\dots,\ M}} F(x_n^1,\ x_n^2,\dots,\ x^i,\ x_{n-1}^{i+1},\dots,\ x_{n-1}^M) \tag{3}$$

with a certain initial approximation $\boldsymbol{x}_0\in D^1\times D^2\times\dots\times D^M$.

Clearly, the objective function $F(\boldsymbol{x})$ does not increases as the process proceeds, and moreover, it might be expected that $F(\boldsymbol{x})$ decreases in strict sense. Although this method thus seems to be promising, unfortunately, the sequence $\{\boldsymbol{x}_n\}$ does not always converge to the minimum point $\check{\boldsymbol{x}}$. Some of the counter examples are given below.

Fig. 1 shows contour lines of a bimodal function of two variables, which has two peaks at point $\boldsymbol{t}$ and $\boldsymbol{s}$. The minimum is at $\boldsymbol{s}$. The sequence beginning with $\boldsymbol{a}$ converges to $\boldsymbol{s}$ although the sequence from $\boldsymbol{b}$ converges to $\boldsymbol{t}$.

Fig. 2 shows contour lines of an inversed pyramid. The minimum is given at $\boldsymbol{t}$. The sequence beginning with the most of the points of the domain degenerates after the first modification and does not reach at $\boldsymbol{t}$. Existence of ridges interrupts the successive modifications.

Expressed in mathematical language, the convergence and the limit point of the sequence defined by the operator system of the form

---

* Obviously, maximization of the function $F(\boldsymbol{x})$ can be obtained by considering minimization of $-F(\boldsymbol{x})$.

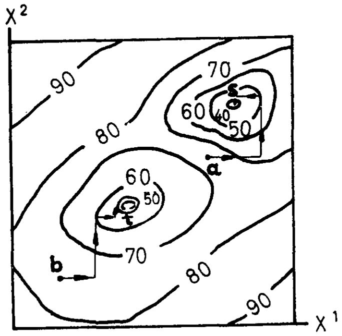

Fig. 1. Contour lines of a bimodal function.

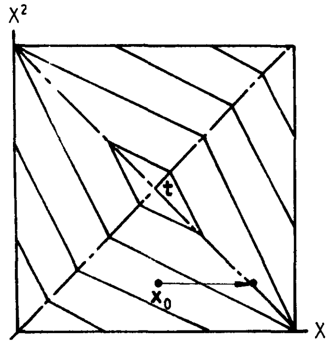

Fig. 2. Contour lines of inversed pyramid.

$$
\begin{aligned}
&y^1=\varphi^1(x^2,\dots,\ x^M)\\
&y^2=\varphi^2(y^1,\ x^2,\dots,\ x^M)\\
&\dots\dots\dots\\
&y^i=\varphi^i(y^1,\dots,\ y^{i-1},\ x^i,\dots,\ x^M)\\
&\dots\dots\dots\\
&y^M=\varphi^M(y^1,\dots,\ y^{M-1},\ x^M)
\end{aligned}
$$

is the problem to be investigated.

Some other examples of this kind of processes may be found elsewhere. Relaxation method of any sort as well as so-called cob-web theory in economics are based on the same idea.

More general form of the process may be given by the operator system

$$
\begin{aligned}
&y^1=\varphi^1(x^1,\ x^2,\dots,\ x^M)\\
&y^2=\varphi^2(y^1,\ x^2,\dots,\ x^M)\\
&\dots\dots\dots\\
&y^i=\varphi^i(y^1,\dots,\ y^{i-1},\ x^i,\dots,\ x^M)\\
&\dots\dots\dots\\
&y^M=\varphi^M(y^1,\dots,\ y^{M-1},\ x^M)
\end{aligned}
$$

which is usually called the Seidel process corresponding to the simple process given by

$$
\begin{aligned}
&y^1=\varphi^1(x^1,\ x^2,\dots,\ x^M)\\
&\dots\dots\dots\\
&y^i=\varphi^i(x^1,\ x^2,\dots,\ x^M)\\
&\dots\dots\dots\\
&y^M=\varphi^M(x^1,\ x^2,\dots,\ x^M).
\end{aligned}
$$

There may be two approaches to this problem: one may be to limit the problem to the linear case and to use the concepts of eigenvalue, eigenvector etc. Although this approach may be fruitful in theoretical sense, and indeed, some results are given by Faddev and others, it not only reduces domain and leaves the important non-linear cases uninvestigated, but also the evaluation of such variables as eigenvalues needs more numerical computation than to solve the equation itself.

Another approach, which will be followed in this paper, is to evaluate another characteristic value associated with the process which is usually called Lipschitz constant. Thus the main object of this paper is to develop a unified algorithm to evaluate the Lipschitz constant from various Lipschitz-constant-like characteristic values associated with the operators which constitute the operator system. Although this approach needs not to narrow the problem, we cannot expect to obtain the exact behavior of the process but only some upper bounds are obtained. Thus, for example, the Lipschitz constant evaluated to be less than unity guarantees the convergence of the process and gives an upper limit as well to the distance between the n-th member of the sequence (i.e. the n-th approximation) and the limit point. But, it is not possible to give exact comparison between the speeds of convergence of two different processes by comparing the evaluated Lipschitz constants.

Nevertheless, the evaluation of the Lipschitz constant, if it is evaluated as small as possible, indicates a good approximate behavior of the process.

Thus again, we need algorithm to evaluate the Lipschitz constant as small (i.e., precise) as possible from all the known characteristic values associated with the operators constituting the system.

Results obtained by following this approach are, so far as the author knows, not many. But it must be mentioned that Sassenfeld ⟨9⟩ gave some results in 1952.

In section II, some fundamental definitions and theorem required in the subsequent sections are given. In sections III-V, methods are given to evaluate the Lipschitz constant from various types of characteristic values associated with the operators constituting the system. Summary and table of the algorithms are given in section IV.

II. Preliminaries

Fundamental definitions and the theorems are given in this section. Although some of the theorems are well known in the field of successive approximation technique in numerical analysis, we recall them here in our notation with their proofs to prevent the readers from refering original papers.

Definition 1 (Lipschitz constant)

Consider an operator $\varphi(x^1)$ mapping a subset F of a metric space $R^1(X^1, \rho^1)$ into a metric space $R^2(X^2, \rho^2)$:

$$\varphi(x^1)\in R^2 \qquad x^1\in F. \tag{1}$$

If there exists a non-negative constant L such that

$$\rho^2(\varphi(x^1).\ \varphi(y^1))\leqq L\rho^1(x^1, y^1), \tag{2}$$

for two arbitrary elements x^1, $y^1\in F$, the constant L is called a Lipschitz constant associated with the operator $\varphi(x)$.

Theorem 1 (Fixed point theorem, Theorem of contraction operator, Collatz ⟨1⟩)

Consider an operator $\varphi(x)$ defined on a complete subset F of a metric space $R(X,\rho)$ mapping uniquely into the space $R(X,\rho)$ itself. If we assume that
(i) there exists a Lipschitz constant L of $\varphi(x)$ and $L<1$,
(ii) there exists x_0 such that $x_1=\varphi(x_0)$ belongs to F,
(iii) the sphere $S\{u:\rho(u, x_1)\leqq(L/1-L)\rho(x_1, x_0)\}$ is contained in F, then the sequence

$$x_0, x_1,\dots, x_n,\dots \tag{3}$$

defined by the recurrence relation

$$x_{n+1}=\varphi(x_n) \tag{4}$$

converges to the unique fixed point of the operator $\varphi(x)$ in F:

$$x=\varphi(x). \tag{5}$$

Proof

First, we prove the continuity of the operator $\varphi(x)$ on F. For any $\varepsilon>0$, take $\delta>0$ such that $0<\delta<\varepsilon/L$, and employing the assumption (i), we see that for any $x_0\in F$ the relation

$$\rho(\varphi(x), \varphi(x_0))<\varepsilon$$

holds for any x_0 and x of F such that

$$\rho(x, x_0)<\delta.$$

In fact,

$$\rho(\varphi(x), \varphi(x_0))\leqq L\rho(x, x_0)<L\delta<\varepsilon.$$

Hence the operator $\varphi(x)$ is continuous.

Next, we prove that the sequence given by the recurrence relation (4) is a fundamental sequence in F. It follows from the assumption (i), that for any integers $m>n$, we have

$$\begin{aligned}\rho(x_n, x_m)&=\rho(\overbrace{\varphi\varphi\dots\varphi}^{n}(x_0), \overbrace{\varphi\varphi\dots\varphi}^{m}(x_0))\leqq(L)^n\rho(x_0, x_{m-n})\\&\leqq(L)^n(\rho(x_0, x_1)+\rho(x_1, x_2)+\dots+\rho(x_{m-n-1}, x_{m-n}))\\&\leqq(L)^n(1+(L)+(L)^2+\dots+(L)^{m-n-1})\rho(x_0, x_1)\\&<((L)^n/(1-L))\rho(x_0, x_1)\end{aligned} \tag{6}$$

so far as $x_m\in F$. In particular, for $n=1$, we have

$$\rho(x_1, x_m)<(L/1-L)\rho(x_0, x_1).$$

Since x_0 is taken as assumed in assumption (ii) and (iii), it follows that x_m belongs to F. From (6) and that $L<1$, we see

$$\rho(x_n, x_m)\to 0 \text{ for } n\to\infty.$$

Hence, the sequence is a fundamental sequence in F.

But, since the set F is complete in R, the sequence converges to a point.

$$\lim_{n\to\infty} x_n = x.$$

And since the sphere S is closed, the limit point x belongs to the set F.

It may be seen that the limit is a fixed point of the operator $\varphi(x)$ in F. In fact, since the operator $\varphi(x)$ continuous,

$$\varphi(x) = \varphi(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} \varphi(x_n) = \lim_{n\to\infty} x_{n+1} = x.$$

It remains to prove the uniqueness of the fixed point of the operator $\varphi(x)$. In fact, if we assume the contrary and take two of the fixed points, x and y:

$$x = \varphi(x), \qquad y = \varphi(y),$$

it follows from the assumption (i), that

$$\rho(x, y) = \rho(\varphi(x), \varphi(y)) \leqq L\rho(x, y).$$

But since, $0 \leq L < 1$, $\rho(x, y) = 0$ i.e., $x = y$ which contradicts to the assumption.

Q. E. D.

Corollary

The distance between the fixed point of the operator $\varphi(x)$ and the n-th term of the sequence (i.e., n-th approximation) given by the recurrence relation (4) for any initial point x_0, for which the assumptions (ii) and (iii) hold, is evaluated by the relation

$$\rho(x_n, x) \leqq ((L)^n/1-L)\,\rho(x_0, x_1). \tag{7}$$

Proof

The relation (7) is readily seen by tending m to infinity in the relation (6).

Q. E. D.

Now let us define an algebraic operator which will enable us to treat a class of operations in a single manner.

Definition 2 (B-operator)

B is an operator which maps sets of finite numbers of non-negative real numbers, $\{a^i\}_{i=1}^N$ to a non-negative real number uniquely:

$$b = B(\{a^i\}_{i=1}^N) \qquad b \geqq 0, \qquad b: \text{real}, \qquad a^i \geqq 0, \qquad 0 \leqq N < \infty \tag{8}$$

for which following four relations hold.

$$B(B(\{a^i\}_{i=1}^N) \cup \{b^j\}_{j=1}^{N'}) = B(\{a^i\}_{i=1}^N \cup \{b^j\}_{j=1}^{N'}) \tag{9—i}$$

$$B(\{\alpha a^i\}_{i=1}^N) = \alpha B(\{a^i\}_{i=1}^N) \qquad \alpha \geq 0 \tag{9—ii}$$

$$B(\{a^i\}_{i=1}^N) \leqq B(\{a'^i\}_{i=1}^N) \text{ if } a^i \leqq a'^i,\ i=1,\dots, N \tag{9—iii}$$

$$B(\{a^i\}_{i=1}^N) \geqq a^i, \qquad i=1,\dots, N \tag{9—iv}$$

Example

Some of the most important B-operators are

$$\sum_{i=1}^{N} a^i, \qquad (v=1, \text{ in } (10\text{—iii})) \tag{10—i}$$

$$\max\{a^i\}_{i=1}^{N}, \qquad (v\to\infty \text{ in } (10\text{—iii})) \tag{10—ii}$$

$$\sum_{i=1}^{N} ((a^i)^v)^{1/v}. \qquad (0\leqq v<\infty). \tag{10—iii}$$

If more than two kinds of B-operators are necessary, we associate them with suffices v, v' etc. to distinguish, and denote by V the set of all possible indeces. Furthermore, in case of the B-operators given above, we associate the numerical values directly with the operators. Thus, for example,

$$B_1(\{a^i\}_{i=1}^{N}) = \sum_{i=1}^{N} a^i.$$

Some of the assertions as to the B-operators are given below.

Assertion 1

It follows from the assumption (9—iv) that the following two relations hold

$$B(\{a^i\}_{i=1}^{N}) < \varepsilon \Rightarrow a^i < \varepsilon, \quad \varepsilon > 0,$$
$$B(A \cup C) \geqq B(A).$$

Assertion 2

It follows from (9—iv) that

$$\max(\{a^i\}_{i=1}^{M}) \leqq B_v(\{a^i\}_{i=1}^{M}) \text{ for all } v \in V.$$

Assertion 3

It follows from the assumption (9—iv) that

$$B_v(\{a^ib^i\}_{i=1}^{M}) \leqq \max((\{a^i\}_{i=1}^{M}) B_v(\{b^i\}_{i=1}^{M}) \leqq B_v(\{a^i\}_{i=1}^{M}) B_v(\{b^i\}_{i=1}^{M}).$$

Moreover, Hölder's inequality can be written in terms of B_v operators as follows

$$B_1(\{a^ib^i\}_{i=1}^{M}) \leqq B_v(\{a^i\}_{i=1}^{M}) B_{\bar{v}}(\{b^i\}_{i=1}^{M}),$$

where

$$1/v + 1/\bar{v} = 1 \qquad v, \ \bar{v} \geqq 1.$$

In what follows, we refer to the operators B_v and $B_{\bar{v}}$ as complementary to each other.

Assertion 4

It follows from the assumptions (9—ii) and (9—iii), that if $a_1 \leqq b_1 + c_1$ and $a_2 \leqq b_2 + c_2$, we have the relation

$$B(a_1, a_2) \leqq B(b_1, b_2) + B(c_1, c_2).$$

In fact, by (9—ii)

$$B(a_1, a_2) = (a_1/b_1) B(b_1, b_1a_2/a_1) \leqq ((b_1+c_1)/b_1) B(b_1, b_1a_2/a_1).$$

By suitable re-numbering and re-naming, we may assume, without loss of generality, that either

$$a_2/a_1 \leqq b_2/b_1, \qquad a_2/a_1 \leqq c_2/c_1, \tag{a}$$

or

$$a_2/a_1 \leqq b_2/b_1, \qquad a_2/a_1 \geqq c_2/c_1, \qquad a_2/a_1 \leqq 1. \tag{b}$$

In the case (a), we see that

$$\begin{aligned}(b_1+c_1)/b_1)B(b_1, b_1a_2/a_1) &= B(b_1, b_1a_2/a_1)+B(c_1, a_2c_1/a_1)\\ &\leqq B(b_1, b_2)+B(c_1, c_2)\end{aligned}$$

by (9—ii) and (9—iii). In the case (b), on the other hand,

$$\begin{aligned}((b_1+c_1)/b_1)B(b_1, b_1a/a_1) &= B(b_1, b_1a_2/a_1)+(a_2/a_1)B(a_1c_1/a_2, c_1)\\ &\leqq B(b_1, b_2)+B(c_1, c_2).\end{aligned}$$

Hence, in either case the above relation holds.

Notation

For the sake of convenience in the use of B-operator, we adopt the notation

$$\begin{aligned}\phi_{ij} &= \text{null set} && i=j\\ &= \text{set of all non-negative real numbers} && i \neq j.\end{aligned}$$

Consider now, M complete sets $F^1, F^2, \ldots, F^M$ each of which belongs to each of the metric spaces $R^1(X^1, \rho^1), R^2(X^2, \rho^2), \ldots, R^M(X^M, \rho^M)$, respectively. Designate the product space of them by

$$\boldsymbol{R}(X) = R^1 \times R^2 \times \ldots \times R^M \tag{11}$$

and its element by

$$\boldsymbol{x} = (x^1, x^2, \ldots, x^M) \tag{12}$$

and the product set of $F^1, F^2, \ldots, F^M$ by

$$\boldsymbol{F} = F^1 \times F^2 \times \ldots \times F^M. \tag{13}$$

Definition 4 (Metrics on product space)

We define the distance on the product space $\boldsymbol{R}(X)$ by

$$\rho_\lambda = B_v(\{w^i \rho^i(x^i, y^i)\}_{i=1}^M), \quad v \in V, \quad 0 < w^i < \infty, \quad i=1, \ldots, M \tag{14}$$

where λ denotes the aggregate

$$\lambda = (v\,;\, w^1, w^2, \ldots, w^M), \quad v \in V, \quad 0 < w^i < \infty, \quad i=1, \ldots, M. \tag{15}$$

For definiteness, we call (14) λ-metric defined on the product space $\boldsymbol{R}(X)$ and designate by Λ the set of all possible λ.

Remark

In fact, the 'distance' ρ_λ given above fulfils the three conditions to be metric on the product space:

$$\rho_\lambda(\boldsymbol{x}, \boldsymbol{y}) = 0 \Longleftrightarrow \boldsymbol{x} = \boldsymbol{y} \tag{i}$$

$$\rho_\lambda(\boldsymbol{x}, \boldsymbol{y}) = \rho_\lambda(\boldsymbol{y}, \boldsymbol{x}) \quad \text{for any } \boldsymbol{x} \text{ and } \boldsymbol{y} \text{ in } \boldsymbol{R}(X) \tag{ii}$$

$$\rho_\lambda(\boldsymbol{x}, \boldsymbol{y}) \leqq \rho_\lambda(\boldsymbol{x}, \boldsymbol{z}) + \rho_\lambda(\boldsymbol{z}, \boldsymbol{y}) \quad \text{for any } \boldsymbol{x}, \boldsymbol{y} \text{ and } \boldsymbol{z} \text{ in } \boldsymbol{R}(X). \tag{iii}$$

Indeed, the first two may be readily seen by (9—ii), (9—iii) and the definition itself. The third, on the other hand, is proved by mathematical induction.

In fact, for $M=2$,

$$\rho_\lambda(\boldsymbol{x}, \boldsymbol{y}) = B_v(\{w^i \rho^i(x^i, y^i)\}_{i=1}^2).$$

But since,

$$w^i \rho^i(x^i, y^i) \leqq w^i \rho^i(x^i, z^i) + w^i \rho^i(z^i, y^i)$$

for $i=1, 2$, we have by Assertion 4 that

$$\begin{aligned} &B_v(\{w^i \rho^i(x^i, y^i)\}_{i=1}^2) \\ &\quad \leqq B_v(\{w^i \rho^i(x^i, z^i)\}_{i=1}^2) + B_v(\{w^i \rho^i(z^i, y^i)\}_{i=1}^2) \end{aligned}$$

which is re-written in the form

$$\rho_\lambda(\boldsymbol{x}, \boldsymbol{y}) \leqq \rho_\lambda(\boldsymbol{x}, \boldsymbol{z}) + \rho_\lambda(\boldsymbol{z}, \boldsymbol{y}).$$

Assume now that the relation

$$\begin{aligned} &B_v(\{w^i \rho^i(x^i, y^i)\}_{i=1}^k) \\ &\quad \leqq B_v(\{w^i \rho^i(x^i, z^i)\}_{i=1}^k) + B_v(\{w^i \rho^i(z^i, y^i)\}_{i=1}^k) \end{aligned}$$

holds for $k=1, 2, \ldots, M$. Then for $M+1$,

$$\begin{aligned} \rho(\boldsymbol{x}, \boldsymbol{y}) &= B_v(\{w^i \rho^i(x^i, y^i)\}_{i=1}^{M+1}) \\ &= B_v(\{w^i \rho^i(x^i, y^i)\}_{i=1}^M \cup \{w^{M+1} \rho^{M+1}(x^{M+1}, y^{M+1})\}) \\ &= B_v(B_v(\{w^i \rho^i(x^i, y^i)\}_{i=1}^M) \cup \{w^{M+1} \rho^{M+1}(x^{M+1}, y^{M+1})\}) \end{aligned}$$

by (9—i). But since,

$$\begin{aligned} &B_v(\{w^i \rho^i(x^i, y^i)\}_{i=1}^M) \\ &\quad \leqq B_v(\{w^i \rho^i(x^i, z^i)\}_{i=1}^M) + B_v(\{w^i \rho^i(y^i, z^i)\}_{i=1}^M) \end{aligned}$$

by inductive hypothesis and

$$\begin{aligned} &w^{M+1} \rho^{M+1}(x^{M+1}, y^{M+1}) \\ &\quad \leqq w^{M+1} \rho^{M+1}(x^{M+1}, z^{M+1}) + w^{M+1} \rho^{M+1}(z^{M+1}, y^{M+1}), \end{aligned}$$

we observe again by Assertion 4 that

$$\begin{aligned} &B_v(\{w^i \rho^i(x^i, y^i)\}_{i=1}^{M+1}) \\ &\quad \leqq B_v(B_v(\{w^i \rho^i(x^i, z^i)\}_{i=1}^M) \cup \{w^{M+1} \rho^{M+1}(x^{M+1}, z^{M+1})\}) \\ &\quad + B_v(B_v(\{w^i \rho^i(z^i, y^i)\}_{i=1}^M) \cup (w^{M+1} \rho^{M+1}(z^{M+1}, y^{M+1})\}), \end{aligned}$$

which is re-written in the form

$$\rho(\boldsymbol{x}, \boldsymbol{y}) \leqq \rho(\boldsymbol{x}, \boldsymbol{z}) + \rho(\boldsymbol{z}, \boldsymbol{y}).$$

Theorem 2 (Completeness of product set)

The product set $F=F^1\times F^2\times\ldots\times F^M$ is complete as to λ-metric ($\lambda\in\Lambda$) defined on the product space $\boldsymbol{R}(\boldsymbol{X})$.

