The Ideal Shape of the Mainsprings for Watches*

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Abstract

The ideal shape of the mainsprings for watches entirely released (outside the barrel) has been analyzed and the central line of the spring was found to be the parallel curve of closoid.

I. Introduction

In a watch, the mainspring is coiled inside the barrel and mainsprings normally have the shape of reverse [1] (resilient) at the state of outside of the barrel.

The central line of this shape of mainsprings was analyzed by J. A. van den Broek and he arrived at double involute curves.[2]

From 1945, carbon-steel mainsprings have been increasingly superseded by springs made of special steels and also of cold-rolled alloys. As a rule, these springs are stainless and have high elastic limits. They are slightly subject to parmanent bending, and there is scarcely any risk of their breaking.[1] And, the more exact analysis of the shape of mainsprings at the state of outside of the barrel is needed.

II. Natural-geometrical Solution

Let r represent the radius to a point of a central line of a mainspring on the tightly coiled about an arbor of diameter d_i, and R the radius of curvature of the central line of the spring at the same point when the spring is entirely released (outside the barrel). As the spring undergoes the change from complete release to a tightly wound state, the change of curvature in the spring is $1/r-1/R$, which equals to M/B, where M is the torque at an arbor of a tightly wound state and B is the bending rigidity of the spring,

For a spring made of a flat leaf of a rectangular cross-section $b \times t$,

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1) G. A. Berner, Dictionnaire Professionnel Illustré de L'Horlogerie, 780—781 (1961), (UBAH).

2) J. A. van den Broek, Trance. A. S. M. E., 53 (18) 247—263 (1931).

$$\frac{1}{r}-\frac{1}{R}=\frac{M}{B}=\frac{2S_m}{tE}=\frac{2\varepsilon_m}{t}, \qquad (1)^{2)}$$

where E is the Young's modulus, S_m the maximum stress, $B=\frac{Ebt^3}{12}$, $M=\frac{Sbt^2}{6}$ and ε_m is the maximum strain and $\varepsilon_m=\frac{S_m}{E}$.

The central line of the spring on the tightly wound state may be considered as an Archimedian spiral, approximately. Let r, φ be the polar co-ordinate of the spiral, respectively, as the origin is at the starting point,

$$r=p\varphi/2\pi. \qquad (2)$$

Afer several windings the length of the curve of the spiral σ measured from the starting point is represented as follows,

$$\sigma=p\varphi^2/4\pi. \qquad (3)$$

where p is a pitch of the spiral.

And the curvature is

$$1/\rho=(2\alpha^2+r^2)/(\alpha^2+r^2)^{3/2}, \qquad (4)$$

where $\alpha=p/2\pi$.

Substituting eqs. (1) and (2) into eq. (4),

$$1/\rho=4\pi(1+2\pi\,\sigma/p)/p\,(1+4\pi\,\sigma/p)^{3/2}.$$

And, after several windings, $\sigma/p\gg 1$,
then,

$$1/\rho=4\pi(2\pi\,\sigma/p)/p\,(4\pi\,\sigma/p)^{3/2}=[\pi/p\sigma]^{1/2}. \qquad (5)$$

Eq. (5) is the approximate natural equation [3] of the Archmedian spiral.

The radius of curvature r_i of the central line at the inner end of the spring fully wound gives

$$r_i=(d_i+t)/2. \qquad (6)$$

If the length of curve measured from the inner end is s,

$$s=\sigma-C,$$

where C is a const. and must be $\pi r_i^2/p$ by using the condition:

$$\text{at}\quad s=0,\qquad \rho=r_i.$$

Therefore,

$$s=\sigma-\pi r_i^2/p. \qquad (7)$$

3) Generally, the eq. between ρ and σ is known as the natural equation of the plane curves, and from the theory of natural geometry, it is known that the plane curve is decided except motion, if the natural equation is given as a continuous function.

Putting eqs. (6) and (7) into eq. (5),

$$1/\rho=[(d_i+p)^2/4+p\,s/\pi]^{-1/2}. \tag{6$'$}$$

This is the natural eq. of the central line of the spring at the fully wound state. Assuming that the central line of the spring is approximately represented by eq. (5), and substituting $r=\rho$ and $p=t$ into eq. (5), we find eq. (1) is written as followes,

$$\frac{1}{R}=\frac{d\varphi}{d\sigma}=\left(\frac{\pi}{t\,\sigma}\right)^{1/2}-\frac{2\varepsilon_m}{t}. \tag{8}$$

Integrating under the condition of $\varphi=0$ at $\sigma=0$,

$$\varphi=2(\pi\sigma/t)^{1/2}-2\varepsilon_m\,\sigma/t.$$

The solution of this is

$$\sqrt{\sigma}=(t/2\varepsilon_m)\{(\pi/t)^{1/2}-(\pi/t-2\varepsilon_m\varphi/t)^{1/2}\}.$$

Substituting this into eq. (8), we obtain

$$\frac{d\varphi}{d\sigma}=\frac{2\varepsilon_m}{t}\left(\frac{\pi}{t}-\frac{2\varepsilon_m\varphi}{t}\right)^{1/2}\Big/\left\{\left(\frac{\pi}{t}\right)^{1/2}-\left(\frac{\pi}{t}-\frac{2\varepsilon_m\varphi}{t}\right)^{1/2}\right\}.$$

Therefore,

$$\frac{d\sigma}{d\varphi}=\left\{\sqrt{\frac{\pi/t}{\left(\frac{\pi}{t}-\frac{2\varepsilon_m\varphi}{t}\right)}}-1\right\}\frac{t}{2\varepsilon_m}.$$

