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On Vibration of Two Circular Cylinders which are Immersed in a Water Region

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Abstract

We consider infinitesimally small vibration of two circular cylinders, which are immersed in a water region of infinite extent. Assuming the water to be an incompressible non-viscous fluid, and treating the problem, as a two-dimensional potential problem, the author estimated analytically the effect of water (in form of "virtual mass") upon the vibration of two circular cylinders. It is noticed that the acceleration of No. 1 cylinder affects the acceleration of No. 2 cylinder, thus causing an effect very much like the phenomenon of ''mutual inductance" in theory of coupled electric circuits.

I. Introduction

Let us consider two circular cylinders which are immersed in a water region of infinite extent. Our aim is to study the effect of surrounding water upon the vibration of these two circular cylinders. For that purpose, we take firstly, the case in which one of these two cylinders is making small oscillatory motion, while the other cylinder is kept at stand still. Assuming the water to be an ideal fluid, and regarding our problem as one of two-dimentional fluid motion, the author has made an analytical calculation about this fluid motion. Then, the amount of effective force acting on each cylinder due to the fluid motion was obtained.

Secondly, we apply the above mentioned result to the study of the case in which two cylinders make vibratory motion. It is pointed out that, in this case, an effect which behaves very much like the mutual inductance for a coupled electric circuit, can be shown to exist.

The method of analysis is based on the use of bipolar coordinates, for the case of two-dimensional fluid motion, about which we do not claim the originality.

II. Notations

The following notations will be used throughout the present paper :— x, y = rectangular coordinate of a point on the xy plane ; ξ, η = bipolar coodinate of the same point ; $z=x+iy$; h = factor of linear element ; ϕ = velocity potential of the (vibratory)

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motion set up in the fluid; R=radius of the cylindrical body; E=central distance of ditto; θ=angular position of a point on the periphery of the circular cylinder; $2C$=distance between the radial centers.

In order to indicate to which one of the two cylinders we are referring, suffixes 1 and 2 will be used, thus R_1, E_2, etc.

Other constants, such as a_n, b_n, λ, ε, etc., will be used, by defining them at the place where they make first appearance.

III. Preliminary discussions

Referring to Fig. 1, two points $(+C, 0)$, $(-C, 0)$ lying on the real axis are taken as radial centers, and we define a system of bipolar coordinates (ξ, η), by means of the relation

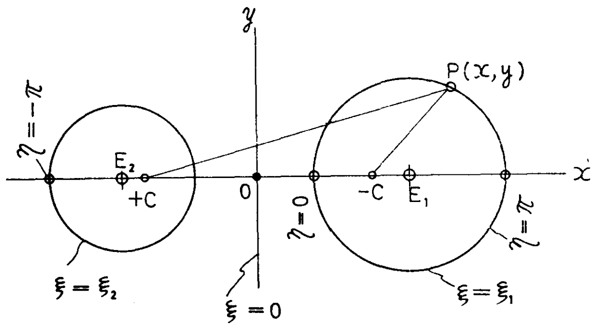

Fig. 1.

$$\xi+i\eta=\log\frac{c+z}{c-z}, \tag{1}$$

where we have $z=x+iy$. From this eq. (1) we obtain,

$$x=\frac{c\,\mathrm{sh}\,\xi}{\mathrm{ch}\,\xi+\cos\eta}, \quad y=\frac{c\sin\eta}{\mathrm{ch}\,\xi+\cos\eta}. \tag{2}$$

The line-element ds is found to be given by,

$$(ds)^2=(dx)^2+(dy)^2=h^2[(d\xi)^2+(d\eta)^2], \tag{3}$$

where we have,

$$h=\frac{c}{\mathrm{ch}\,\xi+\cos\eta}. \tag{4}$$

Therefore, the Laplacian $\Delta\phi$ of the function ϕ is given by,

$$\Delta\phi\equiv\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\equiv\frac{1}{h^2}\left(\frac{\partial^2\phi}{\partial\xi^2}+\frac{\partial^2\phi}{\partial\eta^2}\right) \quad \text{and} \quad \Delta\phi=0. \tag{5}$$

The origin of the (x, y)-plane is transformed into the point $(\xi=0, \eta=0)$. On the other hand, the point at infinity in the (x, y)-plane is transformed into the point

$(\xi=0,\ \eta=\pm\pi)$. The two circles $\xi=\xi_1$ and $\xi=\xi_2$, as shown in Fig. 1, are taken as boundary surfaces of two circular cylinders, which are immersed in a water region extending to infinity.

This water region of infinite extent (outside of two circular boundaries) correspond, in the $(\xi,\ \eta)$-plane, to the inside of a rectangular region as shown in Fig. 2.

Our problem reduces to that of finding a potential function ϕ, which satisfies the Laplace equation (5), and which takes prescribed values of its normal derivatives $\partial\phi/\partial\nu$, along the two circumferences of circular boundaries. This problem, in turn, can be regarded as a boundary value problem for the potential function $\phi(\xi,\ \eta)$, the domain being a rectangle as shown in Fig. 2. It will be seen that, this function $\phi(\xi,\ \eta)$ must be a periodic function of period 2π, with regard to the independent variable η.