Proof

Let $\{\boldsymbol{x}_n\}$ be a fundamental sequence in the product set $\boldsymbol{F}$. For any given $\varepsilon>0$, there exists $N(\varepsilon)$, such that, for any integers $s, r>N(\varepsilon)$,

$$B_\theta(\{w^i\rho^i(x_s^i, x_r^i)\}_{i=1}^M)<\varepsilon.$$

It follows from the last assumption on B-operator (9—iv) (cf. Assertion 1) that

$$w^i\rho^i(x_s^i, x_r^i)<\varepsilon$$

then,

$$\rho^i(x_s^i, x_r^i)<(\varepsilon/\min_i\{w^i\})$$

for $i=1, 2,\ldots, M$ and for any integers $s, r>N(\varepsilon)$. Hence, the sequences $\{x_n^i\}$ are fundamental in the complete sets F^i respectively. Thus we have obtained the convergence of $\{x_n^i\}$:

$$\lim_{n\to\infty} x_n^i = x^i, \qquad i=1, 2,\ldots, M$$

from which we conclude the convergence of the fundamental sequence $\{\boldsymbol{x}_n\}$ on the product set $\boldsymbol{F}\subset\boldsymbol{R}(\boldsymbol{X}, \rho_\lambda)$, $\lambda\in\Lambda$, which means completeness of the product set $\boldsymbol{F}$ as to λ-metric defined on the product space $\boldsymbol{R}(\boldsymbol{X})$.

Q. E. D.

III. Simple process and generalized Seidel process

In this section, definitions of simple process and generalized Seidel process will be given and some conditions of convergence will be examined under the most general assumption as to the system of recurrence relations defined on a product set $\boldsymbol{F}$ of complete subsets F^i of metric spaces which, in turn, is complete as to λ-metric ($\lambda\in\Lambda$).

Consider an operator

$$\boldsymbol{y}=\boldsymbol{\varphi}(\boldsymbol{x}) \tag{1}$$

which maps the product set $\boldsymbol{F}$ into $\boldsymbol{R}(\boldsymbol{X}, \rho_\lambda)$ where

$$\boldsymbol{R}(\boldsymbol{X},\rho_\lambda)=R^1(X^1, \rho^1)\times R^2(X^2, \rho^2)\times\ldots\times R^M(X^M, \rho^M) \tag{2}$$

with element

$$\boldsymbol{x}=(x^1, x^2, \ldots, x^M). \tag{3}$$

Thus, the operator (1) may be rewritten in the form

$$\begin{aligned} y^1&=\varphi^1(x^1, x^2, \ldots, x^M)\\ &\cdots\cdots\cdots\\ y^i&=\varphi^i(x^1, x^2, \ldots, x^M)\\ &\cdots\cdots\cdots\\ y^M&=\varphi^M(x^1, x^2, \ldots, x^M), \end{aligned} \tag{4}$$

where the operator $\varphi^i(X)$ maps $\boldsymbol{F}$ into F^i respectively.

Definition 1 (Simple process and simple operator system)

We call the sequence

$$\boldsymbol{x}_0,\ \boldsymbol{x}_1,\ \dots,\ \boldsymbol{x}_n,\ \dots \tag{5}$$

defined by the recurrence relation

$$\boldsymbol{x}_{n+1}=\boldsymbol{\varphi}(\boldsymbol{x}_n). \tag{6}$$

a *simple process* defined by the operator (1), so far as $\boldsymbol{x}_n$ remains in $\boldsymbol{F}$. And we call the operator (1) a *simple operator system.*

Now, defining the metrics on the product space $\boldsymbol{R}(X)$ as in the previous section, let us assume that there exist Lipschitz constants L_λ^i associated with the individual operators $\varphi^i(\boldsymbol{x})\in R^i(X^i,\ \rho^i)$, $\boldsymbol{x}\in\boldsymbol{F}$, $i=1, 2, \dots, M$, $\lambda\in\Lambda$:

$$\begin{aligned}
&\rho^1(y^1,\ y'^1)\leqq L_\lambda^1\rho(\boldsymbol{x},\ \boldsymbol{x}')\\
&\dots\dots\dots\\
&\rho^i(y^i,\ y'^1)\leqq L_\lambda^i\rho(\boldsymbol{x},\ \boldsymbol{x}')\\
&\dots\dots\dots\\
&\rho^M(y^M,\ y'^M)\leqq L_\lambda^M\rho_\lambda(\boldsymbol{x},\ \boldsymbol{x}').
\end{aligned} \tag{7}$$

With this assumption, we may have the following theorem, which might be regarded as an extension of analogous propositions for recurrence relations given by matrices on a finite dimensional space ⟨8⟩, ⟨1⟩.

Theorem 1

The Lipschitz constant of the operator (1) defined on a complete product set $\boldsymbol{F}$ is evaluated by

$$L_\lambda=B_v(\{w^iL_\lambda^i\}_{i=1}^M) \tag{8}$$

where, v is the index of B-operator corresponding to λ.

If the relation

$$B_v(\{w^iL_\lambda^i\}_{i=1}^M)<1 \tag{9}$$

holds and if there exists a point $\boldsymbol{x}_0\in\boldsymbol{F}$ for which the assumptions

$$\boldsymbol{x}_1=\boldsymbol{\varphi}(\boldsymbol{x}_0)\in\boldsymbol{F} \tag{i}$$

$$S\{\boldsymbol{u}:\ \rho_\lambda(\boldsymbol{u},\ \boldsymbol{x}_1)\leqq(L/1-L)\,\rho_\lambda(\boldsymbol{x}_1,\ \boldsymbol{x}_0)\}\subset\boldsymbol{F} \tag{ii}$$

hold, we may have a process defined by the recurrence relation (6), with the initial point $\boldsymbol{x}_0$ given above, converging to the unique fixed point of the operator $\boldsymbol{\varphi}(\boldsymbol{x})$, $\boldsymbol{x}\in\boldsymbol{F}$.

Proof

It follows from the definition of λ-metric, (9—iii) and (9—ii), **II.** that

$$\begin{aligned}\rho_\lambda(\boldsymbol{y}, \boldsymbol{y}') &= B(\{w^i \rho^i(y^i, y'^i)\}_{i=1}^M)\\ &\leqq B(\{w^i L_\lambda^i\ \rho_\lambda(\boldsymbol{x}, \boldsymbol{x}')\}_{i=1}^M)\\ &= B(\{w^i L^i\}_{i=1}^M)\ \rho_\lambda(\boldsymbol{x}, \boldsymbol{x}')\end{aligned}$$

where,

$$\boldsymbol{y} = \varphi(\boldsymbol{x}), \qquad \boldsymbol{y}' = \varphi(\boldsymbol{x}'), \qquad \boldsymbol{x}, \boldsymbol{x}' \in \boldsymbol{F}.$$

Hence the Lipschitz constant of the operator (1) defined on $\boldsymbol{F}$ is evaluated by (8). With this, the remainder of the theorem follows from Theorem 1, **II.**

Q. E. D.

Example 1

Some examples of Lipschitz constant associated with simple process are given below :

Metric : $\rho_\lambda(\boldsymbol{x}, \boldsymbol{y})$ $\lambda = (v\,;\ w^1, \ldots, w^M)$ *)	Lipschitz constant
(10) $\lambda = (\infty\,;\ 1, \ldots, 1)$ i.e. $\rho_\lambda(\boldsymbol{x}, \boldsymbol{y}) = \max_i(\rho^i(x^i, y^i))$	$\max\{(L_\lambda^i)\}_{i=1}^M$
(11) $\lambda = (v = p\,;\ 1, \ldots, 1)$ i.e. $\rho_\lambda(\boldsymbol{x}, \boldsymbol{y}) = (\sum_i (\rho^i(x^i, y^i))^p)^{1/p}$ where $1 \leqq p < \infty$	$(\sum_{i=1}^M (L^i)^p)^{1/p}$

Example 2

Consider now, a system of M linear operators defined on M-dimensional space $R_M(X, \rho_\lambda)$, $\lambda \in \Lambda$:

$$\begin{aligned}&y^1 = \varphi^1(\boldsymbol{x}) = a_{11}x^1 + a_{12}x^2 + \ldots + a_{1M}x^M\\ &\cdots\cdots\cdots\\ &y^i = \varphi^i(\boldsymbol{x}) = a_{i1}x^1 + a_{i2}x^2 + \ldots + a_{iM}x^M\\ &\cdots\cdots\cdots\\ &y^M = \varphi^M(\boldsymbol{x}) = a_{M1}x^1 + a_{M2}x^2 + \ldots + a_{MM}x^M\end{aligned} \tag{13}$$

or in matrix form,

$$\begin{pmatrix} y^1 \\ \vdots \\ y^i \\ \vdots \\ y^M \end{pmatrix} = \begin{pmatrix} a_{11}, a_{12}, \ldots, a_{1M} \\ \cdots\cdots\cdots \\ a_{i1}, a_{i2}, \ldots, a_{iM} \\ \cdots\cdots\cdots \\ a_{M1}, a_{M2}, \ldots, a_{MM} \end{pmatrix} \cdot \begin{pmatrix} x^1 \\ \vdots \\ x^i \\ \vdots \\ x^M \end{pmatrix}. \tag{14}$$

In this case, the conditions (ii) and (iii) in Theorem 1, **II.** are considered to be

*) cf. (10—ii) and (10—iii) **II.** (10—i) **II.** corresponds to the case of $p=1$, in (11).

satisfied, since the operator $\varphi(x)$ is defined on the whole space $R_M(X, \rho_\lambda)$. Thus the conditions of convergence given in Theorem 1 reduce to the condition that the Lipschitz constant of the operator $\varphi(x)$ is to be less than unity. The Lipschitz constants associated with the individual operators $\varphi^i(x)$ and the operator $\varphi(x)$ are evaluated as follows:

Metric ρ_λ ; $\lambda=(v\,;w^1,\dots,w^M)$	Lipschitz constants of individual operators $\varphi^i(x)$	Lipschitz constant of the operator $\varphi(x)$	
$\lambda=(\infty\,;1,\dots,1)$	$\sum_{j=1}^{M} \|a_{ij}\|$ $i=1,\dots,M$	$\max(\{\sum_{j=1}^{M} \|a_{ij}\|\}_{i=1}^{M})$	(15)
$\lambda=(v=1\,;1,\dots,1)$	$\max(\{\|a_{ij}\|\}_{j=1}^{M})$ $i=1,\dots M$	$\sum_{i=1}^{M}(\max_j\{\|a_{ij}\|\}_{j=1}^{M})$	(16)
$\lambda=(v=p\,;1,\dots,1)$ $p>1$	$(\sum_{j=1}^{M}\|a_{ij}\|^{p/p-1})^{p-1/p}$ *) $i=1,\dots,M$	$(\sum_{i=1}^{M}(\sum_{j=1}^{M} a_{ij}{}^{p/p-1})^{p-1})^{1/p}$	(17)

It may be seen that the condition that Lipschitz constant (15) is less than unity is so called *sum of row* criterion.** And the condition that the Lipschitz constant (17) is less than unity in case $p=2$, is so called Schmidt's criterion ⟨1⟩, ⟨8⟩.

Now, let us define *generalized Seidel operator system* and *generalized* Seidel process.

Definition 2 (Generalized Seidel operator system)

A *generalized Seidel operator system*

$$y=\phi(x), \quad x\in F\subset R(X), \quad \phi(x)\in R(X) \tag{18}$$

Corresponding to the operator (1) is the operator given in the form (Here, also, F is to be defined as a product set):

---

*) These Lipschitz constants associated with the individual operators are obtained readily employing Hölder's inequality (Assertion 3, II.). In fact, if we put

$$y^i = a_{i1}x^1 + a_{i2}x^2 + \dots + a_{iM}x^M,$$
$$y'^i = a_{i1}x'^1 + a_{i2}x'^2 + \dots + a_{iM}x^M$$

then we have

$$|y^i - y'^i| \leqq \sum_{j=1}^{M} |a_{ij}(x^j - x'^j)| \leqq (\sum_{j=1}^{M}|a_{ij}|^{p/p-1})^{p-1/p}(\sum_{j=1}^{M}|x^j - x'^j|^p)^{1/p}$$

In case $p=1$, the Lipschitz constant can be obtained either directly or as the limiting case:

$$(\sum_{i=1}^{M}|a_{ij}|^{p/p-1})^{p-1/p} \to \max(\{|a_{ij}'\}_{j=1}^{M})$$
$$p\to 1.$$

**) The *sum of column* criterion can not be obtained from the arguments of this section. It will be seen in the discussion of VI..

$$
\begin{aligned}
&y^1=\varphi^1(x^1,\ x^2,\ \dots,\ x^M)\\
&y^2=\varphi^2(y^1,\ x^2,\ \dots,\ x^M)\\
&\dots\dots\dots\\
&y^i=\varphi^i(y^1,\ y^2,\ \dots,\ y^{i-1},\ x^i,\ \dots,\ x^M)\\
&\dots\dots\dots\\
&y^M=\varphi^M(y^1,\ y^2,\ \dots,\ y^{M-1},\ x^M)
\end{aligned}
\tag{19}
$$

or, if we denote by ${}^i\boldsymbol{x}$ and ${}_i\boldsymbol{y}$ the vectors

$$
\begin{aligned}
&{}^i\boldsymbol{x}=(x^i,\ x^{i+1},\ \dots,\ x^M)\\
&{}_i\boldsymbol{y}=(y^i,\ y^2,\ \dots,\ y^i),
\end{aligned}
\tag{20}
$$

then, (18) will be re-written in the form :

$$
\begin{aligned}
&y^1=\varphi^1({}^1\boldsymbol{x})\\
&y^2=\varphi^2({}_1\boldsymbol{y},\ {}^2\boldsymbol{x})\\
&\dots\\
&y^i=\varphi^i({}_{i-1}\boldsymbol{y},\ {}^i\boldsymbol{x})\\
&\dots\\
&y^M=\varphi^M({}_{M-1}\boldsymbol{y},\ {}^M\boldsymbol{x}).
\end{aligned}
\tag{21}
$$

Definition 3 (Generalized Seidel process)

A *generalized Seidel process* is a process defined by the recurrence relation given by a generalized Seidel operator system

$$\boldsymbol{x}_{n+1}=\boldsymbol{\phi}(\boldsymbol{x}_n) \qquad \boldsymbol{x}_0\in\boldsymbol{F} \tag{22}$$

so far as $\boldsymbol{x}_n$ remain in $\boldsymbol{F}$.

In what follows, we will omit, in most cases, the term *generalized*, if there would be no fear of confusion. Instead, we will call in some cases the Seidel operator system, as to whose individual operators, only the Lipschitz constants are known, *a Seidel operator system of the general kind* in order to distinguish them from the other kinds of Seidel operator system which will be described in subsequent sections.

Now, let us give the evaluation of Lipschitz constant associated with Seidel operator system of the general kind.

Theorem 2

The Lipschitz constant $\mathscr{L}_\lambda$ of the Seidel operator system of the general kind (18) (also (19) or (21)) defined on a complete product set $\boldsymbol{F}\subset\boldsymbol{R}(\boldsymbol{X},\ \rho_\lambda)$ is evaluated as follows

$$\mathscr{L}_\lambda=B_v(\{w^iC^i\}_{i=1}^M) \qquad \lambda=(v\ ;\ w^1,\ w^2,\ \dots,\ w^M) \tag{24}$$

where

$$C^i=(L_\lambda^i)B_v(\{w^r C^r\}_{r=1}^{i-1}\cup\{1\}), \qquad C^1=B_v(L_\lambda^1\}) \tag{25}$$

so that if we assume that

$$\text{(i)} \qquad \mathscr{L}_\lambda=B_v(\{w^i C^i\}_{i=1}^M)<1, \qquad \lambda=(v\ ;\ w^1,\ \dots,\ w^M), \tag{26—i}$$

(ii) there exists $\boldsymbol{x}_0$ such that $\boldsymbol{x}_1=\boldsymbol{\phi}(\boldsymbol{x}_0)$ belongs to $\boldsymbol{F}$, (26—ii)

(iii) the sphere $S\{\boldsymbol{u}:\rho_\lambda(\boldsymbol{u},\boldsymbol{x}_1)\leqq(\mathscr{L}_\lambda/1-\mathscr{L}_\lambda)\,\rho_\lambda(\boldsymbol{x}_0,\boldsymbol{x}_1)\}$ is contained in $\boldsymbol{F}$, (26—iii)

then the Seidel process given by the operator (18) is convergent to its unique fixed point in $\boldsymbol{F}$.*)

Proof

From the definition of the Seidel operator (cf (19) or (21)) it follows for any $\boldsymbol{x}, \boldsymbol{x}'\in\boldsymbol{F}$ that

$$\begin{aligned}
&y^1=\varphi^1(\boldsymbol{x})\\
&y^2=\varphi^2(\varphi^1(\boldsymbol{x}),{}^2\boldsymbol{x})\\
&y^3=\varphi^3(\varphi^1(\boldsymbol{x}),\ \varphi^2(\varphi^1(\boldsymbol{x}),\ {}^2\boldsymbol{x}),\ {}^3\boldsymbol{x})\\
&\cdots\cdots\cdots
\end{aligned}\tag{27}$$

and

$$\begin{aligned}
&y'^1=\varphi^1(\boldsymbol{x}')\\
&y'^2=\varphi^2(\varphi^1(\boldsymbol{x}'),\ {}^2\boldsymbol{x}')\\
&y'^3=\varphi^3(\varphi^1(\boldsymbol{x}'),\ \varphi^2(\varphi^1(\boldsymbol{x}'),\ {}^2\boldsymbol{x}'),\ {}^3\boldsymbol{x}'),\\
&\cdots\cdots\cdots\ .
\end{aligned}\tag{28}$$

Now, let us show that the relations

$$\rho^i(y^i,\ y'^i)\leqq(C^i)\,\rho_\lambda(\boldsymbol{x},\ \boldsymbol{x}')\tag{29}$$

hold for $i=1, 2, \dots, M$ by means of mathematical induction.