Putting

$$\frac{\pi}{t}-\frac{2\varepsilon_m\varphi}{t}=\frac{2\varepsilon_m\theta}{t};\quad d\varphi=-d\theta,$$

$$\frac{d\sigma}{d\theta}=\frac{t}{2\varepsilon_m}\left\{1-\sqrt{\frac{\pi}{2\varepsilon_m\theta}}\right\}. \tag{9}$$

From the expression (7), $\sigma=s+\pi r_i/t$; $d\sigma=ds$,

$$\rho=\frac{ds}{d\theta}=\frac{t}{2\varepsilon_m}-\sqrt{\frac{\pi t^2}{8\varepsilon_m{}^3\theta}}, \tag{10}$$

where

$$\theta=\frac{\pi}{2\varepsilon_m}-\varphi.$$

Eq. (10) is the natural eq. of the ideal central line of the mainspring for watches entirely released, because the spring has a uniform strength and the stored energy is maximum.[2)]

III. Analysis of the Ideal Curve

The ideal curve represented by eq. (10) is the parallel curve of the clodoid;

$$\rho=\frac{ds}{d\theta}=-\sqrt{\frac{\pi t^2}{8\varepsilon_m{}^3}}\frac{1}{\sqrt{\theta}}. \tag{11}$$

Integrating this under the condition of $\theta=0$ at $s=0$, we obtain

$$s=-2\sqrt{\frac{\pi t^2}{8\varepsilon_m{}^3}}\sqrt{\theta},$$

or,

$$\frac{1}{\sqrt{\theta}}=-2\sqrt{\frac{\pi t^2}{8\varepsilon_m{}^3}}\frac{1}{s}.$$

Substituting this into eq. (11),

$$\frac{ds}{d\theta}=\rho=\frac{\pi t^2}{4\varepsilon_m{}^3}\frac{1}{s}. \tag{12}$$

From eq. (12), it is known that the curvature $1/\rho$ of the curve of eq. (11) is proportional to the length of the curve s.

Letting the yield strain of the material be ε_y, it may be assumed from the Baushinger's effect that

$$\varepsilon_m \leqq 2\varepsilon_y. \tag{13}$$

IV. Numerial Example

According to this theory, the numerical example may be shown by using values as follows; the cross-sectional dimentions of the spring

$$b = 1.70\,\text{mm},$$

$$t = 0.115\,\text{mm},$$

the length of the spring

$$l = 270\,\text{mm},$$

the diameter of the arbor

$$d_i = 3.00\,\text{mm}$$

and the maximum strain of the material

$$\varepsilon_m = 0.02.$$

Putting $\beta=\dfrac{\sqrt{\pi}\,t}{2\sqrt{\varepsilon_m{}^3}}$ in eq. (12),

$$\rho=\frac{\beta^2}{s}. \tag{14}$$

When $\beta=1/\sqrt{\pi}$, the x, y co-ordinates of the curve (14) are Fresnel integral $C(u)$, $S(u)$. And so, the co-ordinates of the curve (14) are,

$$\left.\begin{aligned} x&=\beta\sqrt{\pi}\;C(u)=\frac{\pi t}{2\sqrt{\varepsilon_m{}^3}}C(u)\\ y&=\beta\sqrt{\pi}\;S(u)=\frac{\pi t}{2\sqrt{\varepsilon_m{}^3}}S(u)\end{aligned}\right\}, \tag{15}$$

where

$$\frac{\pi t}{2\sqrt{\varepsilon_m{}^3}}=\frac{\pi t}{4\sqrt{2}}10^3=\frac{\pi\times 0.115}{5.656}10^3=63.8\ \mathrm{mm}.$$

With the aid of a Table of the Fresnel integral,[4] we can find the closoid of eq. (12), as shown with the brocken line in Fig. 1.

After some calculations, we can derive,

$$u=\frac{1}{\sqrt{\varepsilon_m}}\left\{1-\frac{2\varepsilon_m}{t}\rho\right\}. \tag{16}$$

At the inner end, from eq. (6),

$$1/\rho_i=1/r_i=2/(d_i+t)=0.642\ \mathrm{mm}^{-1}.$$

$$\therefore\quad u_i=\frac{10}{\sqrt{2}}\left\{1-\frac{0.04}{0.115\times 0.642}\right\}=3.23$$

At the outer end, from eq. (6)′

$$1/\rho_0=[(d_i+t)^2/4+tl/\pi]^{-\frac{1}{2}}=0.285\ \mathrm{mm}^{-1}$$

$$\therefore\quad u_0=\frac{10}{\sqrt{2}}\left\{1-\frac{0.04}{0.115\times 0.285}\right\}=-1.55.$$

The end points of the brocken line in Fig. 1. are corresponding to the u_i and u_0.

The full line in Fig. 1 is the ideal curve of the central line of spring, the eq. (10).

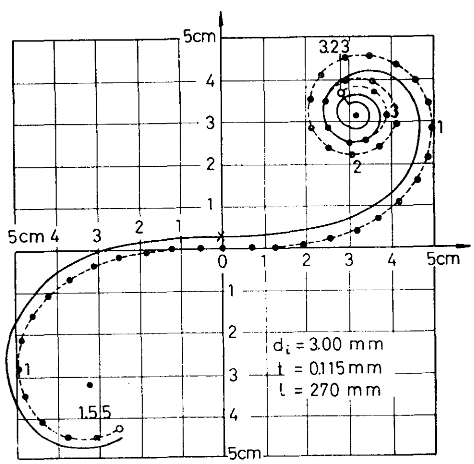

Fig. 1.

4) For example,
T. Pearcey, Table of the Fresnel Integral to Six Decimal Places, (1956), Cambridge Univ. Press.