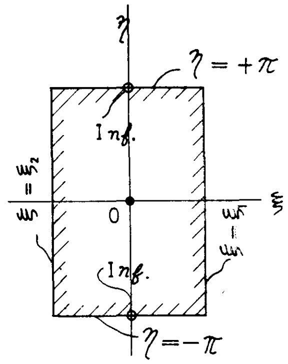

Fig. 2.

Along the circumference of one circle for which $\xi=\xi_1$, we have,

$$R_1 e^{i\theta}=x+iy-E_1=c\frac{\text{sh}\ \xi_1+i\sin\eta}{\text{ch}\ \xi_1+\cos\eta}-E_1,$$

so that we have,

$$\left.\begin{aligned} R_1\cos\theta&=c\frac{\text{sh}\ \xi_1}{\text{ch}\ \xi_1+\cos\eta}-E_1,\\ R_1\sin\theta&=c\frac{\sin\eta}{\text{ch}\ \xi_1+\cos\eta}. \end{aligned}\right\} \tag{6}$$

Next, we observe that, the values

$$\frac{R_1\cos\theta}{\text{ch}\ \xi_1+\cos\eta}\quad\text{and}\quad\frac{R_1\sin\theta}{\text{ch}\ \xi_1+\cos\eta},$$

are periodic functions (of period 2π) with regard to the variable η.

Thus, we may assume the following form of Fourier series, for these quantities;

$$\left.\begin{aligned} \frac{R_1\cos\theta}{\text{ch}\ \xi_1+\cos\eta}&=\sum a_n\sin n\eta+\sum b_n\cos n\eta,\\ \frac{R_1\sin\theta}{\text{ch}\ \xi_1+\cos\eta}&=\sum c_n\sin n\eta+\sum d_n\cos n\eta. \end{aligned}\right\} \tag{7}$$

Actual values of Fourier coefficients $a_n,\ b_n,\ c_n,\ d_n$ are to be found by usual method, viz., by putting the expressions (6) into (7), and by multiplication by $\sin n\eta$ or $\cos n\eta$, and by integration from $\eta=-\pi$ to $\eta=+\pi$. It is deduced that, in the present case, the evaluation of $a_n,\ b_n$, etc., can be reduced to the integration of difinite integrals of following form;

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi}\frac{\cos m\eta+i\sin m\eta}{[\lambda+\cos\eta]^s}\,d\eta,$$

and

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi}\frac{\sin\eta\,[\cos m\eta+i\sin m\eta]}{[\lambda+\cos\eta]^s}\,d\eta\,,$$

wherein we have put,

$$\lambda=\text{ch}\,\xi_1 \qquad s=1,\ \text{or}\ 2.$$

The evaluation of these definite integrals can, in turn, be reduced to the evaluation of contour integrals (in the complex z-plane) of following forms : —

For $s=1$,

$$I_1=\frac{1}{2\pi i}\int_C \frac{z^m}{\left[\lambda+\frac{1}{2}\left(z+\frac{1}{z}\right)\right]}\frac{dz}{z}$$

$$=\frac{1}{2\pi i}\int_C \frac{2z^m}{\left(z+\varepsilon\right)\left(z+\frac{1}{\varepsilon}\right)}\,dz.$$

For $s=2$,

$$I_2=\frac{1}{2\pi i}\int_C \frac{4z^{m+1}}{\left(z+\varepsilon\right)^2\left(z+\frac{1}{\varepsilon}\right)^2}\,dz,$$

$-\varepsilon$ being a root of quadratic equation

$$z^2+2\lambda z+1=0,$$

such that $0<\varepsilon<1$, we have

$$\varepsilon=\frac{1}{\lambda+\sqrt{\lambda^2-1}}$$

In these expressions I_1 and I_2, m are positive integers 1, 2, 3, ……. The contour C of integration is to be understood as a whole circle $z=e^{i\eta}$ (of radius 1, and center at the origin) of the complex z-plane. The result of calculation may be summarized as follows : —

By putting (for $m=0$, 1, 2, ……),

$$\left.\begin{aligned} K_m^{(s)}&=\frac{1}{2\pi}\int_{-\pi}^{+\pi}\frac{\cos m\eta}{(\text{ch}\,\xi_1+\cos\eta)^s}\,d\eta\,,\\ S_m^{(s)}&=\frac{1}{2\pi}\int_{-\pi}^{+\pi}\frac{\sin m\eta}{(\text{ch}\,\xi_1+\cos\lambda)^s}\,d\eta\,,\end{aligned}\right\}\tag{8}$$

we have

$$K_m^{(1)}=\frac{(-)^m\,2\varepsilon^{m+1}}{(1-\varepsilon^2)}\,,$$

$$K_m^{(2)}=\frac{(-)^m 4(m+1)\varepsilon^{m+2}}{(1-\varepsilon^2)^2}+\frac{(-)^m 8\varepsilon^{m+4}}{(1-\varepsilon^2)^3},$$