For $i=1$, it is obvious from (9—iv), **II.** that (29) holds. Assume now that the relations (29) holds for $i=1, \dots, r-1$ then for $i=r$ we have by assumptions as to B-operator

$$\begin{aligned}
\rho^r(y^r,\ y'^r)&\leqq L_\lambda^r\rho_\lambda((_{r-1}\boldsymbol{y},\ {}^r\boldsymbol{x}),\ (_{r-1}\boldsymbol{y}',\ {}^r\boldsymbol{x}'))\\
&=L_\lambda^r\,B_v(\{w^i\rho^i(y^i,\ y'^i)\}_{i=1}^{r-1}\cup\{w^i\rho^i(x^i,\ x'^i)\}_{i=r}^{M})\\
&\leqq L_\lambda^r\,B_v(\{w^iC^i\rho_\lambda(\boldsymbol{x},\ \boldsymbol{x}')\}_{i=1}^{r-1}\cup\{B_v(\{w^i\rho^i(x^i,\ x'^i)\}_{i=r}^{M})\})\\
&\leqq L_\lambda^r\,B_v(\{w^iC^i\rho_\lambda(\boldsymbol{x},\ \boldsymbol{x}')\}_{i=1}^{r-1}\cup\{\rho_\lambda(\boldsymbol{x},\ \boldsymbol{x}')\})\\
&=L_\lambda^r\,B_v(\{w^iC^i\}_{i=1}^{r-1}\cup\{1\})\cdot\rho_\lambda(\boldsymbol{x},\ \boldsymbol{x}')\\
&=C^r\rho_\lambda(\boldsymbol{x},\ \boldsymbol{x}').
\end{aligned}$$

Thus we have (29) for $i=1, 2, \dots, M$. These relations and Theorem 1 show that the Seidel operator (18) has a Lipschitz constant of the form

$$\mathscr{L}_\lambda=B_v(\{w^iC^i\}_{i=1}^{M}).\tag{24—bis}$$

The remainder of the theorem follows from Theorem 1, **II.**

Q. E. D.

*) It may be readily observed that a simple process and the corresponding Seidel process have the same limit point, so far as their convergence are asserted by Theorems 1 and 2. Indeed, the relation $\boldsymbol{x}=\varphi(\boldsymbol{x})$ holds at either of their limit points. And the uniqueness of such fixed points is asserted by the either of the theorems.

Remark 1

The Lipschitz constant $\mathscr{L}_\lambda$ given in equation (24) is evaluated in terms of the Lipschitz constants $L_\lambda^i(\lambda\in\Lambda)$ themselves in the form

$$\begin{aligned}\mathscr{L}_\lambda=B_v(\{w^iC^i\}_{i=1}^M)=B_v(\{w^iL_\lambda^i\}_{i=1}^M \cup \{w^{i_1}L_\lambda^{i_1}w^{i_2}L_\lambda^{i_2}\}_{1\leqq i_1<i_2\leqq M}\\ \cup\{w^{i_1}L_\lambda^{i_1}w^{i_2}L_\lambda^{i_2}w^{i_3}L_\lambda^{i_3}\}_{1\leqq i_1<i_2<i_3\leqq M}\cup\cdots\\ \cup\{w^1L_\lambda^1w^2L_\lambda^2\ldots w^ML_\lambda^M\}).\end{aligned}\tag{30}$$

This relation follows from the relation (24) and

$$\begin{aligned}w^iC^i=B_v(\{w^iL_\lambda^i\}\cup\{w^iL_\lambda^iw^rL_\lambda^r\}_{r=1}^{i-1}\cup\{w^iL_\lambda^iw^{r_1}L_\lambda^{r_1}w^{r_2}L_\lambda^{r_2}\}_{1\leqq r_1<r_2\leqq i-1}\\ \cup\{w^iL_\lambda^iw^{r_1}L_\lambda^{r_1}w^{r_2}L_\lambda^{r_2}w^{r_3}L_\lambda^{r_3}\}_{1\leqq r_1<r_2<r_3\leqq i-1}\cup\cdots\\ \cup\{w^1L_\lambda^1w^2L_\lambda^2\ldots w^iL_\lambda^i\})\qquad i=1,2,\ldots,M.\end{aligned}\tag{31}$$

The relations (30) and (31) can be shown simultaneously by means of mathematical induction as to M. For $M=1$, it is obvious that the relations (30) and (31) hold:

$$B_v(\{w^1C^1\})=B_v(\{w^1L_\lambda^1\})$$
$$w^1C^1=B_v(\{w^1L_\lambda^1\}).$$

If we now, assume that the relation (30) and (31) hold for $M=1,\ldots,s-1$, then for $M=s$, from (25) it follows that,

$$\begin{aligned}w^sC^s&=w^sL_\lambda^sB_v(\{w^iC^i\}_{i=1}^{s-1}\cup\{1\})\\ &=w^sL_\lambda^sB_v(B_v(\{w^iC^i\}_{i=1}^{s-1})\cup\{1\})\\ &=B_v(w^sL_\lambda^sB_v(\{w^iC^i\}_{i=1}^{s-1})\cup\{w^sL_\lambda^s\})\\ &=B_v(\{w^sL_\lambda^s\}\cup\{w^sL_\lambda^sw^iL^i\}_{i=1}^{s-1}\cup\{w^sL_\lambda^sw^{i_1}L_\lambda^{i_1}w^{i_2}L_\lambda^{i_2}\}_{1\leqq i_1<i_2\leqq s-1}\\ &\cup\ldots\cup\{w^1L_\lambda^1w^2L_\lambda^2\ldots w^sL_\lambda^s\}).\end{aligned}$$

This shows that the relation (31) holds also for $M=s$. On the other hand, from this we have

$$\begin{aligned}B_v(\{w^iC^i\}_{i=1}^s)&=B_v(B_v(\{w^iC^i\}_{i=1}^{s-1})\cup\{w^sC^s\})\\ &=B_v(\{w^iL_\lambda^i\}_{i=1}^{s-1}\cup\{w^{i_1}L_\lambda^{i_1}w^{i_2}L_\lambda^{i_2}\}_{1\leqq i_1<i_2\leqq s-1}\\ &\cup\ldots\cup\{w^1L_\lambda^1w^2L_\lambda^2\ldots w^{s-1}L_\lambda^{s-1}\}\cup\{w^sL_\lambda^s\}\\ &\cup\{w^sL_\lambda^sw^iL_\lambda^i\}_{i=1}^{s-1}\cup\{w^sL_\lambda^sw^{i_1}L_\lambda^{i_1}w^{i_2}L_\lambda^{i_2}\}_{1\leqq i_1<i_2\leqq s-1}\\ &\cup\ldots\cup\{w^1L_\lambda^1w^2L_\lambda^2\ldots w^sL_\lambda^s\})\\ &=B_v(\{w^iL_\lambda^i\}_{i=1}^s\cup\{w^{i_1}L_\lambda^{i_1}w^{i_2}L_\lambda^{i_2}\}_{1\leqq i_1<i_2\leqq s}\\ &\cup\ldots\cup\{w^1L_\lambda^1w^2L_\lambda^2\ldots w^sL_\lambda^s\}).\end{aligned}$$

Thus, we have obtained the relation (30) for $M=s$, which shows that the relation (30) hold for any $M>1$. Some examples of the relation (30) are given below.

$$\lambda=(v=\infty\,;\,1,\ldots,1)\ \text{ i.e. }\ \rho_\lambda(x,y)=\max_i(\rho^i(x^i,y^i))$$

$$\mathscr{L}_\lambda=\max(\{L_\lambda^{i_1}\},\{(L_\lambda^{i_1}L_\lambda^{i_2})\},\ldots,\{(L_\lambda^{i_1}L_\lambda^{i_2}\ldots L_\lambda^{i_r})\},\ldots(L_\lambda^1L_\lambda^2\ldots L_\lambda^M))\tag{32}$$
$$i_s=1,\ldots,M\,;\quad i_r<i_s\ \text{ if }\ r<s$$

$$\lambda=(v=p\,;\,1,\ldots,1)\ \text{ i.e. }\ \rho_\lambda(x,y)=(\sum_i(\rho^i(x^i,y^i))^p)^{1/p},\ p>1$$

$$\mathscr{L}_\lambda=(\sum_{i=1}^{M}(L_\lambda^i)^p+\sum_{i_1<i_2}^{M}(L_\lambda^{i_1}L_\lambda^{i_2})^p+\dots+\sum_{i_1<i_2<\dots<i_r}^{M}(L_\lambda^{i_1}L_\lambda^{i_2}\dots L_\lambda^{i_r})^p+\dots$$
$$+(L_\lambda^1 L_\lambda^2\dots L_\lambda^M)^p)^{1/p}. \qquad (33)$$

Remark 2

Comparing the Lipschitz constant evaluated by (8) of Theorem 1 of a simple operator system and that evaluated by (24) of Theorem 2 of the corresponding Seidel operator system of the general kind, we see that

$$L_\lambda \leqq \mathscr{L}_\lambda.$$

In fact, we see that

$$w^i L^i \leqq w^i C^i \qquad i=1, \dots, M$$

from the recurrence relation giving C^i, from which above relation follows directly.

It seems that this assertion contradicts to our empirical knowledge that in most cases, the sequence given by Seidel process converges faster than the sequence given by the corresponding simple operator. But, so far as our information is limited to Lipschitz constants of the individual operators constituting the system, it is not possible to evaluate the Lipschitz constant of the Seidel operator system so as to be less than that of the corresponding simple operator system. This, however, indicates the necessity to investigate to utilize other forms of information about the operator system. In what follows, various methods are evolved to utilize various informations about the operator system in order to evaluate the Lipschitz constant of the Seidel operator system as small (i.e precise) as possible.

The remainder of the section will be devoted to the evaluation of the Lipschitz constant associated with some particular type of the Seidel operator which might be considered to be important in application.

Consider a class of Seidel operators which have the following form.

$$\begin{aligned}
&y^1=\varphi^1(x^2, \dots, x^M)\\
&y^2=\varphi^2(y^1, x^2, \dots, x^M)\\
&\dots\dots\dots\\
&y^i=\varphi^i(y^1, \dots, y^{i-1}, x^i, \dots, x^M) \qquad (34)\\
&\dots\dots\dots\\
&y^M=\varphi^M(y^1, \dots\ y^{M-1}, x^M),
\end{aligned}$$

i.e. the first operator does not include x^1 in its arguments.*)

In this case, however, it is possible to consider a new Seidel operator which maps

*) In case, some x^i beside x^1 lacks together in both 1 st and i th relations in (34), we may reduce it to the form of (34) by reordering the i-th element and the i-th relation so as to be the 2 nd element and the 2-nd relation and putting the pair (x^1, x^i) the first of the renumbered system of elements with a suitable metrization of the product space $R^1(X^1, \rho^1)\times R^i(X^i, \rho^i)$.

a produce set $F' \subset R^2 \times R^3 \times \ldots \times R^M$ into the space $R^2 \times R^3 \times \ldots \times R^M$ in the form

$$
\begin{aligned}
&y^2 = \varphi^2(\varphi^1({}^2\boldsymbol{x}),\ {}^2\boldsymbol{x}) \\
&y^3 = \varphi^3(\varphi^1({}^2\boldsymbol{x}),\ y^2,\ {}^3\boldsymbol{x}) \\
&\ldots\ldots\ldots \\
&y^i = \varphi^i(\varphi^1({}^2\boldsymbol{x}),\ y^2,\ y^3,\ \ldots,\ y^{i-1},\ {}^i\boldsymbol{x}) \\
&\ldots\ldots\ldots \\
&y^M = \varphi^M(\varphi^1({}^2\boldsymbol{x}),\ y^2,\ y^3,\ \ldots,\ y^{M-1},\ x^M).
\end{aligned}
\tag{35}
$$

If we define metric λ on the product space $R^2 \times R^3 \times \ldots \times R^M$, corresponding to the metric λ defined on the product space $R^1 \times R^2 \times R^3 \times \ldots \times R^M$, $\lambda \in \Lambda$ we may evaluate the Lipschitz constant of the Seidel operator of this type employing Theorem 2 and the remark given above.

Corollary of Theorem 2

There are two methods to evaluate the Lipschitz constant of the Seidel operator system (35) defined on a product set $F' \subset R^2 \times R^3 \times R^M$:

(*a*) Since the formal evaluation of the Lipschitz constants of the operators constituting the system is

$$(L^i) B_v(1,\ w^1 L^i), \qquad i = 2, 3, \ldots, M \tag{36}$$

we may substitute these values into (24) to obtain the Lipschcitz constant of the system, which we denote $\mathscr{L}^*$.

(*b*) Considered as a simple operator system, we have

$$\rho^i(y^i,\ y'^i) \leqq C^i \rho_\lambda(\boldsymbol{x},\ \boldsymbol{x}') = C^i B_v(\{\rho^i(x^i,\ x'^i)\}_{i=2}^M), \tag{37}$$
$$i = 2,\ 3,\ \ldots,\ M$$

where C^i is defined by recurrence relation

$$C^i = L^i B_v(\{w^r C^r\}_{r=1}^{i-1},\ 1) \tag{38}$$

with

$$C^1 = L^1.$$

Substituting these C^i, $i = 2, 3, \ldots, M$ into (8), we obtain the Lipschitz constant of the system (35) as

$$\mathscr{L}^{**} = B_v(\{w^i C^i\}_{i=2}^M). \tag{39}$$

Moreover, we see that

$$\mathscr{L}^{**} \leqq \mathscr{L}^*. \tag{40}$$

Proof

(*a*) Considering the simple operator system corresponding to (35), the Lipschitz constant of the *i*-th relation is evaluated as follows. Putting

$$
\begin{aligned}
&y^i = \varphi^i(\varphi^1({}^2\boldsymbol{x}),\ {}^2\boldsymbol{x}) \\
&y'^i = \varphi^i(\varphi^2({}^2\boldsymbol{x}'),\ {}^2\boldsymbol{x}'), \qquad i = 2,\ 3,\ \ldots,\ M
\end{aligned}
$$

we have,

$$\begin{aligned}\rho^i(y^i, y'^i) &\leqq L_\lambda^i B_v(\{w^1\rho^1(\varphi^1({}^2x'), \varphi^1({}^2x'))\} \cup \{w^i\rho^i(x^i, x'^i)\}_{i=2}^M) \\ &\leqq L_\lambda^i B_v(\{\rho_\lambda({}^2x, {}^2x')\} \cup \{w^1 L_\lambda^1 \rho({}^2x, {}^2x')\})^{*)} \\ &= L_\lambda^i B_v(\{1\} \cup \{w^1 L_\lambda^1\})\, \rho_\lambda({}^2x, {}^2x).\end{aligned}$$

Thus we have obtained (36).

Now, substituting (36) into (30) we obtain

$$\begin{aligned}\mathscr{L}_\lambda^* &= B_v(\{w^i C^i\}_{i=2}^M) \\ &= B_v(\{w^1 L_\lambda^i B_v(1, w^1 L_{\lambda'}^1)\}_{i=2}^M \\ &\cup \{w^{i_1} L_\lambda^{i_1} B_v(1, w^1 L_{\lambda'}^1) w^{i_2} L_\lambda^{i_2} B_v(1, w^1 L_{\lambda'}^1)\}_{2 \leqq i_1 < i_2 \leqq M} \\ &\cup \dots \cup \{w^2 L_\lambda^2 B_v(1, w^1 L_{\lambda'}^1) w^3 L_\lambda^3 B_v(1, w^1 L_{\lambda'}^1) \dots w^M L_\lambda^M B_v(1, w^1 L_{\lambda'}^1)\}).\end{aligned} \tag{41}$$

Considering here, the binomial expansion of B-operator,**) we have

$$\begin{aligned}\mathscr{L}_\lambda^* = B_v(&\{w^i L_\lambda^i\}_{i=2}^M \cup \{w^i L_\lambda^i w^1 L_\lambda^1\}_{i=2}^M \cup \{w^{i_1} L_\lambda^{i_1} w^{i_2} L_\lambda^{i_2}\}_{2 \leqq i_1 < i_2 \leqq M} \\ &\cup \{w^1 L_\lambda^1\}\, w^{i_1} L_\lambda^{i_1} w^{i_2} L_\lambda^{i_2}\}_{2 \leqq i_1 < i_2 \leqq M} \cup \dots \\ &\cup \{w^1 L_\lambda^1 w^2 L_\lambda^2 \dots w^M L_\lambda^M\} \cup \dots \cup \{(w^1 L_\lambda^1)^{M-2} w^2 L_\lambda^2 w^3 L_\lambda^3 \dots w^M L_\lambda^M\}).\end{aligned} \tag{42}$$

---

*) The Lipschitz constant L_λ^1 must be defined as a positive number which satisfies the relation

$$\rho^1(y^1, y'^1) \leqq L_{\lambda'}^1 \rho_{\lambda'}({}^2x, {}^2x') = L_{\lambda'}^1 B_v(\{w^i \rho^i(x^i, x'^i)\}_{i=2}^M).$$

$$\lambda = (v\,;\, w^1, w^2, \dots, w^M) \qquad \lambda' = (v\,;\, w^2, \dots, w^M)$$

If there should be no fear of confusion we would not distinguish λ and λ' later.

**) Binomial expansion as to B-operator

$$(\#)\quad B((B(a,b))^n = B(a^n, \underbrace{a^{n-1}b, \dots, a^{n-1}b}_{\binom{n}{1}}, \dots, \underbrace{a^{n-r}b^r, \dots, a^{n-r}b^r}_{\binom{n}{r}}, \dots, b^n)$$

$n=1, 2\dots$

can be shown by mathematical induction. It is obvious that the retation (#) holds for $n=1$. Assume now, that the relations (#) hold for $n=1, 2, \dots, N$. Then for $n=N$, we have

$$\begin{aligned}&B((B(a,b))^N) = B(B(a,b)(B(a,b))^{N-1}) \\ &= B(B(a,b)\{\underbrace{a^{N-1}, a^{N-2}b, \dots,}_{\binom{N-1}{1}} \underbrace{a^{N-r}b^{r-1}, \dots, a^{N-r}b^{r-1},}_{\binom{N-1}{r-1}} \underbrace{a^{N-1-r}b^r, \dots, a^{N-1-r}b^r,}_{\binom{N-1}{r}} \dots, b^{N-1}\}) \\ &= B(\{a^N, \underbrace{a^{N-1}b, \dots, a^{N-1}b}_{\binom{N-1}{1}}, \dots, \underbrace{a^{N-r}b^r, \dots, a^{N-r}b^r}_{\binom{N-1}{r}}, \dots, ab^{N-1}\} \\ &\cup \{a^{N-1}b, \underbrace{a^{N-2}b^2, \dots, a^{N-2}b^2}_{\binom{N-1}{1}}, \dots, \underbrace{a^{N-r}b^r, \dots, a^{N-r}b^r}_{\binom{N-1}{r-1}}, \dots, ab^N\}) \\ &= B(a^N, \underbrace{a^{N-1}b, \dots, a^{N-1}b}_{\binom{N}{1}}, \dots, \underbrace{a^{N-r}b^r, \dots, a^{N-r}b^r}_{\binom{N}{r}}, \dots, b^N)\end{aligned}$$

Thus we have obtained the relation (#).

(*b*) On the other hand, (37) is obtained by mathematical in induction as follows. For $i=2$, we have

$$\begin{aligned} w^2\rho^2(y^2, y'^2) &\leqq w^2L^2B_v(\{w^1\rho^1(\varphi^1({}^2\boldsymbol{x}), \varphi^1({}^2\boldsymbol{x}'))\} \\ &\quad \cup \{w^r\rho^r(x^r, x'^r)\}_{r=2}^{M}) \\ &\leqq w^2L^2B_v(\{w^1L^1B_v(\{w^r\rho^r(x^r, x'^r)\}_{r=2}^{M})\} \\ &\quad \cup B_v(\{w^r\rho^r(x^r, x'^r)\}_{r=2}^{M})) \\ &= w^2L^2B_v(w^1L^1, 1)B_v(\{w^r\rho^r(x^r, x'^r)\}_{r=2}^{M}) \\ &= w^2C^2\rho(\boldsymbol{x}, \boldsymbol{x}'). \end{aligned}$$

Assume now that the relations (37) hold for $r=2, 3,\dots, i-1$. Then, by inductive hypothesis,

$$\begin{aligned} w^i\rho^i(y^i, y'^i) &\leqq w^iL^iB_v(\{w^1\rho^1(\varphi^1({}^2\boldsymbol{x}), \varphi^1({}^2\boldsymbol{x}'))\} \\ &\quad \cup (\{w^r\rho^r(y^r, y'^r)\}_{r=2}^{i-1}) \\ &\quad \cup (\{w^r\rho^r(x^r, x'^r)\}_{r=i}^{M})) \\ &\leqq w^iL^iB_v(\{w^1L^1\rho(\boldsymbol{x}, \boldsymbol{x}')\} \\ &\quad \cup (\{w^rC^r\rho(\boldsymbol{x}, \boldsymbol{x}')\}_{r=2}^{i-1}) \\ &\quad \cup \{B_v(\{w^r\rho^r(x^r, x'^r)\}_{r=i}^{M})\}) \\ &\leqq w^iL^iB_v(\{w^1L^1\rho(\boldsymbol{x}, \boldsymbol{x}')\} \\ &\quad \cup (\{w^rC^r\rho(\boldsymbol{x}, \boldsymbol{x}')\}_{r=2}^{i-1}) \\ &\quad \cup (\{\rho(\boldsymbol{x}, \boldsymbol{x}')\}) \\ &= w^iL^iB_v((\{w^rC^r\rho(\boldsymbol{x}, \boldsymbol{x}')\}_{r=1}^{i-1}) \cup \{1\}) \cdot \rho(\boldsymbol{x}, \boldsymbol{x}'). \end{aligned}$$

Thus (37) holds for $i=2, 3,\dots, M$.

Referring to (30) we see that

$$\begin{aligned} \mathscr{L}_\lambda{}^{**} &= B_v(\{w^iC^i\}_{i=2}^{M}) = B_v(\{w^iL_\lambda^i\}_{i=2}^{M} \cup \{w^{i_1}L_\lambda^{i_1}w^{i_2}L_\lambda^{i_2}\}_{1\leqq i_1<i_2\leqq M} \cup \dots \\ &\quad \cup \{w^1L_\lambda^1w^2L_\lambda^2 \dots w^ML_\lambda^M\}). \end{aligned} \tag{43}$$

Observe here that (30) differs from (43) in which the term w^1L^1 is lacking.

Moreover, we see that the operand set under the B-operator in (43) is included, in the strict sence, by the operand set under the B-operator of (42), from which it follows

$$\mathscr{L}^* \leqq \mathscr{L}^{**}. \tag{44}$$

Q. E. D.

Remark 3

Observe that the set under the B-operator in the expression (37) is not symmetric as to $L_\lambda^1,\dots, L_\lambda^M$. From this we may have different evaluations of Lipschitz constant of Seidel operator if some of x^i lacks in the i-th relation, by reordering the elements and the relations so that such an x^i becomes the first member of the elements.

Example 3

It may seem somewhat trivial, but as a most simple illustration, let us consider the case, in which the operator is represented as

$$\begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix} = \begin{pmatrix} 0.0 & 1.0 & 0.3 \\ 0.2 & 0.0 & 0.2 \\ 0.2 & 0.2 & 0.0 \end{pmatrix} \cdot \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}, \tag{45}$$

a matrix operator which maps 3-dimensional space onto itself. It may be recognized that it is also possible to apply the corollary beside Theorem 2 to evaluate the Lipschitz constants of the corresponding Seidel operator. Let us now, evaluate and compare the Lipschitz constants of the operator both as a simple operator and corresponding Seidel operator employing methods given in this section.

1° $\lambda=(v=\infty\,;\ 1,\dots,\ 1)$

Operator	i	1	2	3	Lipschitz constant
Simple operator	L^i_λ	1.30	0.40	0.40	$L_\lambda=1.30>1$
Seidel operator	C^i	1.30	0.52	0.52	$\mathscr{L}_\lambda=1.30>1$
	$\max(\{C^r\}_{r=1}^{i-1},\ 1)$		1.30	1.30	$(\mathscr{L}_\lambda{}^{**}=0.52<1)$
	$L^i_\lambda \max L^1_\lambda,\ 1)$		0.52	0.52	$\mathscr{L}_\lambda{}^{*}=0.52<1$
	C'^i		0.52	0.52	
	$\max(\{C'^r\}_{r=2}^{i-1},\ 1)$			1.00	

2° $\lambda=(v=1\,;\ 1,\dots,\ 1)$

Operator	i	1	2	3	Lipschitz constant
Simple operator	L^i_λ	1.00	0.20	0.20	$L_\lambda=1.40$
Seidel operator	C^i	1.00	0.40	0.48	$\mathscr{L}_\lambda=1.88>1$
	$(\sum_{r=1}^{i-1} C^r)+1$		2.00	2.40	$(\mathscr{L}_\lambda{}^{**}=0.88<1)$
	$(L^i_\lambda)(L^1_\lambda+1)$		0.40	0.40	$\mathscr{L}_\lambda{}^{*}=0.96<1$
	C^i		0.40	0.56	
	$(\sum_{r=2}^{i=1} C'^r)+1$			1.40	

3° $\lambda=(v=2\,;\ 1,\dots,\ 1)$

Operator	i	1	2	3	$\left(\text{Lipschitz constant}\right)^2$
Simple operator	$(L^i_\lambda)^2$	1.09	0.08	0.08	$L^2_\lambda=1.25$
Seidel operator	$(C^i)^2$	1.09	0.1672	0.1806*)	$\mathscr{L}^2_\lambda=1.4378>1$
	$(\sum_{r=1}^{i-1}(C^r)^2)+1$		2.09	2.2572	$((\mathscr{L}_\lambda{}^{**})^2=0.3478<1)$
	$(L^i_\lambda)^2((L^1_\lambda)^2+1)$		0.1672	0.1672	$(\mathscr{L}_\lambda{}^{*})^2=0.3624<1$
	$(C'^i)^2$		0.1672	0.1952	
	$(\sum_{r=2}^{i-1}(C'^r)^2)+1$			1.1672	

*) If rounding off is necessary, always add 1 to the last digit, reminding that our theorems give sufficient condition for convergence.

Thus, we observe that the evaluations of the Lipschitz constants associated with the simple operator are greater than unity while those of Seidel operator are less than unity. In fact, we know that the processes (either simple or Seidel) converge, because the eigenvalues of the matrix in (45) are $0<0.1978$, 0.4180, $0.6172<1$.*) Indeed the first few terms of the processes beginning with the initial vector $\boldsymbol{x}_0=(1.0, 1.0, 1.0)$ are as follows.

Simple process

x_1	x_2	x_3
1.0	1.0	1.0
1.3	0.4	0.4
0.52	0.34	0.34
0.44	0.17	0.17
0.22	0.122	0.122

Seidel process

x_1	x_2	x_3
	1.0	1.0
1.3		
	0.46	
		0.352
0.530		
	0.176	
		0.141
0.218		
	0.072	
		0.058
0.089		

Observe also in the above example that the relation

$$\mathscr{L}_\lambda^* \geqq \mathscr{L}_\lambda^{**}$$

holds.

Example 4

Consider now, the Seidel operator consisting of operators each of which is associated with the same Lipschitz constant $\bar{L}_\lambda$. Let us evaluate the Lipschitz constant of the Seidel operator $\mathscr{L}_\lambda$ corresponding to $\bar{L}_\lambda$ and the value $\tilde{L}_\lambda$ of $\bar{L}_\lambda$ which makes the Lipschitz constant equal to unity for some metrics $\lambda \in \Lambda$. If we have the Lipschitz constant $\bar{L}_\lambda$ less than this value, then we see, considering the monotone increasing property of the B-operator, that the first and the main condition for convergence is fulfilled.

1° $\lambda=(\infty\,;\, 1,\dots, 1)$ $\qquad B_v(\{a^i\}_{i=1}^M)=\max(\{a^i\}_{i=1}^M)$

It follows from (32) that

$$\begin{aligned} \mathscr{L}_\lambda &= \bar{L}_\lambda & \bar{L}_\lambda \leqq 1 \\ &= (\bar{L}_\lambda)^M & \bar{L}_\lambda \geqq 1 \end{aligned} \tag{46}$$

Thus we have

$$\tilde{L}_\lambda = 1 \tag{47}$$

2° $\lambda=(v=p\geqslant 1\,;\, 1,\dots, 1)$ $\qquad \cdot\ B_v(\{a^i\}_{i=1}^M)=(\sum_{i=1}^M (a^i)^p)^{1/p}$

It follows from (33) that

---

*) It is known that a simple or a Seidel process given by a matrix converges if the eigenvalues are less than unity.

$$\mathscr{L}_\lambda = ((1+(\bar{L}_\lambda)^p)^M - 1)^{1/p} \tag{48}$$

from which we see that

$$\tilde{L}_\lambda = (2^{1/M} - 1)^{1/p} \tag{49}$$

Observe here that the left hand side of the above expression, (49) is positive for all finite M and p.

IV. Pseudo linear operator system of the first kind

This and the subsequent sections will be devoted to investigate to what extent we are able make precise (smaller) the evaluations of the Lipschitz constant of Seidel operator system, if we are given some constants similar but different from Lipschitz constants as to the individual operators constituting the system. In this section, we consider the case of what is to be detined as *the pseudo linear operator system of the first kind*. The result obtained in this section includes Sassenfeld's criterion ⟨9⟩ for the linear (matrix) Seidel operator system as a part.

Most of the proofs are omitted in what follows. But they are quite similar to those given in the preceeding section.

Definition 1 (Pseudo linear operator system of the first kind)

If there exists a set of non-negative constant $L_\lambda^i (i=1,\dots, M)$ and $M_\lambda^i (i=2,\dots, M)$ $\lambda \in \Lambda$ as to the operators

$$\begin{aligned} &y^1 = \varphi^1(\boldsymbol{x}) \\ &\cdots\cdots\cdots \\ &y^i = \varphi^i(\boldsymbol{x}) \\ &\cdots\cdots\cdots \\ &y^M = \varphi^M(\boldsymbol{x}) \end{aligned} \tag{1}$$

mapping a product set $\boldsymbol{F} = F^1 \times F^2 \times \dots \times F^M (F^i \subset R^i,\ i=1,\ 2,\dots,\ M)$ into R^i respectively, such that for any two elements

$$\rho^i(y^i, y'^i) \leqq B_{v'}(\dot{\varphi}_{i1} \cap M_\lambda^i \rho({}_1\boldsymbol{x}, {}_1\boldsymbol{x}'),\ L_\lambda^i \rho({}^i\boldsymbol{x}, {}^i\boldsymbol{x}')) \tag{2}^{*)}$$
$$i = 1,\ 2,\dots,\ M$$
$$\lambda = (v\ ;\ w^1,\ w^2,\dots,\ w^M)$$
$$v,\ v' \in V$$

where

$$y^i = \varphi^i(x^1, x^2, \dots, x^M)$$
$$y'^i = \varphi^i(x'^1, x'^2, \dots, x'^M),$$

we call (1) v'-λ-pseudo linear operator system of the first kind according to the index of the B-operator and the metric defined on the space $\boldsymbol{R}$.

---

*) Observe here that in general the B-operator B_{v}' in (2) differs from that defining the metric on R.

Theorem 1

The Lipschitz constant of v'-λ-pseudo linear *simple* operator system of the first kind defined on a complete set $\boldsymbol{F} \subset \boldsymbol{R}$ is evaluated in the forms

$$L_\lambda = B_v(B_v'\{w^1 L^1\} \cup \{w^i B_v'(M^i, L^i)\}_{i=2}^{M}) \tag{3}$$

$$L_\lambda = B_v(B_v'\{w^1 L^1\} \cup \{w^i \max(M^i, L^i)\}_{i=2}^{M}) \qquad v = v' \tag{4}$$

If the relation

$$L_\lambda < 1$$

holds and if there exists a point $\boldsymbol{x}_0 \in \boldsymbol{F}$ for which the assumptions

$$\boldsymbol{x}_1 = \boldsymbol{\varphi}(\boldsymbol{x}_0) \in \boldsymbol{F} \tag{i}$$

$$S\{\boldsymbol{u}: \rho_\lambda(\boldsymbol{u}, \boldsymbol{x}_1) \leqq (L_\lambda/1 - L_\lambda)\rho_\lambda(\boldsymbol{x}_1, \boldsymbol{x}_0)\} \subset \boldsymbol{F} \tag{ii}$$

hold, we have a process defined by the recurrence relation

$$\boldsymbol{x}_{n+1} = \boldsymbol{\varphi}(\boldsymbol{x}_n) \tag{5}$$

with the initial point $\boldsymbol{x}_0$ given above, coverging to the unique fixed point of the v'-λ-pseudo linear simple operator of the first kind, $\boldsymbol{\varphi}(\boldsymbol{x}) = (\varphi^1(\boldsymbol{x}), \varphi^2(\boldsymbol{x}), \ldots, \varphi^M(\boldsymbol{x}))$.

Proof

Observing that the individual operators are associated with Lipschitz constants of the form

$$L_\lambda^{i*} = B_v'(\phi_{1i} \cap M_\lambda^i, L_\lambda^i) \qquad i = 1, 2, \ldots, M \tag{6}$$

$$L_\lambda^{i*} = \max(\phi_{1i} \cap M_\lambda^i, L_\lambda^i) \qquad v = v' \tag{7}$$

the theorem follows from Theorem 1, **III**.

Although we may obtain the Lipschitz constant of v'-λ-pseudo linear Seidel operator of the first kind corresponding to (1) employing the Lipschitz constants (6) and (7) of the individual operators constituting the system via Theorem 2, **III**, here we give a method to obtain the Lipschitz constant directly employing the relations in (2).

Theorem 2

The Lipschitz constant $\mathscr{L}_\lambda$ of v'-λ-pseudo linear Seidel operator of the first kind corresponding to (1) mapping a complete product set $\boldsymbol{F} \subset \boldsymbol{R}$ into itself is evaluated as

$$\mathscr{L}_\lambda = B_v(\{w^i C^i\}_{i=1}^{M}), \qquad \lambda = (v; w^1, \ldots, w^M) \tag{8}$$

where

$$C^i = B_v'(M_\lambda^i B_v(\{w^r C^r\}_{r=1}^{i-1}), L_\lambda^i) \qquad C^1 = B_v'(\{L_\lambda^1\}), \tag{9}$$

so that if we assume that

$$\mathscr{L}_\lambda = B_v(\{w^i C^i\}_{i=1}^{M}) < 1 \tag{10—i}$$

(10—ii) there exists $\boldsymbol{x}_0$ such that $\boldsymbol{x}_1 = \boldsymbol{\phi}(\boldsymbol{x}_0)$ belongs to $\boldsymbol{F}$ where $\boldsymbol{\phi}(\boldsymbol{x})$ is the Seidel operator system corresponding to (1), (10—iii) the sphere $S\{\boldsymbol{u}: \rho(\boldsymbol{u}, \boldsymbol{x}_1) \leqq (\mathscr{L}_\lambda/1 - \mathscr{L}_\lambda)\rho_\lambda(\boldsymbol{x}_0, \boldsymbol{x}_1)$ is contained in $\boldsymbol{F}$, then the v'-λ-pseudo linear Seidel operator system of the first kind corresponding to (1) converges to the unique fixed point of the operator in $\boldsymbol{F}$.

Remark 1

The Lipschitz constant $\mathscr{L}_\lambda$ given in (8) is evaluated in terms of the constants L_λ^i, $i=1,\dots, M$ and M_λ^i, $i=2,\dots M,\dots$, $(\lambda\in\Lambda)$ themselves if $v=v'$ in the form

$$\begin{aligned} B_v(\{w^iC^i\}_{i=1}^M) = B_v(w^iL_\lambda^i\}_{i=1}^M &\cup \{w^{i_1}L_\lambda^{i_1}w^{i_2}M_\lambda^{i_2}\}_{1\leqq i_1<i_2\leqq M} \\ &\cup\dots\cup\{w^{i_1}L_\lambda^{i_1}w^{i_2}M_\lambda^{i_2}\dots w^{i_r}M_\lambda^{i_r}\}_{1\leqq i_1<\dots<i_r\leqq M} \\ &\cup\dots\cup\{w^1L_\lambda^1w^2M_\lambda^2\dots w^MM_\lambda^M\}) \end{aligned} \tag{11}$$

This relation follows from the relation (8) and

$$\begin{aligned} w^iC^i = B_v(\{w^iL_\lambda^i\} &\cup \{w^iM_\lambda^iw^rL_\lambda^r\}_{r=1}^{i-1}\cup\{w^iM_\lambda^iw^{r_1}L_\lambda^{r_1}w^{r_2}M_\lambda^{r_2}\}_{1\leqq r_1<r_2\leqq i-1} \\ &\cup\{w^iM_\lambda^iw^{r_1}L_\lambda^{r_1}w^{r_2}M_\lambda^{r_2}w^{r_3}M_\lambda^{r_3}\}_{1\leqq r_1<r_2<r_3\leqq i-1} \\ &\cup\{w^iM_\lambda^iw^1L_\lambda^1w^2M_\lambda^2\dots w^{i-1}M_\lambda^{i-1}\}), \qquad i=1,2,\dots, M. \end{aligned} \tag{12}$$

The relations (11) and (12) can be shown simultaneously by means of mathematical induction as to i.

Remark 2

As it might be expected, the evaluation of the Lipschitz constant $\mathscr{L}_\lambda{}^*$ of the Seidel operator system of the general kind corresponding to (1) employing the Lipschitz constants (6) and (7) associated with the individual operators constituting the system by Theorem 2, **III** is, in general, not smaller than the evaluation $\mathscr{L}_\lambda$ by Theorem 2 of this section.

$$\mathscr{L}_\lambda \leq \mathscr{L}_\lambda{}^*$$

Example 1 (Sassenfeld's criterion ⟨9⟩)

Consider now, a system of M linear operator defined on M-dimensional space $R_M(X, \rho_\lambda)$, $\lambda\in\Lambda$:

$$\begin{aligned} y^1 &= \varphi^1(\boldsymbol{x}) = a_{11}x^1+a_{12}x^2+\dots+a_{1M}x^M \\ &\dots\dots\dots \\ y^i &= \varphi^i(\boldsymbol{x}) = a_{i1}x^1+a_{i2}x^2+\dots+a_{iM}x^M \\ &\dots\dots\dots \\ y^M &= \varphi^M(\boldsymbol{x}) = a_{M1}x^1+a_{M2}x^2+\dots+a_{MM}x^M \end{aligned} \tag{14}$$

or in matrix form,

$$\begin{pmatrix} y^1 \\ \dots \\ \dots \\ y^i \\ \dots \\ \dots \\ y^M \end{pmatrix} = \begin{pmatrix} a_{11}, a_{12}, \dots, a_{1M} \\ \dots\dots\dots \\ \\ a_{i1}, a_{i2}, \dots, a_{iM} \\ \dots\dots\dots \\ \\ a_{M1}, a_{M2}, \dots, a_{MM} \end{pmatrix} \cdot \begin{pmatrix} x^1 \\ \dots \\ \dots \\ x^i \\ \dots \\ \dots \\ x^M \end{pmatrix}. \tag{15}$$

It may be clear that this is a pseudo linear Seidel operator of the first kind as to some pairs of (λ, v). Some of their examples are given below with the relations (8) and (9).

Metric & B_v'	Constants L_λ^i & M_λ^i	Lipschitz Constant
$\lambda=(\infty; 1,\dots, 1)$ $B_v'(\{a^i\}_{i=1}^M)$ $=\sum_{i=1}^M a^i$	$L_\lambda^i=\sum_{r=i}^M \lvert a_{ir}\rvert$ $i=1, 2,\dots, M$ $M_\lambda^i=\sum_{r=1}^{i-1} \lvert a_{ir}\rvert$ $i=2,\dots, M$	(16) Lipschitz constant $=\max(\{C^i\}_{i=1}^M)$ $C^i=((\sum_{r=1}^{i-1}\lvert a_{ir}\rvert)\max(\{C^r\}_{r=1}^{i-1})+\sum_{r=i}^M\lvert a_{ir}\rvert)$ $i=1, 2,\dots, M$
$\lambda=(v=1; 1,\dots,1)$ $B_v'(\{a^i\}_{i=1}^M)$ $=\sum_{i=1}^M a^i$	$L_\lambda^i=\max(\{\lvert a_{ir}\rvert\}_{r=i}^M)$ $i=1,\dots, M$ $M_\lambda^i=\max(\{\lvert a_{ir}\rvert\}_{r=1}^{i-1})$ $i=2,\dots, M$	(17) Lipschitz constant $=\sum_{i=1}^M C^i$ $C^i=(\max(\{\lvert a_{ir}\rvert\}_{r=1}^{i-1})(\sum_{r=1}^{i-1} C^r)$ $+\max(\{\lvert a_{ir}\rvert\}_{r=i}^M))$ $i=1, 2,\dots, M$
$\lambda=(v=2; 1,\dots,1)$ $B_v'(\{a^i\}_{i=1}^M)$ $=(\sum_{i=1}^M (a^i)^2)^{1/2}$	$(L_\lambda^i)^2=\sum_{r=i}^M (a_{ir})^2$ $i=1, 2,\dots, M$ $(M_\lambda^i)^2=\sum_{r=1}^{i-1}(a_{ir})^2$ $i=1, 2,\dots, M$	(18) Lipschitz constant $=(\sum_{i=1}^M(C^i)^2)^{1/2}$ $(C^i)^2=\sum_{r=1}^{i-1}(a_{ir})^2\cdot(\sum_{r=1}^{i-1}(C^r)^2)$ $+\sum_{r=i}^M(a_{ir})^2$ $i=1, 2,\dots, M$

Here, (16) is the Lipschitz constant given in the second criterion of Sassenfeld (Kriterium II) ⟨9⟩.

Example 2

Consider now, a matrix operator

$$\begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = \begin{pmatrix} 0.05 & 0.10 \\ 7.00 & 0.20 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}.$$

Since the matrix is positive definite (the eigenvalues are 1.43 and 1.93) the corresponding Seidel process is convergent ⟨10⟩. Obviously, usual norms of the matrix is greater than unity which do not assure the convergence of the process. ⟨8⟩. Let us try to evaluate the Lipschitz constant of Seidel operator corresponding to this matrix employing Theorems 2, **III** and 2, **IV** and compare them with the corresponding evaluations by Theorem 1, **IV**.

1° $\lambda=(\infty; 1,\dots, 1)$ $\quad B_v(\{a^i\}_{i=1}^M)=\max(\{a^i\}_{i=1}^M)$

$$B_{v'}(\{a^i\}_{i=1}^M)=\sum_{i=1}^M(a^i) \qquad v\neq v'$$

i	1	2	Lipschitz Constant
L^i_λ	0.15	0.20	$\mathscr{L}_\lambda=1.25>1$ (Theorem 2, **IV**)
M^i_λ		7.00	
$\max(\{C^r\}_{r=1}^{i-1})$		0.15	
$M^i \max(\{C^r\}_{r=1}^{i-1})$		1.05	
C^i	0.15	1.25	
$B_v'(L^i_\lambda, M^i_\lambda)$	0.15	7.20	$\mathscr{L}_\lambda^*=7.20>1$ (Theorem 2, **III**)
$\max(\{C^{*r}\}_{r=1}^{i-1}, 1)$		1.00	
C^{*i}	0.15	7.20	

The corresponding evaluation of the Lipschitz constant by Theorem 1. **IV** gives the value $7.20 > 1$.

2° $\lambda=(v=1\,;\,1,\dots,\,1)$

$$B_v'(\{a^i\}_{i=1}^M)=\sum_{i=1}^M(a^i) \qquad v=v'$$

i	1	2	Lipschitz Constant
L^i_λ	0.10	0.20	$\mathscr{L}_\lambda=1.00$ (Theorem 2, **IV**)
M^i_λ		7.00	
$\sum_{r=1}^{i-1} C^r$		0.10	
$M^i_\lambda \sum_{r=1}^{i-1} C^r$		0.70	
C^i	0.10	0.90	$\mathscr{L}^i_\lambda=7.80>1$ (Theorm 2, **III**)
$\max(L^i_\lambda, M^i_\lambda)$	0.10	7.00	
$(\sum_{r=1}^{i-1} C^{*r})+1$		1.10	
C^{*i}	0.10	7.70	

The corresponding evaluation of the Lipschitz constant by Theorem 1, **IV** gives the value $7.10 > 1$.

3° $\lambda=(v=2\,;\,1,\dots,\,1)$

$$B_v'(\{a^i\}_{i=1}^M)=(\sum_{i=1}^M (a^i)^2)^{1/2} \qquad v=v$$

i	1	2	Lipschitz Constant
$(L^i_\lambda)^2$	0.0125	0.04	$\mathscr{L}_\lambda=\sqrt{0.665}<1$ $(=0.8155)$ (Theorem 2, **IV**)
$(M^i_\lambda)^2$		49.00	
$\sum_{r=1}^{i-1}(C^r)^2$		0.0125	
$M^i \sum_{r=1}^{i-1}(C^r)^2$		0.6125	
$(C^i)^2$	0.0125	0.6525	$\mathscr{L}_\lambda^*=\sqrt{49.6250}>1$ (Theorem 2, **III**)
$\max((L^i_\lambda(^2,)\,M^i_\lambda)^2)$	0.0125	49.00	
$\sum_{r=1}^{i-1}(C^{*r})^2+1$		1.0125	
$(C^{*r})^2$	0.0125	49.6125	

The corresponding evaluation of the Lipschitz constant by Theorem 1, **IV** gives the value $(49.0125)^{1/2}>1$.

The remainder of this section will be devoted to the evaluation of the Lipschitz constant associated with the particular type of the pseudo linear Seidel operator of the first kind as in the previous section:

$$\begin{aligned} y^1 &= \varphi^1(x^2, \ldots, x^M) \\ y^2 &= \varphi^2(y^1, x^2, \ldots, x^M) \\ &\cdots\cdots\cdots \\ y^i &= \varphi^i(y^1, \ldots, y^{i-1}, x^i, \ldots, x^M) \\ &\cdots\cdots\cdots \\ y^M &= \varphi^M(y^1, \ldots, y^{M-1}, x^M), \end{aligned} \tag{19}$$

i.e. the first operator does not include x^1 in the arguments. In this case, as was observed in the previous section, (23) can be considered to be a new operator which maps a set $F' \subset R^2 \times R^3 \times \ldots \times R^M$ into the space $R^2 \times R^3 \times \ldots \times R^M$ in the form

$$\begin{aligned} y^2 &= \varphi^2(\varphi^1({}^2\boldsymbol{x}), {}^2\boldsymbol{x}) \\ y^3 &= \varphi^3(\varphi^1({}^2\boldsymbol{x}), y^2, {}^3\boldsymbol{x}) \\ &\cdots\cdots\cdots \\ y^i &= \varphi^i(\varphi^1({}^2\boldsymbol{x}), y^2, \ldots, y^{i-1}, {}^i\boldsymbol{x}) \\ &\cdots\cdots\cdots \\ y^M &= \varphi^M(\varphi^1({}^2\boldsymbol{x}), y^2, \ldots, y^{M-1}, {}^M\boldsymbol{x}), \end{aligned} \tag{20}$$

with the metric λ on the product space $R^2 \times R^3 \times \ldots \times R^M$ correspoding to the metric defined on the product space $R^1 \times R^2 \times \ldots \times R^M$, $\lambda \in \Lambda$.

There may be two possibilities in evaluating the Lipschitz constant of the Seidel operator of (19). 1° One method is to consider (19) as a Seidel operator of general kind in the form of (20) (not as a pseudolinear Seidel operator of the first kind), and to apply Theorem 2, of **III**. 2 Another method is to consider (19) again as a Seidel operator system of general kind, with Lipschitz constants associated with the individual operators given in (6) and (7), and to evaluate the Lipschitz constant of the Seidel operator system of the type above, employing Corollary of Theorem 2 of **III**.

As for the first method, we have

Theorem 3

The Lipschitz constant $\mathscr{L}^{**}$ of Seidel operator (20) defined on a set $F \subset R^2 \times R^3 \times \ldots \times R^M$ can be obtained by applying Theorem 2, **III** to the Seidel operator consisting of $M-1$ operators with Lipschitz constants

$$L^{**i} = B_v'(M_\lambda^2 B_v(w^1 L_\lambda^1), L_\lambda^2) \qquad i=2 \tag{21}$$

$$L^{**i} = B_v'(M_\lambda^i B_v(w^1 L_\lambda^1, 1), L_\lambda^i) \quad i=3,\ldots, M \quad L^{**i} = B_v'(M_\lambda^i \max (M_\lambda^i L_\lambda^i)) \tag{22}$$

$$\lambda = (v\,;\, w^1, \ldots, w^M). \qquad v=v$$

(cf. footnote of Corollary of Theorem 2, **III**)

Remarks 3

Let us denote by $\mathscr{L}_\lambda{}^*(=B_v(\{w^iC^{*i}\}_{i=2}^M)$ the evaluation of the Lipschitz constant, recognizing (19) as a Seidel operator system of the general kind. Note that there is no definite inequality between $\mathscr{L}_\lambda{}^*$ and $\mathscr{L}_\lambda{}^{**}(=B_v(\{w^iC^{**i}\}_{i=2}^M))$ the evaluation of the Lipschitz constant by Theorem 2, of this section. This can be easily shown by examples.

Example 3

Consider again the matrix operator

$$\begin{bmatrix} y^1 \\ y^2 \\ y^3 \end{bmatrix} = \begin{bmatrix} 0.0 & 1.0 & 0.3 \\ 0.2 & 0.0 & 0.2 \\ 0.2 & 0.2 & 0.0 \end{bmatrix} \cdot \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix}$$

which was considered in Example 3 III. Let us try to evaluate the Lipschitz constant of the Seidel operator corresponding to this matrix operator employing Theorems 2 and 3 of this section. We can also compare the results with the corresponding evaluations given in the Example 3, III.

1° $\lambda=(\infty\,;\,1,\ldots,\,1)$ $\quad B_v(\{a^i\}_{i=1}^M)=\max\,(\{a^i\}_{i=1}^M)$

$v'=1$ $\quad B_v'(\{a^i\}_{i=1}^M)=\sum_{i=1}^M a^i$

i	1	2	3	Lipschitz Constant
L_λ^i	1.30	0.20	0.00	
M_λ^i		0.20	0.40	$\mathscr{L}_\lambda=1.30$
$\max(\{C^r\}_{r=1}^{i-1})$		1.30	1 30	(Theorem 2, IV)
$M^i\max(\{C^r\}_{r=1}^{i-1})$		0.26	0 52	
C^i	1.30	0.46	0.52	
$L_\lambda{}^{**i}$		0.66	0.52	$\mathscr{L}_\lambda{}^{**}=0.66$
C^{**i}		0.66	0.52	(Theorem 3, IV)
$\max(\{C^{**r}\}_{r=2}^{i-1}, 1)$			1.00	
$L_\lambda{}^{*i}$	1.30	0.40	0.40	
C^{*i}	1.30	0.52	0.52	$\mathscr{L}_\lambda{}^*=0.52$
$\max(\{C^{*r}\}_{r=1}^{i-1}, 1)$		1.30	1.30	(Corollary of Theorem 2, III)

2° $\lambda=(v=1\,;\,1,\ldots,\,1)$ $\quad B_v(\{a^i\}_{i=1}^M)=\sum_{i=1}^M a^i$

$v'=v=1$

i	1	2	3	Lipschitz Constant
L_λ^i	1.00	0.20	0.00	
M_λ^i		0.20	0.20	
$\sum_{r=1}^{i-1} C^r$		1.00	1.40	$\mathscr{L}_\lambda=1.68$
$M_\lambda^i \sum_{r=1}^{i-1} C^r$		0.20	0.20	(Theorem 2, IV)
C^i	1.00	0.40	0.28	
$L_\lambda{}^{**i}$		0.40	0.20	
C^{**i}		0.40	0.28	$\mathscr{L}_\lambda{}^{**}=0.68$
$\sum_{r=1}^{i-1} C^{**r}+1$			1.40	(Theorem 3, IV)
$L_\lambda{}^{*i}$	1.00	0.20	0.20	
C^{*i}	1.00	0.40	0.48	$\mathscr{L}_\lambda{}^*=0.88$
$\sum_{r=1}^{i-1} C^{*r}+1$		2.00	2.40	(Corollary of Theorem 2, III)

3° $\lambda=(v=2\,;1,\dots,\,1)$ $\quad B_v(\{a^i\}_{i=1}^M)=(\sum_{i=1}^M(a^i)^2)^{1/2}$

$v=v'$ $\quad B_v'(\{a^i\}_{i=1}^M)=(\sum_{i=1}^M(a^i)^2)^{1/2}$

i	1	2	3	Lipschitz Constant
$(L_\lambda^i)^2$	1.09	0.04	0.00	
$(M_\lambda^i)^2$		0.04	0.08	
$\sum_{r=1}^{i-1}(C^r)^2$		1.09	1.174	$\mathscr{L}_\lambda=\sqrt{1.268}=1.126$ (Theorem 2, IV)
$(M_\lambda^i)^2\sum_{r=1}^{i-1}(C^r)^2$		0.044	0.094	
$(C^i)^2$	1.09	0.084	0.094	
$(L_\lambda^{**i})^2$		0.080	0.080	
$(C^{**i})^2$		0.080	0.0864	$\mathscr{L}_\lambda^{**}=\sqrt{0.1664}$
$\sum_{r=1}^{i-1}(C^{**r})^2+1$			1.080	(Theorem 3, IV)
$(L_\lambda^{*i})^2$	1.09	0.04	0.04	
$(C^{*i})^2$	1.09	0.0836	0.08694	$\mathscr{L}_\lambda=\sqrt{0.170}$
$\sum_{r=1}^{i-1}(C^{*r})^2+1$		2.09	2.1736	(Corollary of Theorem 2, III)

V. Pseudolinear operator system of the 2-nd kind

This section is devoted to the evaluation of the Lipschitz constant of what we call *the pseudo linear operator system of the 2-nd kind*. This is the case we are informed slightly more about the properties of the individual operators constituting the system than the pseudo linear operator system of the 1-st kind. Although the method of evaluation given in this section is essentially the same as evaluation given in the preceding sections, it will be possible, with the result of this section to evaluate the Lipschitz constant associated with the deduced system which consists of repeated arrangement of individual operators constituting the original system. This evaluation can be a measure of the speed of convergence of the deduced process which would be required in computing practice.

Definition 1 (Pseudo linear operator system of the 2-nd kind)

If there exist a set of non-negative constants M_λ^i $(i=2,\,3,\dots,\,M)$, L_λ^i $(i=1,\,2,\dots,\,M)$ and K_λ^i $(i=1,\,2,\dots,\,M-1)$, $\lambda\in\Lambda$, as to the operators

$$\begin{aligned}y^1&=\varphi^1(\boldsymbol{x})\\ &\cdots\cdots\cdots\\ y^i&=\varphi^i(\boldsymbol{x})\\ &\cdots\cdots\cdots\\ y^M&=\varphi^M(\boldsymbol{x})\end{aligned}\tag{1}$$

defined on a product set $\boldsymbol{F}=F^1\times F^2\times\dots\times F^M$, $F^i\subset R^i$, $i=1,\dots,\,M$ and each of which maps the set $\boldsymbol{F}$ into F^i, such that for any two elements $\boldsymbol{x},\,\boldsymbol{x}'\in\boldsymbol{F}$,

$$\rho^i(y^i,\,y'^i)\leqq B_v'(\{\phi_{1i}\cap\{M_\lambda^i\rho_\lambda({}_{i-1}\boldsymbol{x},\,{}_{i-1}\boldsymbol{x}')\}\}\cup\{L_\lambda^i w^i\rho^i(x^i,\,x'^i)\}\\ \cup\{\phi_{Mi}\cap\{K_\lambda^i\rho_\lambda({}^{i+1}\boldsymbol{x},\,{}^{i+1}\boldsymbol{x}')\}\})\tag{2}$$

where,

$$y^i=\varphi^i(x^1,\,x^2,\dots,\,x^M)$$

$$y'^i=\varphi^i(x'^1,\,x'^2,\dots,\,x'^M)$$

and

$$\lambda=(v\,;\,w^1,\dots,\,w^M)$$

we call (1) v'-λ-pseudolinear operator system of the 2-nd kind corresponding to the index of the B-operator and the metric defined on the product space $\boldsymbol{R}$.

Theorem 1

The Lipschitz constant of v'-λ-pseudo linear simple operator of the 2-nd kind defined on a complete set $\boldsymbol{F}\subset \boldsymbol{R}(\boldsymbol{X}, \rho_\lambda)$ is evaluated as

$$L=B_v(\{w^i B_v'((\phi_{1i}\cap M_\lambda^i), L_\lambda^i(\phi_{Mi}\cap K_\lambda^i)\}_{i=1}^M) \tag{3}$$

$$L=B_v(\{w^i \max((\phi_{1i}\cap M_\lambda^i), L_\lambda^i, (\phi_{Mi}\cap K_\lambda^i))\}_{i=1}^M) \qquad v=v' \tag{3'}$$

where v is the index of B-operator corresponding to λ.

If the relation

$$L_\lambda < 1 \tag{4}$$

holds and if there exists a point $\boldsymbol{x}_0 \in \boldsymbol{F}$ for which the assumptions

$$\boldsymbol{x}^1=\boldsymbol{\varphi}(\boldsymbol{x}_0)\in \boldsymbol{F} \tag{i}$$

$$S\{\boldsymbol{u} : \rho_\lambda(\boldsymbol{u}, \boldsymbol{x}_1)\leqq (L_\lambda/1-L_\lambda)\rho_\lambda(\boldsymbol{x}_1, \boldsymbol{x}_0)\}\subset \boldsymbol{F} \tag{ii}$$

hold, we may have a process defined by the recurrence relation

$$\boldsymbol{x}_{n+1}=\boldsymbol{\varphi}(\boldsymbol{x}_n) \tag{5}$$

with the initial point $\boldsymbol{x}_0$ given above, converging to the unique fixed point of the v'-λ-pseudolinear simple operator of the 2-nd kind, $\boldsymbol{\varphi}(\boldsymbol{x})=(\varphi^1(\boldsymbol{x}), \varphi^2(\boldsymbol{x}),\ldots, \varphi^M(\boldsymbol{x}))$.

Proof

Observe that each of the individual operators (1) has the Lipschitz constant of the form

$$B_v'((\phi_{1i}\cap M_\lambda^i), L_\lambda^i, (\phi_{Mi}\cap K_\lambda^i))$$

or

$$\max((\phi_{1i}\cap M_\lambda^i), L_\lambda^i, (\phi_{Mi}\cap K_\lambda^i)) \qquad v=v'$$

which follows from the relations (2) directly. And the remainder follows from Theorm 1 of III.

Q. E. D.

Theorem 2

The Lipschitz constant of v'-λ-pseudolinear Seidel operator system of the 2-nd kind $\boldsymbol{\phi}(\boldsymbol{x})$ corresponding to (1) mapping a complete product set $\boldsymbol{F}=F^1\times F^2\times\ldots\times F^M$, $F^i\subset \boldsymbol{R}^i$, $i=1, 2,\ldots, M$ into itself is evaluated as

$$\mathscr{L}_\lambda=B_v(\{w^i C^i\}_{i=1}^M) \qquad \lambda=(v ; w^1, w^2,\ldots, w^M) \tag{6}$$

where

$$C^i=B_v'(\phi_{1i}\cap M_\lambda^i B_v(\{w^r C^r\}_{r=1}^{i-1}), B_v'(L_\lambda^i, \phi_{Mi}\cap K_\lambda^i)) \tag{7}$$

so that if we assume that

(i) $$\mathscr{L}_\lambda=B_v(\{w^i C^i\}_{i=1}^M)<1 \tag{8—i}$$

(ii) there exist x_0 such that $x_1=\phi(x_0)$ belongs to F (8—ii)

(iii) $S\{u : \rho_\lambda(u, x_1) \leqq (\mathscr{L}_\lambda/1-\mathscr{L}_\lambda)\, \rho_\lambda(x_0, x_1)\} \subset F$ (8—iii)

then the Seidel process corresponding to (1) converges to the unique fixed point of the v'-λ-pseudo linear Seidel operator system of the 2-nd kind in F.

In case, $v=v'$, we have

$$C^i=B_v(\phi_{1i}\cap M_\lambda^i B_v(\{w^r C^r\}_{r=1}^{i-1}), \max(L_\lambda^i, (\phi_{Mi}\cap K_\lambda^i)) \tag{9}$$

instead of (7). The evaluation via (9) is, in general, not greater than that given via (7).

Proof

The proof is quite similar to those given in III.

Remark 1

We may also evaluate the Lipschitz constant of the pseudo linear Seidel operator system of the 2-nd kind recognizing it as a pseudo-linear operator system of the 1-st kind Let L_λ^{*i} and M_λ^{*i} represent the constants associated with the pseudolinear operator system of the 1-st kind. These constants can be represented in terms of the constants M_λ^i, L_λ^i, K_λ^i associated with the pseudo-linear operator system of the 2-nd kind as follows:

$$L_\lambda^{*i}=B_v'(L_\lambda^i, \phi_{Mi}\cap K_\lambda^i) \tag{13}$$

$$L_\lambda^{*i}=\max(L_\lambda^i, \phi_{Mi}\cap K_\lambda^i) \qquad v=v' \tag{13'}$$

$$M_\lambda^{*i}=\phi_{1i}\cap M_\lambda^i. \tag{14}$$

If we now substitute the representations (13), (13') and (14) into the recurrence relations (9) of Theorem 2 of IV, we obtain the same recurrence relation as (7) and (9) of Theorem 2 of this section. Hence, the resulting Lipschitz constant is the same as given in Theorem 2 of IV. This shows that Theorem 2 of this section is essentially the same as Theorem 2 of IV.

Remark 2

In another way, we may also evaluate the Lipschitz constant of the pseudo linear Seidel operator system of the 2-nd kind by recognizing it as a Seidel operator system of the general kind. Let L_λ^{**i} represent the Lipschitz constants associated with the individual operators constituting the operator system. These Lipschitz constants can be represented in terms of the constants M_λ^i, L_λ^i, K_λ^i associated with the pseudo linear oprator system of the 2-nd kind as follows.

$$L_\lambda^{**i}=B_v{}^i(\phi_{1i}\cap M_\lambda^i, L_\lambda^i, \phi_{Mi}\cap K_\lambda^i) \tag{15}$$

$$L_\lambda^{**i}=\max(\phi_{1i}\cap M_\lambda^i, L_\lambda^i, \phi_{Mi}\cap K_\lambda^i), \qquad v=v'. \tag{15'}$$

Hence, if we consider the operator of the 2-nd kind (1) as a pseudolinear Seidel operator system of the 1-st kind with constants (13), (13') and (14) and consider it again as a Seidel operator system of the general kind with Lipschitz constants associated with the individual operators constituting the system

$$B_v'(\phi_{i1} \cap w^i M_\lambda^{*i},\ w^i L_\lambda^i)$$

or if $v=v'$

$$\max(\phi_{i1} \cap w^i M_\lambda^i,\ w^i L_\lambda^{*i})$$

(cf. Remark 1, IV), we obtain the same result as the method above.

Example 1

Consider now the case, where

$$B_v'(\{a^i\}_{i=1}^M) = \sum_{i=1}^M a^i$$

and

$$B_v(\{a^i\}_{i=1}^M) = \max(\{a^i\}_{i=1}^M)$$

and

$$M^i = K^i = b, \qquad L^i = a, \qquad i=1,\dots, M$$

In this case, we have

$$C^1 = a+b$$
$$C^2 = b(a+b) + (a+b) = (b+1)(a+b)$$
$$C^3 = b(b+1)(a+b) + (a+b) = (b^2+b+1)(a+b)$$
$$\cdots\cdots\cdots$$
$$C^M = (1+b+b^2+\dots+b^{M-1})(a+b)$$

thus

$$\mathscr{L}_\lambda = (1+b+b^2+\dots+b^{M-1})(a+b) = (1-b^M)(a+b)/(1-b) \qquad b \neq 1$$

so that if $L^i = a = 0$, we have

$$\mathscr{L}_\lambda = b(1-b^M)/(1-b).$$

From this, we see that if

$$b < b_M^*$$

we have

$$\mathscr{L}_\lambda < 1,$$

the main criterion of the convergence, where

$$b_M^*(1-(b_M^*)^M)/(1-(b_M^*)) = 1.$$

As a rough estimate, it may be readily seen that

$$1 > b_M^* > 1/2$$

for all finite M. And it is also observed that b_M^* increase as M.

Example 2

Consider now, a system of M linear operators defined on M-dimensional space $R_M(X, \rho_\lambda)$:

$$\boldsymbol{y}=\boldsymbol{\varphi}(\boldsymbol{x})=\boldsymbol{A}\boldsymbol{x}=\begin{pmatrix} a_{11},\dots, & a_{1M} \\ \dots\dots\dots & \\ \dots\dots\dots & \\ \dots\dots\dots & \\ a_{M1},\dots, & a_{MM} \end{pmatrix}\begin{pmatrix} x_1 \\ \dots \\ \dots \\ \dots \\ x_M \end{pmatrix}. \tag{16}$$

It may be clear that this is a pseudo linear operator system of the 2-nd kind as to some pairs of (λ, v'). But if we take

$$B_v(\{a^i\}_{i=1}^M)=\max(\{a^i\}_{i=1}^M)$$
$$B_{v'}(\{a^i\}_{i=1}^M)=\sum_{i=1}^M a^i$$

or

$$B_v(\{a^i\}_{i=1}^M)=\sum_{i=1}^M a^i \text{ and } v=v'$$

we obtain the same result as (20) or (21) of IV respectively.

On the other hand, if we take

$$B_v(\{a^i\}_{i=1}^M)=(\sum_{i=1}^M (a^i)^2)^{1/2} \text{ and } v=v'$$

we have, instead of (22) of IV,

$$(C^i)^2=\sum_{r=1}^{i-1}(a_{ir})^2(\sum_{r=1}^{i-1}(C^r)^2)+\max((a_{ii})^2, \sum_{r=i+1}^{M}(a_{ir})^2) \tag{17}$$

which is, in general, not greater than (22) of IV.

Let us now turn to consider a class of operator systems deduced from the pseudo linear Seidel operator system of the 2-nd kind which has the following form :

$$\begin{aligned}
&y_1^1=\varphi^1(x^1, x^2,\dots, x^M)\\
&y_2^1=\varphi^1(y_1^1, x^2,\dots, x^M)\\
&\dots\dots\dots\\
&y_{n1}^1=\varphi^1(y_{n1-1}^1, x^2,\dots, x^M)\\
y^1=&\varphi^1(y_{n1}^1, x^2,\dots, x^M)\\
&y_1^2=\varphi^2(y^1, x^2,\dots, x^M)\\
&y_2^2=\varphi^2(y^1, y_1^2,\dots, x^M)\\
&\dots\dots\dots\\
&y_{n2}^2=\varphi^2(y^1, y_{n2-1}^2,\dots, x^M)\\
y^2=&\varphi^2(y^1, y_{n2}^2,\dots, x^M)\\
&\dots\dots\dots\\
\dots\dots\dots&\\
&y_1^i=\varphi^i(y^1, y^2,\dots, y^{i-1}, x^i,\dots, x^M)\\
&y_2^i=\varphi^i(y^1, y^2,\dots, y^{i-1}, y_1^i, x^{i+1}, \dots, x^M)\\
&\dots\dots\dots\\
&y_{ni}^i=\varphi^i(y^1, y^2,\dots, y^{i-1}, y_{ni-1}^i, x^{i+1},\dots, x^M)\\
y^i=&\varphi^i(y^1, y^2,\dots, y_{n1}^i, x^{i+1},\dots, x^M)\\
&\dots\dots\dots
\end{aligned} \tag{18}$$

$$\begin{aligned}
&\cdots\cdots\cdots\\
&\quad y_1^M=\varphi^M(y^1, y^2,\dots, y^{M-1}, x^M)\\
&\quad y_2^M=\varphi^M(y^1, y^2,\dots, y^{M-1}, y_1^M)\\
&\quad \cdots\cdots\cdots\\
&\quad y_{n\,M}^M=\varphi^M(y^1, y^2,\dots, y^{M-1}, y_{n\,M-1}^M)\\
&y^M=\varphi^M(y^1, y^2,\dots, y_{n\,M}^M)
\end{aligned}$$

That is, the i-th operator in the original operator system is repeated n_{i+1} times in the deduced operator system. Observe here that the fixed point of both the original and the deduced operator system coincide whenever they exist.

Now, let us evaluate the Lipschitz constant of the deduced operator system (18).

Theorem 3

The Lipschitz constant $\mathscr{L}_\lambda'$ of the operator system (18) defined on a product set $\boldsymbol{F}'\subset\boldsymbol{R}$ is eveluated as

$$\mathscr{L}_\lambda'=B_v(\{w^iC^i\}_{i=1}^M) \tag{19}$$

where

$$C^i=B_{v'}(\phi_{1i}\cap M_\lambda^iB_v(\{w^rC^r\}_{r=1}^{i-1}),\ L_\lambda'^i,\ \phi_{iM}\cap K_\lambda^i) \tag{20}$$
$$i=1,\dots, M$$

and

$$L_\lambda'^i=D_{ni}^iL_\lambda^i \tag{21}$$

where

$$D_r^i=B_{v'}(\phi_{1i}\cap M_\lambda^iB_v(\{w^sC^s\}_{s=1}^{i-1}),\ D_{r-1}^iL_\lambda^i,\ \phi_{iM}\cap K_\lambda^i) \tag{22}$$
$$r=1,\dots, n_i$$
$$D_0^i=1 \tag{23}$$

Proof

In fact, we have the following two relations:

$$\rho^i(y^i, y'^i)\leqq C^i\rho_\lambda(\boldsymbol{x}, \boldsymbol{x}')\qquad i=1, 2,\dots, M \tag{24}$$

$$\rho^i(y_s^i, y_s'^i\leqq D_s^i\rho_\lambda(\boldsymbol{x}, \boldsymbol{x}')\qquad s=1, 2,\dots, n_i,\ i=1, 2,\dots, M \tag{25}$$

It may be enough with the proof of the relation (25). If we assume that these relations hold for 1, 2,..., $i-1$, we see that

$$\begin{aligned}
\rho^i(y_1^i, y_1'^i)\leqq B_{v'}(&\phi_{1i}\cap M_\lambda^i\rho_\lambda({}_{i-1}\boldsymbol{y}, {}_{i-1}\boldsymbol{y}'),\ L_\lambda^iw^i\rho_\lambda^i(x^i, x'^i),\\
&\phi_{Mi}\cap K_\lambda^i\rho_\lambda({}^{i+1}\boldsymbol{x}, {}^{i+1}\boldsymbol{x}'))\\
\leqq B_{v'}(&\phi_{1i}\cap M_\lambda^iB_v(\{w^rC^r\}_{r=1}^{i-1}),\ L_\lambda^i,\ \phi_{Mi}\cap K_\lambda^i)\,\rho_\lambda(\boldsymbol{x}, \boldsymbol{x}')
\end{aligned}$$

Thus we have (25) for $s=1$. If we assume that (25) hold for $s=1, 2,\dots, r-1$. then

$$\begin{aligned}
\rho^i(y_r^i, y_r'^r)\leqq B_{v'}(&\phi_{1i}\cap M_\lambda^i\rho({}_{i-1}\boldsymbol{y}, {}_{i-1}\boldsymbol{y}'),\ L_\lambda^iw^i\rho_\lambda^i(y_{r-1}^i, y_r'^i{}_{-1}),\\
&\phi_{Mi}\cap K_\lambda^i\rho_\lambda({}^{i+1}\boldsymbol{x}, {}^{i+1}\boldsymbol{x}'))\\
\leqq B_{v'}(&\phi_{1i}\cap M_\lambda^iB_v(\{w^rC^r\}_{r=1}^{i-1})\,\rho(\boldsymbol{x}, \boldsymbol{x}'),\ L_\lambda^iw^iD_{r-1}^i\rho_\lambda^i(y^i,y'^i),\\
&\phi_{Mi}\cap K_\lambda^i\rho_\lambda({}^{i+1}\boldsymbol{x}, {}^{i+1}\boldsymbol{x}'))\\
\leqq B_{v'}(&\phi_{1i}\cap M_\lambda^i,\ L_\lambda^iD_{r-1}^i,\ \phi_{Mi}\cap K_\lambda^i)\,\rho_\lambda(\boldsymbol{x}, \boldsymbol{x}').
\end{aligned}$$

Thus we have obtained the relation (25). The evaluation (19) follows from (24) and (25) immediately.

Corollary of Theorem 3

In case $v=v'$, we have

$$C'^{i}=B_v(\phi_{1i}\cap M^i_\lambda B_v(\{w^s C'^s\}_{s=1}^{i-1}),\ \max(L^i_\lambda,\ \phi_{iM}\cap K^i_\lambda)) \tag{26}$$

and

$$D^i_r=B_v(\phi_{1i}\cap M^i_\lambda B_v(\{w^s C'^s\}_{s=1}^{i-1}),\ \max(D^i_{r=1}L^i_\lambda,\ \phi_{iM}\cap K^i_\lambda)) \tag{27}$$

instead of (24) and (25).

Remarks 3

Observe here, that if $D^i_1\leqq 1$ for $i=1,\dots,M$, the resulting Lipschitz constant of the operator system (18) is evaluated not to be greater than the evaluation of the Lipschitz constant of the original Seidel operator system. Of course we can also deduce analogous Seidel operator system as (18) from Seidel operator systems of general kind or pseudo linear Seidel operator system of the 1-st kind. But it is not possible to establish the evaluation of Lipschitz constant not greater than those given to the original operator system.

Examples 3

Let us now evaluate the Lipschitz constant of the operator system

$$\begin{aligned} y^1_1&=0.05\,x^1+0.10\,x^2\\ y^1&=0.05\,y^1_1+0.10\,x^2\\ y^2&=7.00\,y^1+0\ 20\,x^2 \end{aligned} \tag{28}$$

deduced from the operator system discussed in Example 2 of IV, as to some metric defined on the product space $R^1\times R^2$ and compare them with the Lipschitz constants of the original operator system evaluated in Example 2 of IV. Observe here that

$$M=2$$
$$n_1=1$$
$$n_2=0.$$

Some of the evaluations of the Lipschitz constants are given below.

1° $\lambda=(\infty;1,\dots,1)$, $v'=1$ $\quad B_v(\{a^i\}_{i=1}^M)=\max(\{a^i\}_{i=1}^M)$, $B_v'(\{a^i\}_{i=1}^M)=\sum_{i=1}^M a^i$
$v\neq v'$

i	1	2	Lipschitz constant
M^i_λ		7.0	
L^i_λ	0.05	0.20	
K^i_λ	0.10		
$M^i_\lambda \max(\{C^{\cdot}\}_{s=1}^{i-1})$		0.7525	$\mathscr{L}'_\lambda=0.9525$
D^i_1	0.15		
$D^i_1L^i_\lambda$	0.0075		
C^i	0.1075	0.9525	

2° $\lambda=(v=1;1,\dots,1)$ $B_v(\{a^i\}_{i=1}^M)=B_{v'}(\{a^i\}_{i=1}^M)=\sum_{i=1}^M a^i$
$v=v'$

i	1	2	Lipschitz constant
M_λ^i		7.00	
L_λ^i	0.05	0.20	
K_λ^i	0 10		
$M_\lambda^i\sum_{s=1}^{i-1}C^i$		0.70	$\mathscr{L}'_\lambda=1.00$
D_1^i	0.10		
$D_1^iL_\lambda^i$	0.005		
C^i	0.10	0.90	

3° $\lambda=(v=2;1,\dots,1)$ $B_v(\{a^i\}_{i=1}^M)=B_{v'}(\{a^i\}_{i=1}^M)=(\sum_{i=1}^M(a^i)^2)^{1/2}$

$v=v'$

i	1	2	Lipschitz constant
$(M_\lambda^i)^2$		49.0	
$(L_\lambda^i)^2$	0.0025	0 04	
$(K_\lambda^i)^2$	0.01		
$(M_\lambda^i)^2\sum_{s=1}^{i-1}(C^i)^2$		0.4912	$(\mathscr{L}'_\lambda)^2=0.5053$ $\mathscr{L}'_\lambda=0.7108$
$(D_1^i)^2$	0.01		
$(D_1^iL_\lambda^i)^2$	0.000025		
$(C^i)^2$	0.010025	0.495225	

The remainder of this section will be devoted to the evaluation of the Lipschitz constants associated to the particular type of the pseudolinear Seidel operator system of the 2-kind as we have examined in the previous sections.

$$
\begin{aligned}
&y^1=\varphi^1(x^2,\dots,x^M)\\
&y^2=\varphi^2(y^1,x^2,\dots,x^M)\\
&\dots\dots\dots\\
&y^i=\varphi^i(y^1,\dots,y^{i-1},x^i,\dots,x^M)\\
&\dots\dots\dots\\
&y^M=\varphi^M(y^1,\dots,y^{M-1},x^M)
\end{aligned}
\tag{29}
$$

i.e., the first operator does not include x^1 among the arguments.

There may be 3 ways to evaluate the Lipschitz constant of the operator of this type.

1° Put $L_\lambda^1=0$ and apply the method given in Theorm 2 of this section.

2° Since (29) is also an operator system of the general kind with Lipschitz constants associated with the individual operators

$$
\begin{aligned}
&L_\lambda^{*i}=B_{v'}(\phi_{1i}\cap M_\lambda^i,\ L_\lambda^i,\ \phi_{Mi}\cap K_\lambda^i) \qquad i=2,\dots,M\\
&L_\lambda^{*1}=B_{v'}(K_\lambda^1)
\end{aligned}
\tag{30}
$$

or in case of $v=v'$

$$L_\lambda^{*i}=\max(\phi_{1i}\cap M_\lambda^i,\ L_\lambda^i,\ \phi_{Mi}\cap K_\lambda^i) \qquad i=2,\dots,M \tag{31}$$
$$L_\lambda^{*1}=K_\lambda^1$$

we may apply Corollary of Theorem 2 of III.

3° As it was observed in the preceeding section, (29) can be considered to be a new operator system which maps a product set $F'\subset R^2\times R^3\times\dots\times R^M$ into itself of the form

$$\begin{aligned} y^2&=\varphi^2(\varphi^1({}^2x),\ {}^2x)\\ y^3&=\varphi^3(\varphi^1({}^2x),\ y^2,\ {}^3x)\\ &\dots\dots\dots\\ y^i&=\varphi^i(\varphi^1({}^2x),\ y^2,\dots,\ y^{i-1},\ {}^ix)\\ &\dots\dots\dots\\ y^M&=\varphi^M(\varphi^1({}^2x),\ y^2,\dots,\ y^{M-1},\ {}^Mx). \end{aligned} \tag{32}$$

This is also a Seidel operator system of the general kind with Lipschitz constants associated with the individual operators

$$L^{**i}=B_v(M_\lambda^i B_v(B_{v'}(K_\lambda^1),1),\ L_\lambda^i,\ \phi_{Mi}\cap K_\lambda^i) \qquad i=2,\dots,M \tag{33}$$

or in case of $v=v'$,

$$L^{**i}=B_v(M_\lambda^i K_\lambda^1,\ \max(M_\lambda^i,\ L_\lambda^i,\ \phi_{Mi}\cap K_\lambda^i)) \qquad i=2,\dots,M \tag{34}$$

Unfortunately, we can not establish any definite inequality relationship among the evaluations given by the method 1°, 2° and 3°. Indeed, it is not difficult to give some examples to show this fact.

VI. Pseudolinear operator system of the 3-rd kind

This section is devoted to the evaluation of the Lipschitz constant of what we define here *the pseudo linear operator system of the 3-rd kind*. This is the case, in which we have the most detailed knowledge about the operators than we have discussed before. Although we would not assume here the strict linearity of the operators, we can derive, indeed, the sum of column criterion and Sassenfeld's 1-st criterion among the results of this section.

Definition 1 (Pseudolinear operator system of the 3-rd kind)

If there exists a set of non-negative constants L_λ^{ij}, $(i,j=1,\dots,M,\ \lambda\in\Lambda)$ as to the operators

$$\begin{aligned} y^1&=\varphi^1(x)\\ &\dots\dots\dots\\ y^i&=\varphi^i(x)\\ &\dots\dots\dots\\ y^M&=\varphi^M(x) \end{aligned} \tag{1}$$

mapping a product set $F=F^1\times F^2\times\ldots\times F^M\subset R(X,\ \rho_\lambda)$, $(F^i\subset R^i,\ i=1,\ 2,\ldots,\ M)$ into R^i respectively, such that for any two elements,

$$\rho^i(y^i,\ y'^i)\leqq B_{v'}(\{L_\lambda^{ij}w^j\rho^j(x^j,\ x'^j)\}_{j=1}^M) \tag{2}$$

where

$$y^i=\varphi^i(x^1,\ x^2,\ldots,\ x^M)$$

$$y'^i=\varphi^i(x'^1,\ x'^2,\ldots,\ x'^M)$$

we call (1) the v'-λ-pseudo linear operator system of the 3-rd kind.

Theorem 1

The Lipschitz constant of v'-λ-pseudo linear simple operator system of the 3-rd kind defined on a complete set $F\subset R$ is evaluated as

$$L_\lambda=B_v(\{w^iB_{v'}(\{L_\lambda^{ij}\}_{j=1}^M)\}_{i=1}^M) \tag{3}$$

$$L_\lambda=\max_j(\{B_v(\{w^iL_\lambda^{ij}\}_{i=1}^M)\}_{j=1}^M) \qquad v=v' \tag{4}$$

where v is the index of B-operator corresponding to λ :

$$\lambda=(v\ ;\ w^1,\ w^2,\ldots,\ w^M).$$

If the relation

$$L_\lambda<1 \tag{5}$$

holds and if there exists a point $x_1\in F$ for which the assumptions

(i) $$x_1=\varphi(x_0)$$

(ii) $$S\{u:\ \rho_\lambda(u,\ x_1)\leqq(L_\lambda/1-L_\lambda)\,\rho_\lambda(x_1,\ x_0)\}\subset F$$

holds, we may have a process defined by the recurrence relation

$$x_{n+1}=\varphi(x_n) \tag{6}$$

with the initial point x_0 given above, converging to the unique fixed point of the v'-λ-pseudo linear simple operator system of the 3rd kind,

$$\varphi(x)=(\varphi^1(x),\ \varphi^2(x),\ldots,\ \varphi^M(x)).$$

Proof

The proof is quite similar to those given in the preceding sections and will be omitted here.

Remark 1

In case $v=v'$, the evaluation (4) is, in general, not greater than the evaluation (3). In fact, it follows from (3) that

$$\begin{aligned}B_v(\{w^iB_v(\{L_\lambda^{ij}\}_{j=1}^M)\}_{i=1}^M)&=B_v(\{w^iL_\lambda^{ij}\}_{i=1,\ j=1}^M)\\&=B_v(\{B_v(\{w^iL_\lambda^{ij})_{i=1}^M)\}_{j=1}^M)\\&\geqq\max_j(\{B_v(\{w^iL_\lambda^{ij}\}_{i=1}^M)\}_{j=1}^M).\end{aligned}$$

Corollary of Theorem 1

$v>1$

In case $v'=1$, we can evaluate the Lipschitz constant of the operator system (1) as follows

$$B_v(\{w^i \cdot B_v^-(\{L_\lambda^{ij}\}_{j=1}^M)\}_{i=1}^M) \tag{7}$$

where B_v^- is the complementary operator of the operator B_v.

Proof

We can show this easily by Hölder's inequality.

Remark 2

In case $v=v'=1$, the evaluation of the Lipschitz constant by (4) is in general not greater than the evaluation by (7).

In fact, in this case (4) and (7) becomes

$$\max_j (\{\sum_{i=1}^M (w^i L_\lambda^{ij})\}_{j=1}^M) \tag{4'}$$

and

$$\sum_{i=1}^M \{w^i(\max_j \{L_\lambda^{ij}\}_{j=1}^M)\} \tag{7'}$$

respectively and from which the above statement follows immediately.

Example 1

Consider now, a system of M linear operators defined on M-dimensional space $R_M(X, \rho_\lambda)$

$$\begin{aligned} y^1 &= \varphi^1(\boldsymbol{x}) = a_{11}x^1 + a_{12}x^2 + \ldots + a_{1M}x^M \\ &\cdots\cdots\cdots \\ y^i &= \varphi^i(\boldsymbol{x}) = a_{i1}x^1 + a_{i2}x^2 + \ldots + a_{iM}x^M \\ &\cdots\cdots\cdots \\ y^M &= \varphi^M(\boldsymbol{x}) = a_{M1}x^1 + a_{M2}x^2 + \ldots + a_{MM}x^M \end{aligned} \tag{8}$$

or in vector-matrix form

$$\boldsymbol{y} = \begin{pmatrix} y^1 \\ \vdots \\ y^M \end{pmatrix} = \boldsymbol{\varphi}(\boldsymbol{x}) = \begin{pmatrix} \varphi^1(\boldsymbol{x}) \\ \vdots \\ \varphi^M(\boldsymbol{x}) \end{pmatrix} = \begin{pmatrix} a_{11} & \ldots & a_{1M} \\ \vdots & & \vdots \\ a_{M1} & \ldots & a_{MM} \end{pmatrix} \cdot \begin{pmatrix} x^1 \\ \vdots \\ x^2 \end{pmatrix}.$$

Reminding that in this case,

$$v'=1 \qquad \text{i.e. } B_{v'}(\{a^i\}_{i=1}^M) = \sum_{i=1}^M a^i,$$

we may evaluate the Lipschitz constant by the theorem above as in the table on the next page.

Remark that (9) and (11) are the same as (15) and (17), III and (10) is in general not greater than (16). The convergence criterion associated with (10) is called sum of column criterion ⟨1⟩, ⟨8⟩. Remark also that this criterion can be obtained only

Metric	Lipschitz constant	
$\lambda=(v=\infty\,;\ 1,\ldots,\ 1)$ $B_v(\{a^i\}_{i=1}^M)=\max(\{a^i\}_{i=1}^M)$	$L_\lambda=\max_i(\{\sum_{j=1}^M \lvert a_{ij}\rvert\}_{i=1}^M)$	(9)
$\lambda=(v=1\,;\ 1,\ldots,\ 1)$ $B_v(\{a^i\}_{i=1}^M)=\sum_{i=1}^M a^i$	$L_\lambda=\max_j(\{\sum_{i=1}^M \lvert a_{ij}\rvert\}_{j=1}^M)$	(10)
$\lambda=(v>1\,;\ 1,\ldots,\ 1)$ $B_v(\{a^i\}_{i=1}^M)=\sum_{i=1}^M a^i$	$L_\lambda=(\sum_{i=1}^M(\sum_{j=1}^M \lvert a_{ij}\rvert^{v/v-1})^{v-1})^{1/v}$	(11)

under the assumption of the pseudolinearity of the 3-rd kind.

Example 2

Consider now, the pseudolinear operator system of the 3-rd kind with which a single constant $L_\lambda^{ij}=\bar{L}$ is associated. Let us evaluate the Lipschitz constant L_λ of the pseudolinear operator system corresponding to $\bar{L}$ and the value $\tilde{L}$ of $\bar{L}$ which makes the Lipschitz constant L_λ equal to unity for some metrics $\lambda\in\Lambda$. If we have the Lipschitz constant less than this value, then we see, considering the monotone increasing property of B-operator, that the first and the main condition for convergence is fulfilled.

1° $\lambda=(\infty\,;\ 1,\ldots,\ 1)$ $v'=1$ i.e., $B_v(\{a^i\}_{i=1}^M)=\max\{a^i\}_{i=1}^M$

$$B_{v'}(\{a^i\}_{i=1}^M)=\sum_{i=1}^M a^i$$

$$L_\lambda=M\bar{L},\quad \tilde{L}=1/M \tag{12}$$

2° $\lambda=(v>1\,;\ 1,\ldots,\ 1)$ $v'=1$ i.e., $B_v(\{a^i\}_{i=1}^M)=(\sum_{i=1}^M(a^i)^v)^{1/v}$,

$$B_{v'}(\{a^i\}_{i=1}^M)=(\sum_{i=1}^M(a^i))$$

$$L=M\bar{L},\quad \tilde{L}=1/M \tag{13}$$

3° $\lambda=(1\,;\ 1,\ldots,\ 1)$ $v'=1$ i.e., $B_v(\{a^i\}_{i=1}^M)=\sum_{i=1}^M a^i$

$$L=M\bar{L},\quad \tilde{L}=1/M \tag{14}$$

Now, let us turn to the evaluation of the Lipschitz constant of the pseudolinear Seidel operator system of the 3rd kind $\boldsymbol{\phi}(\boldsymbol{x})$.

Theorem 2

The Lipschitz constant $\mathscr{L}_\lambda$ of v'-λ-pseudo linear Seidel operator system of the 3rd kind $\boldsymbol{\phi}(\boldsymbol{x})$ corresponding to (1) mapping a complete product set $\boldsymbol{F}=F^1\times F^2\times\ldots\times F^M$, $F^i\subset R^i$, $i=1,\ 2,\ldots,\ M$ into itself is evaluated as follows:

$$\mathscr{L}_\lambda=B_v(\{w^iC^i\}_{i=1}^M),\qquad \lambda=(v\,;\ w^1,\ w^2,\ldots,\ w^M) \tag{15}$$

where

$$C^i = B_{v'}(\phi_{i1} \cap \{L_\lambda^{ij} w^j C^j\}_{j=1}^{i-1}, \{L^{ij}\}_{j=1}^{M}) \tag{16}$$
$$C^1 = B_{v'}(\{L_\lambda^{1j}\}_{j=1}^{M})$$

so that if we assume that

$$\mathscr{L}_\lambda = B_v(\{w^i C^i\}_{i=1}^{M}) < 1, \tag{17—i}$$

there exist x_0 such that $x_1 = \phi(x_0)$ belongs to F, (17—ii)

$S\{u : \rho_\lambda(u, x_1) \leqq (\mathscr{L}_\lambda / 1 - \mathscr{L}_\lambda)\, \rho_\lambda(x_0, x_1)$ is contained in F, (17—iii)

then the Seidel process corresponding to (1) converges to the unique fixed point of the operator system in F.

In case $v = v'$, we have

$$C^{i\prime} = B_v(\phi_{i1} \cap \{L_\lambda^{ij} w^j C^j\}_{j=1}^{i-1}, \max \{L_\lambda^{ij}\}_{j=i}^{M}) \tag{18}$$
$$C^{1\prime} = B_v \max (\{L_\lambda^{1jj}\}_{j=1}^{M})$$

instead of (16). And the evaluation via (18) is in general not greater than that given via (16).

Proof

The proof is similar to those given in the preceding sections and will be omitted here.

Corollary of Theorem 2

$v \geq 1$

In case $v' = 1$, we may also use the constants

$$C''^i = B_1(\phi_{i1} \cap \{L^{ij} w^j C^j\}_{j=1}^{i-1} \cup B_{\bar{v}}(\{L^{ij}\}_{j=1}^{M})) \tag{21}$$
$$C''^i = B_{\bar{v}}(\{L^{ij}\}_{j=1}^{M})$$

instead of (16). The evaluation of the Lipschitz constant via (21) is in general not greater than the evaluation via (16).

Remark 3

We may also evaluate the Lipschitz constant of the pseudolinear Seidel operator system of the 3-rd kind reconsidering it as a pseudolinear Seidel operator of the 2-nd, 1-st and as the general kind.

1° First, let us consider it again as an operator system of the 2-nd kind. Let M_λ^{*i}, L_λ^{*i}, and K_λ^{*i} represent the constants associated with the pseudolinear operator system of the 2-nd kind. These constants can be represented in terms of the constants L_λ^{ij} associated with the pseudolinear operator system of the 3-rd kind as follows

$$M_\lambda^{*i} = B_v'(\{L_\lambda^{ij}\}_{j=1}^{i-1}) \tag{22}$$

$$L_\lambda^{*i} = L_\lambda^{ii} \tag{23}$$

$$K_\lambda^{*i} = B_v'(\{L_\lambda^{ij}\}_{j=i+1}^{M}). \tag{24}$$

In case $v = v'$, we have also the relations

$$L_\lambda^{*'i} = \max(\{L_\lambda^{ij}\}_{j=1}^{i-1}) \tag{22'}$$

$$K_\lambda^{*'i} = \max(\{L^{ij}\}_{j=i+1}^{M}) \tag{24'}$$

instead of the relations (22) and (24). (22′) (24′) are, in general, not greater than (22) and (24). These relations can be derived by a similar argument as above.

It can be shown also by a similar argument as above, that we may use the relations as follows, in case $v'=1$.

$$L_\lambda^{*''i} = B_{\bar{v}}(\{L_\lambda^{ij}\}_{j=1}^{i-1}) \tag{22''}$$

$$K_\lambda^{*''i} = B_{\bar{v}}(\{L_\lambda^{ij}\}_{j=i+1}^{M}) \tag{24''}$$

If we now substitute the relations (22), (23) and (24) or (22′), (23) and (24′) into (7) or (9) of V and represent by C^{*i}, $C^{*'i}$ and $\mathscr{L}_\lambda{}^*$, $\mathscr{L}_\lambda{}^{*'}$ the resulting constants, we have the relations

$$C^{*i} \geqq C^i \tag{25}$$

and

$$C^{*'i} \geqq C^{\,i}, \qquad v=v' \tag{25'}$$

thus

$$\mathscr{L}_\lambda{}^* \geqq \mathscr{L}_\lambda \tag{26}$$

and

$$\mathscr{L}_\lambda{}^{*'} \geqq \mathscr{L}_\lambda{}'. \qquad v=v' \tag{26'}$$

In fact,

$$\begin{aligned} C^{*1} &= B_v'(L_\lambda^{*1}, K_\lambda^{*1}) = B_v'(L_\lambda^{11}, B_v'(\{L_\lambda^{ij}\}_{j=2}^{M})) \\ &= B_v'(\{L_\lambda^{1j}\}_{j=1}^{M}) = C^1 \end{aligned}$$

and

$$\begin{aligned} C^{*'1} &= B_v(\max(L_\lambda^{11}, \max(\{L_\lambda^{ij}\}_{j=2}^{M}))) \\ &= B_v(\max(\{L_\lambda^{1j}\}_{j=1}^{M})) = C^1 \end{aligned}$$

Assuming that (25) and (25′) hold for $1,\dots, i-1$, we see for i,

$$\begin{aligned} C^{*i} = B_v'(\phi_{i1} &\cap B_v'(\{L_\lambda^{ij}\}_{j=1}^{i-1}) B_v(\{w^r C^{*r}\}_{r=1}^{i-1}) \\ &\cup B_v'(L_\lambda^{ii} \cup \phi_{iM} \cap B_{v'}(\{L_\lambda^{ij}\}_{j=i+1}^{M}))) \\ = B_v'(\phi_{i1} &\cap B_v'(\{L_\lambda^{ij}\}_{j=1}^{i-1}) B_v(\{w^r C^{*r}\}_{r=1}^{i-1}) \cup \{L_\lambda^{ij}\}_{j=i}^{M}) \end{aligned}$$

and in case $v=v'$

$$\begin{aligned} C^{*'i} = B_v(\phi_{i1} &\cap \max(\{L_\lambda^{ij}\}_{j=1}^{i-1}) B_v(\{w^r C^{*'r}\}_{r=1}^{i-1}) \\ &\cup \max(L_\lambda^{ii} \cup \phi_{iM} \cap \max(\{L_\lambda^{ij}\}_{j=i+1}^{M}))) \\ = B_v(\phi_{i1} &\cap \max(\{L_\lambda^{ij}\}_{j=1}^{i-1}) B_v(\{w^r C^{*'r}\}_{r=1}^{i-1}) \max(\{L_\lambda^{ij}\}_{j=1}^{M})). \end{aligned}$$

Reminding that

$$\begin{aligned} B_v'(\{L_\lambda^{ij}\}_{j=1}^{i-1}) B_v(\{w^r C^r\}_{r=1}^{i-1}) &\geqq \max(\{L_\lambda^{ij}\}_{j=1}^{i-1}) B_v(\{w^r C^r\}_{r=1}^{i-1}) \\ &\geqq B_v'(\{L_\lambda^{ij} w^j C^j\}_{j=1}^{i-1}), \end{aligned} \tag{27}$$

we see that (25) and (25′) hold for $i=1, 2,\dots, M$. Thus we have the relations (26) and (26′).

In case $v'=1$, $v>1$, substitute (22″), (23) and (24″) into (7) of V and represent the resulting constants by $C^{*\prime\prime}$ and $\mathscr{L}_\lambda^{*\prime\prime}$. Then we have the relations

$$C^{*\prime\prime i} \geqq C^i \tag{25″}$$

and

$$\mathscr{L}_\lambda^{*\prime\prime} \geqq \mathscr{L}_\lambda. \tag{26″}$$

In fact, for $i=1$,

$$C^{*\prime\prime 1}=B_1(L_\lambda^{11},\ B_v^-(\{L_\lambda^{1j}\}_{j=2}^M))$$

On the other hand, since

$$C^{\prime\prime 1}=B_1(L_\lambda^{11},\ B_1(\{L_\lambda^{1j}\}_{j=2}^M))$$

We see readily

$$C^{*\prime\prime 1} \leqq C^{\prime\prime 1}.$$

Assume now, that the relations (25″) hold for 1, 2,..., $i-1$. Then, for i,

$$C^{*\prime\prime i}=B_1(\phi_{1i}\cap B_v^-(\{L_\lambda^{ij}\}_{j=1}^{i-1})B_v(\{w^r C^{\prime\prime *r}\}_{r=1}^{i-1})\cup B_v(L_\lambda^{ij},\ \phi_{Mi}\cap B_v^-(\{L_\lambda^{ij}\}_{j=i+1}^M)))$$

On the other hand, it follows from (21),

$$C^{\prime\prime i}=B_1(\phi_{1i}\cap \{L_\lambda^{ij}w^j C^{\prime\prime j}\}_{j=1}^{i-1}\cup B_v^-(\{L_\lambda^{ij}\}_{j=i}^M))$$

$$=B_1(\phi_{1i}\cap B_1(\{L_\lambda^{ij}w^j C^{\prime\prime j}\}_{j=1}^{i-1})\cup B_v^-(L_\lambda^{ii},\ \phi_{Mi}\cap B_v^-(\{L_\lambda^{ij}\}_{j=i+1}^M)))$$

Comparing these two relations, the relation (25″) thus (26″) follow according to Hölder's inequality.

2° Next let us consider the pseudo linear Seidel operator system of the 3-rd kind as a pseudo linear Seidel operator system of the 1-st kind. Let M_λ^{**i} and L_λ^{**i} represent the constants associated with the pseudo linear operator system of the 1-st kind. These constants can be expressed in terms of the constants L_λ^{ij} associated with the pseudo linear operator system of the 3-rd kind as follows.

$$M_\lambda^{**i}=B_v'(\{L_\lambda^{ij}\}_{j=1}^{i-1}) \tag{28}$$

$$M_\lambda^{**i'}=\max(\{L_\lambda^{ij}\}_{j=1}^{i-1}) \qquad v=v' \tag{28'}$$

$$L_\lambda^{**i}=B_v'(\{L_\lambda^{ij}\}_{j=1}^{i-1}) \tag{29}$$

$$L_\lambda^{**i'}=\max(\{L_\lambda^{ij}\}_{j=1}^{M}). \qquad v=v' \tag{29'}$$

But here, it may be readily seen that if we evaluate the constants M_λ^{**i} and L_λ^{**i} according to the Remark 1 of V, via (22), (22′), (23), (24) and (24′), we would obtain the same results as (28), (28′), (29) and (29) respectively. Then, by 1°, we see that the evaluation of the Lipschitz constant by the method given in Theorem 2, IV is in general not smaller than the evaluation in Theorem 2 of this section.

3° Finally, let us consider the pseudolinear Seidel operator system of the 3-rd kind as a pseudolinar operator system of the general kind. Let L_λ^{***i} represent the Lipschitz constants associated with the individual operators constituting the system. L_λ^{***i} in terms of the constants L_λ^{ij} of the pseudolinear operator system of the 3-rd kind, may be expressed as follows.

$$L_\lambda^{***i}=B_{v'}(\{L_\lambda^{ij}\}_{j=1}^M) \tag{30}$$

$$L_\lambda^{***'i}=\max(\{L_\lambda^{ij}\}_{j=1}^M) \qquad v=v' \tag{30'}$$

$$L_\lambda^{***''i}=B_v^-(\{L_\lambda^{ij}\}_{j=1}^M), \qquad v'=1, \quad v\geqq 1. \tag{30''}$$

If we now, substitute (30), (30′) and (30″) into (24) and (25) of III and represent the resulting constants by C^{***i} and $\mathscr{L}_\lambda^{***}$ we have the relations

$$C^{***i} \geqq C^i \tag{31}$$

$$C^{***'i} \geqq C'^i \tag{31'}$$

$$C^{***''i} \geqq C''^i \tag{31''}$$

which follow by analogous inductions as in 1° and thus

$$\mathscr{L}_\lambda^{***} \geqq \mathscr{L}_\lambda \tag{32}$$

$$\mathscr{L}_\lambda^{***'} \geqq \mathscr{L}_\lambda' \tag{32'}$$

$$\mathscr{L}_\lambda^{***''} \geqq \mathscr{L}_\lambda''. \tag{32''}$$

Example 2

Consider a Seidel operator system corresponding to

$$\begin{aligned} y^1&=1/2\min(x^1, x^2)+3 \\ y^2&=1/3\max(x^1, x^2)+2 \end{aligned} \tag{33}$$

for which it will be seen immediately that the following two relations hold:

$$\begin{aligned} \rho(y^1, y'^1)&\leqq\max(1/2\rho(x^1, x'^1),\ 1/2\rho(x^2, x'^2)) \\ \rho(y^2, y'^2)&\leqq\max(1/3\rho(x^1, x'^1),\ 1/3\rho(x^2, x'^3)), \end{aligned} \tag{34}$$

thus

$$\begin{aligned} L^{11}&=L^{12}=1/2 \\ L^{21}&=L^{22}=1/3. \end{aligned} \qquad v'=\infty \qquad w^1=w^2=\ldots=w^M=1 \tag{35}$$

The evaluations of the Lipschitz constant of the Seidel operator system corresponding to (33) are as follows.

v	C^i	Lipschitz constant
∞	$C^1=\max(1/2,\ 1/2)=1/2$ $C^2=\max(1/3\cdot1/2,\ 1/3)=1/3$	$\mathscr{L}=\max(1/2,\ 1/3)=1/2$
2	$C^1=\max(1/2,\ 1/2)=1/2$ $C^2=\max(1/3\cdot1/2,\ 1/3)=1/3$	$\mathscr{L}=((1/2)^2+(1/3)^2)^{1/2}=\frac{\sqrt{13}}{6}$
1	$C^1=\max(1/2,\ 1/2)=1/2$ $C^2=\max(1/3\cdot1/2,\ 1/3)=1/3$	$\mathscr{L}=\max(1/2,\ 1/3)=1/2$

From the evaluations above, we see that there exist a unique solution of the system of equations

$$\begin{aligned} x^1&=1/2\min(x^1, x^2)+3 \\ x^2&=1/3\max(x^1, x^2)+2. \end{aligned} \tag{36}$$

Example 3

Consider now, the system of M linear operators defined on M-dimensional space $R_M(X, \rho_\lambda)$

$$\begin{aligned} y^1 &= \varphi^1(x) = a_{11}x^1 + a_{12}x^2 + \ldots + a_{1M}x^M \\ &\cdots\cdots\cdots \\ y^i &= \varphi^i(x) = a_{i1}x^1 + a_{i2}x^2 + \ldots + a_{iM}x^M \\ &\cdots\cdots\cdots \\ y^M &= \varphi^M(x) = a_{M1}x^1 + a_{M2}x^2 + \ldots + a_{MM}x^M \end{aligned} \tag{37}$$

or in matrix form

$$\begin{pmatrix} y^1 \\ \cdots \\ \cdots \\ y^i \\ \cdots \\ \cdots \\ y^M \end{pmatrix} = \begin{pmatrix} a_{11}, & a_{12}, & \ldots, & a_{1M} \\ \cdots\cdots\cdots \\ a_{i1}, & a_{i2}, & \ldots, & a_{iM} \\ \cdots\cdots\cdots \\ a_{M1}, & a_{M2}, & \ldots, & a_{MM} \end{pmatrix} \begin{pmatrix} x^1 \\ \cdots \\ \cdots \\ x^i \\ \cdots \\ \cdots \\ x^M \end{pmatrix} \tag{38}$$

It may be clear that corresponding Seidel operator system is a pseudo linear Seidel operator system of 3-rd kind as to the pairs of the form $(\lambda, v'=1)$ with the constants $L_\lambda^{ij} = |a^{ij}|$. Some of the evaluations of the Lipschitz constant of the Seidel operator system are given in the following. (cf. Corollary of Theorem 2 of this section.)

metrics	Lipschitz constant
$\lambda = (v=\infty\,;\ 1, \ldots, 1)$	(39)* $\mathscr{L} = \max(\{C^i\}_{i=1}^M)$ $C^i = \sum_{j=1}^{i-1} \lvert a_{ij}\rvert C^j + \sum_{j=i}^{M} \lvert a_{ij}\rvert$ $C^1 = \sum_{j=1}^{M} \lvert a_{1j}\rvert$
$\lambda = (v=1\,;\ 1, \ldots, 1)$	(40) $\mathscr{L} = \sum_{i=1}^{M} C^i$ $C^i = \sum_{j=1}^{i-1} \lvert a_{ij}\rvert C^j + \max\{\lvert a_{ij}\rvert\}_{j=1}^M$ $C^1 = \max(\lvert a_{1j}\rvert_{j=1}^M)$
$\lambda = (v=2\,;\ 1, \ldots, 1)$	(41) $\mathscr{L} = (\sum_{i=1}^{M} (C^i)^2)^{1/2}$ $C^i = \sum_{j=1}^{i-1} \lvert a_{ij}\rvert C^j + (\sum_{j=i}^{M} (a_{ij})^2)^{1/2}$ $C^1 = (\sum_{j=1}^{M} (a_{1j})^2)^{1/2}$

Let us now turn to consider a class of operator systems deduced from the pseudo linear Seidel operator systems of the 3-rd kind which have the following form.

* This is the Lipschitz constant evaluated in the 1st criterion of Sassenfeld (Kriterium I, ⟨9⟩)

$$
\begin{aligned}
&\quad y_1^1=\varphi^1(x^1,\ x^2,\dots,\ x^M)\\
&\quad y_2^1=\varphi^1(y_1^1,\ x^2,\dots,\ x^M)\\
&\quad \cdots\cdots\cdots\\
&\quad y_{n_1}^1=\varphi^1(y_{n_1-1}^1,\ x^2,\dots,\ x^M)\\
&y^1=\varphi^1(y_{n_1}^1,\ x^2,\dots,\ x^M)\\
&\quad y_1^2=\varphi^2(y^1,\ x^2,\ x^3,\dots,\ x^M)\\
&\quad y_2^2=\varphi^2(y^1,\ y_1^2,\ x^3,\dots,\ x^M)\\
&\quad \cdots\cdots\cdots\\
&\quad y_{n_2}^2=\varphi^2(y^1,\ y_{n_2-1}^2,\ x^3,\dots,\ x^M)\\
&\quad \cdots\cdots\cdots\\
&\cdots\cdots\cdots\\
&\quad y_1^i=\varphi^i(y^1,\ y^2,\dots,\ y^{i-1},\ x^i,\dots,\ x^M)\\
&\quad y_2^i=\varphi^i(y^1,\ y^2,\dots,\ y^{i-1},\ y_1^i,\ x^{i+1},\dots,\ x^M)\\
&\quad \cdots\cdots\cdots\\
&\quad y_{n_i}^i=\varphi^i(y^1,\ y^2,\dots,\ y^{i-1},\ y_{n_i-1}^i,\ x^{i+1},\dots,\ x^M)\\
&y^i=\varphi^i(y^1,\ y^2,\dots,\ y^{i-1},\ y_{n_i}^i,\ x^{i+1},\dots,\ x^M)\\
&\quad \cdots\cdots\cdots\\
&\cdots\cdots\cdots\\
&\quad y_1^M=\varphi^M(y^1,\ y^2,\dots,\ y^{M-1},\ x^M)\\
&\quad y_2^M=\varphi^M(y^1,\ y^2,\dots,\ y^{M-1},\ y_1^M)\\
&\quad \cdots\cdots\cdots\\
&\quad y_{n_M}^M=\varphi^M(y^1,\ y^2,\dots,\ y^{M-1},\ y_{n_M-1}^M)\\
&y^M=\varphi^M(y^1,\ y^2,\dots,\ y_{n_M}^M).
\end{aligned}
\tag{42}
$$

That is, the i th operator in the original operator system is repeated n_i+1 times in the deduced operator system. Observe here that the fixed points of both the original and the deduced operator system coincide if they exist.

Now, let us evaluate the Lipschitz constant of the deduced operator system (42).

Theorem 3

The Lipschitz constant $\mathscr{L}_\lambda$ of the operator system (42) defined on a product set $F'\subset \boldsymbol{R}$ is evaluated as follows

$$\mathscr{L}_\lambda=B_v(\{w^iC^i\}_{i=1}^M) \tag{44}$$

where

$$C^i=B_{v'}(\phi_{1i}\cap\{L_\lambda^{ij}w^jC^j\}_{j=1}^{i-1}\cup\{L_\lambda'^{ii}\}\cup\{L_\lambda^{ij}\}_{j=1}^M)) \tag{45}$$

$$i=1,\ 2,\dots,\ M$$

and

$$L_\lambda'^{ii}=D_{n_i}^iL_\lambda^{ii} \tag{46}$$

where

$$D_r^i=B_{v'}(\phi_{1i}\cap\{L_\lambda^{is}w^sC^s\}_{s=1}^{i-1}\cup\{D_{r-1}^iL_\lambda^{ii}\}\cup\{L_\lambda^{ij}\}_{j=i+1}^M) \tag{47}$$

$$D_0^i=1. \tag{48}$$

Proof

The proof is quite similar to that of Theorem 3, V and omitted here.

Corollary of Theorem 3

In case $v=v'$, we have

$$C'^{i}=B_{v}'(\phi_{1i}\cap\{L_{\lambda}^{ij}w^{j}C'^{i}\}_{j=1}^{i-1}\cup\max(L_{\lambda}'^{ii},\ \{L_{\lambda}^{ij}\}_{j=i+1}^{M})) \tag{49}$$

and

$$D'^{i}=B_{v}'(\phi_{1i}\cap\{L_{\lambda}^{ij}w^{j}C'^{j}\}_{j=1}^{i-1}\cup\max(D_{r-1}'^{i}L_{\lambda}^{ii},\ \{L_{\lambda}^{ij}\}_{j=i+1}^{M})) \tag{50}$$

instead of (45) and (47).

In case $v'=1$, $v\geqq 1$, we have

$$C''^{i}=B_{1}(\phi_{1i}\cap\{L_{\lambda}^{ij}w^{j}C''^{j}\}_{j=1}^{i-1}\cup B_{v}^{-}(L^{ii''},\ \{L_{\lambda}^{ij}\}_{j=i+1}^{M})) \tag{51}$$

$$D''^{i}=B_{1}(\phi_{1i}\cap\{L_{\lambda}^{ij}w^{j}C''^{j}\}_{j=1}^{i-1}\cup B_{v}^{-}(D_{r-1}^{i}L^{ii''},\ \{L_{\lambda}^{ij}\}_{j=i+1}^{M})) \tag{52}$$

instead of (45) and (47).

The remainder of this section will be devoted to the evaluation of the Lipschitz constants associated with the particular type of the pseudolinear Seidel operator of the 3-rd kind as in the previous sections.

$$\begin{aligned}
&y^{1}=\varphi^{1}(x^{2},\ x^{3},\dots,\ x^{M})\\
&y^{2}=\varphi^{2}(y^{1},\ x^{2},\dots,\ x^{M})\\
&\dots\dots\dots\\
&y^{i}=\varphi^{i}(y^{1},\dots,\ y^{i-1},\ x^{i},\dots,\ x^{M})\\
&\dots\dots\dots\\
&y^{M}=\varphi^{M}(y^{1},\dots,\ y^{M-1},\ x^{M}),
\end{aligned} \tag{53}$$

i.e. the first operator does not include x^{1} in its arguments.

There may be 2 ways to evaluate the Lipschitz constant of the Seidel operator of this type.

1° Put $L^{11}=0$, and apply the methods given in Theorem 2 or its corollary in case $v'=1$, $v\geqq 1$.

2° As it was observed in the preceding sections, (53) can be considered to be a new operator system which maps a product set $F'\subset R^{2}\times R^{3}\times\dots\times R^{M}$ into itself of the form

$$\begin{aligned}
&y^{2}=\varphi^{2}(\varphi^{1}({}^{2}\boldsymbol{x}),\ {}^{2}\boldsymbol{x})\\
&y^{3}=\varphi^{3}(\varphi^{1}({}^{2}\boldsymbol{x}),\ y^{2},\ {}^{3}\boldsymbol{x})\\
&\dots\dots\dots\\
&y^{i}=\varphi^{i}(\varphi^{1}({}^{2}\boldsymbol{x}),\ y^{2},\dots,\ y^{i-1},\ {}^{i}\boldsymbol{x})\\
&\dots\dots\dots\\
&y^{M}=\varphi^{M}(\varphi^{1}({}^{2}\boldsymbol{x}),\ y^{2},\dots,\ y^{M-1},\ {}^{M}\boldsymbol{x}).
\end{aligned}$$

As it is easily shown, this is also a Seidel operator system of the 3-rd kind with constants

$$L_\lambda^{*ij}=B_v'(L_\lambda^{ij}\cup w^1L_\lambda^{i1}L_\lambda^{1j}) \qquad i,\ j=2,\ 3,\dots,\ M$$

With these constants, we may evaluate the Lipschitz constant of the Seidel operator system of this type by Theorem 2 of this section or its corollary in case $v'=1$ and $v\geqq 1$.

Unfortutately, we can not establish any definite inequality relationships between the evaluations of the Lipschitz constant by methods 1° and 2°. This fact can be shown by some examples.

VII. Summary and General Remark

So far, we have investigated the methods of evaluation of the Lipschitz constants of operator systems of various kinds and forms. And we saw, that we can obtain the Lipschitz constants by evaluating the representation

$$B_v(\{w^iC^i\}_{i=1}^M) \tag{1}$$

where C^i is the constants given by recurrence relations which correspond to the construction of the operator system and the kinds of the constants given as to the individual operators constituting the system. And we have assumed so far in most cases that the operator systems, so to say, consist of the operators of a single kind. But it is also possible to consider *Seidel system of the mixed kind.* In this case, the constant C^i can not be given by a single recurrence relation but by different forms of recurrence relations of the individual operators and the associated constants. And moreover, it is even possible to convert the construction of the operator, if there exist a set of possibilities to consider the i th operator to be an operator of particular kind. In other words, if it is possible to give C^i in form

$$C^i=f_\alpha(C^1,\ C^2,\dots,\ C^{i-1}) \qquad \alpha\in A$$

where α correspoeds to the kind of the operators to whom the i th operator considered to belong, then it might be advisable to convert the system for C^i so as to take the minimum as to α, in order to obtain smaller evaluations of the Lipschitz constants :

$$C^i=\min f_\alpha(C^1,\ C^2,\dots,\ C^{i-1}).$$

Observe that it follows from this arguments that the Seidel operator system of the general kind is not advantageous than the corresponding simple operator so far the Lipschitz constants are concerned.

It may be also a problem to permute the order of the arguments $x^1,\ x^2,\dots,\ x^M$ to have smaller evaluations of the Lipschitz constant of the system. But unfortunately we could not reached at any simple algorithms to have the most advantageous ordering of the arguments except to evaluate the Lipschitz constant corresponding to every permutation, although we may esimate more advantageous ordering of the arguments by inspecting the recurrence relations and given constants.

The main recurrence relations evolved in this article is summarized in a table in page 51.

Acknowledgement

The author is much indebted to the constant encouragement and many sugesstions given by the professors of Department of Administration Engineering of Keio University, especially to Profs. Z. Yamauchi, H. Sakamoto, S. Ura and T. Sekine. Also acknowledgement must be shown to Prof. Hitotsumatsu of Rikkyo University and Prof. Iri of Tokyo Univ. to their invaluable advices and suggestions.

References

⟨1⟩ Collatz, L. Einige Anwendungen funktionalanalytischer Methoden in der praktischen Analysis, ZAMP, Bd. IV, 1953, ss 327—357

⟨2⟩ Collatz, L. Iterationsverfahren für komplexe Nullstellen algebraischer Gleichungen, ZAMM Bd. 30, Nr 4, April 1950

⟨3⟩ Sassenfeld, H. Zur Iteration bei linearen Gleichungen, ZAMM Bd. 30. Nr. 8/9 Aug. /Sept. 1950

⟨4⟩ Collatz, L. Zur Herleitung von Kovergenzkriterien für Iterationsverfahren bei linearen Gleichungssystemen ZAMM Bd. 30. Nr. 8/9 Aug. /Sept. 1950

⟨5⟩ Bodewig, E. Konvergenztypen und das Verhalten von Approximation in der Nähe einer mehrfachen Wurzel einer Gleichung, ZAMM, Bd. 29. Nr. 1/2. J. / Feb. 1949

⟨6⟩ Moriguti, S. Further Results in the Theory of Numerical Convergence, Recent Developments in Information and Decision Process, Ed. by Machol, R. E. & 1. Gray, Macmillan, 1962

⟨7⟩ Urabe, M. Convergence of Numerical Iteration in Solution of Equations, Journal of Science of Hiroshima Uuiv. Series A Vol. 19 No. 3, Jan. 1956

⟨8⟩ Демидович, Б. П. и И. А. Марон, Основы Вычислительной Математики, Физматгиз, Москва, 1963

⟨9⟩ Sassenfeld, H. Ein hinreichendes Konvergenzkritrium und Fehlerabschätzung für die Iteration in Einzelschritten bei linearen Gleichungen ZAMM Bd. 31, Nr. 3, (März) 1951

⟨10⟩ Forsythe, G. E. aud W. R. Wasow, Finite Difference Methds for Partial Differential Equation J. Wiley & Sons, 1960

⟨11⟩ Bialy, H. Iterative Lösung von Funktionalgleichungen erster Art, ZAMM Bd. 38 Nr. 7/8 Juli/Aug. 1958

Evaluation of Lipschitz constants of operator systems

kind of operator system	Constants associated with the individual operators	Evaluations of Lipschitz constants of operator system	
general kind	$\rho^i(y^i, y'^i) \leqq L^i_\lambda \rho_\lambda(x, x')$	*Simple system*	$L_\lambda = B_v(\{w^i C^i\}^M_{i=1})$ $C^i = L^i_\lambda$
		Seidel system	$\mathscr{L}_\lambda = B_v(\{w^i C^i\}^M_{i=1})$ $C^i = B_v(\phi_{1i} \cap \{w^j C^j L^i_\lambda\}^{i-1}_{j=1}), L^i_\lambda)$
pseudo linear operator system of the 1 st kind	$\rho^i(y^i, y'^i) \leqq B_v'(M^i_\lambda \rho_\lambda({}_{i-1}x, {}_{i-1}x'),$ $L^i_\lambda \rho_\lambda({}^i x, {}^i x'))$	*Simple system*	$L_\lambda = B_v(\{w^i C^i\}^M_{i=1})$ $C^i = B_v'(\phi_{1i} \cap M^i_\lambda, L^i_\lambda)$ $C^i = \max(\phi_{i1} \cap M^i_\lambda, L^i_\lambda)$ $v = v'$
		Seidel system	$\mathscr{L}_\lambda = B_v(\{w^i C^i\}^M_{i=1})$ $C^i = B_v'(\phi_{i1} \cap M^i_\lambda B_v(\{w^j C^j\}^{i-1}_{j=1}), L^i_\lambda)$
pseudo linear operator system of the 2 nd kind	$\rho^i(y^i, y'^i) \leqq B_v'((\phi_{1i} \cap M^i_\lambda \rho({}_{i-1}x, {}_{i-1}x'))$ $\cup \{L^i_\lambda w^i \rho^i(x^i, x'^i)\}$ $\cup \{\phi_{Mi} \cap K^i_\lambda \rho_\lambda({}^{i+1}x, {}^{i+1}x')\})$	*Simple system*	$L_\lambda = B_v(\{w^i C^i\}^M_{i=1})$ $C^i = B_v'((\phi_{1i} \cap M^i_\lambda), L^i, (\phi_{Mi} \cap K^i_\lambda))$ $C^i = \max((\phi_{1i} \cap M^i_\lambda), L^i, (\phi_{Mi} \cap K^i_\lambda))$ $v = v'$
		Seidel system	$\mathscr{L}_\lambda = B_v(\{w^i C^i\}^M_{i=1})$ $C^i = B_v'(\phi_{1i} \cap M^i_\lambda B_v(\{w^j C^j\}^{i-1}_{j=1}), L^i_\lambda, (\phi_{Mi} \cap K^i_\lambda))$ $C^i = B_v(\phi_{1i} \cap M^i_\lambda B_v(\{w^j C^j\}^{i-1}_{j=1}), \max(L^i_\lambda, \phi_{Mi} \cap K^i_\lambda))$ $v = v'$
pseudo linear operator system of the 3 rd kind	$\rho^i(y^i, y'^i) \leqq B_v'(\{L^{ij}_\lambda \rho^j(x^j, x'^j)\}^M_{j=1})$	*Simple system*	$L_\lambda = B_v(\{w^i B_v'(\{L^{ij}_\lambda\}^M_{j=1})\}^M_{i=1})$ $L_\lambda = \max(\{B_v(\{w^i L^{ij}_\lambda\}^M_{i=1})\}^M_{j=1})$ $v = v'$ $L_\lambda = B_v(\{w^i B_{\bar{v}}(\{K^{ij}_\lambda\}^M_{i=1})\}^M_{i=1})$ $v \geqq 1,\ v' = 1$
		Seidel system	$\mathscr{L}_\lambda = B_v(\{w^i C^i\}^M_{i=1})$ $C^i = B_v'(\phi_{i1} \cap \{L^{ij}_\lambda w C^j\}^{i-1}_{j=1} \cup \{L^{ij}_\lambda\}^M_{j=1})$ $C^i = B_v(\phi_{i1} \cap \{L^{ij}_\lambda w^j C^j\}^{i-1}_{j=1}) \cup \max(\{L^{ij}_\lambda\}^M_{j=1}))$ $v = v'$ $C^i = B_1(\phi_{i1} \cap \{L^{ij}_\lambda w^j C^j\}^{i-1}_{j=1} \cup B_{\bar{v}}(\{L^{ij}_\lambda\}^M_{j=1}))$ $v \geqq 1,\ v' = 1$