$$S_m^{(s)}=0, \quad (s=1,\ 2,\ \cdots\cdots)$$

Using these values, the Fourier coefficients a_n, b_n, etc., in the expression (7) are written in the following form :—

$$\left.\begin{aligned} b_0&=c\sqrt{\lambda^2-1}\ K_0^{(2)}-E_1 K_0^{(1)} \\ b_n&=2c\sqrt{\lambda^2-1}\ K_n^{(2)}-2E_1 K_n^{(1)} \\ c_n&=cK_{n-1}^2-cK_{n+1}^2 \end{aligned}\right\} \tag{9}$$

where $n=1,\ 2,\ \cdots\cdots$, and other coefficients a_n, d_0, d_n being equal to zero.

Lastly, we observe that the values of R_1 and E_1 can be derived from the general formula (2), and we obtain,

$$\left.\begin{aligned} E_1&=\frac{c\,\mathrm{sh}\,\xi_1\,\mathrm{ch}\,\xi_1}{(\mathrm{ch}\,\xi_1)^2-1} \\ R_1&=\frac{c\,\mathrm{sh}\,\xi_1}{(\mathrm{ch}\,\xi_1)^2-1} \end{aligned}\right\} \tag{10}$$

The value of b_0 is actually equal to zero, as we see in the calculation in Appendix A.

IV. Velocity potential ϕ for the vibratory motion of No. 1 cylinder, while No. 2 cylinder is kept stand still

The velocity potential ϕ of the fluid motion must satisfy the Laplace equation (5). A solution of this equation which is a periodic function of η with period 2π, may be written;

$$\phi=\sum_{n=1}^{\infty}[\mathrm{ch}\,n\xi+A_n\,\mathrm{sh}\,n\xi]\cdot[B_n\cos n\eta+F_n\sin n\eta]\cdot f'(t) \tag{11}$$

A_n, B_n, F_n are arbitrary constants which are to be determined later. $f(t)$ is a function of time t, which is introduced to represent the vibratory motion set up in the fluid. In this section, we shall treat the case in which the No. 1 circular cylinder is vibrating, while the No. 2 circular cylinder is kept at stand-still. The normal velocity of fluid motion, being given by,

$$V_\nu=\frac{1}{h}\frac{\partial\phi}{\partial\xi},$$

for any one of cylindrical surfaces, the above-mentioned conditions may be expressed as follows :—

(*a*). for $\xi=\xi_1$ (since ξ decreases outwardly)

$$-\frac{1}{h}\frac{\partial\phi}{\xi\xi}=[\cos\varphi_0\cos\theta+\sin\varphi_0\sin\theta]f'(t) \tag{12}$$

where φ_0 represents the direction of vibratory motion of No. 1 cylinder.

(*b*). for $\xi=\xi_2$

$$\frac{1}{h}\frac{\partial\phi}{\partial\xi}=0 \tag{13}$$

This condition (11) is satisfied, if we take,

$$A_n=-\tanh n\xi_2 . \tag{14}$$

It is noted that ξ_2 has a negative value, while ξ_1 is positive.

(*c*). for $\eta=\pm\pi$, we have $h\to\infty$.

So that the fluid motion represented by the velocity potential ϕ will be such that, at infinity of the (x, y)-plane, the fluid is at rest.

Lastly, in order that condition (12) is satisfied, we must have;

$$-\sum_{n=1}^{\infty} n[\mathrm{sh}\, n\xi+A_n\,\mathrm{ch}\, n\xi]\cdot[B_n\cos n\eta+F_n\sin n\eta]$$
$$=\frac{c(\cos\varphi_0\cos\theta+\sin\varphi_0\sin\theta)}{(\mathrm{ch}\,\xi+\cos\eta)} \tag{15}$$

This condition (14) is satisfied, according to what we have obtained in equations (7) and (9), if we take the coefficients B_n, F_n, as follows;

$$\left.\begin{aligned} B_n&=-\frac{c}{R_1}\frac{b_n}{nG_n}\cos\varphi_0 ,\\ F_n&=-\frac{c}{R_1}\frac{c_n}{nG_n}\sin\varphi_0 , \end{aligned}\right\} \tag{16}$$

where we have put, for convenience,

$$G_n=\mathrm{sh}\, n\xi_1+A_n\,\mathrm{ch}\, n\xi_1 , \tag{17}$$

and, we take $n=1, 2, \cdots\cdots$.

V. Estimation of hydraulic forces acting upon two cylinders

For a given value of the velocity potential ϕ, the values of hydraulic pressure is given by,

$$p=-\rho\left[\frac{\partial\phi}{\partial t}+\frac{1}{2}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right\}\right] \tag{18}$$

When the velocity of motion of the fluid is infinitesimally small, we can take approximately, as follows: —

$$p=-\rho\,\frac{\partial\phi}{\partial t} \tag{19}$$

For our solution (11), we have, therefore,

$$p=-\rho\sum_{n=1}^{\infty}[\mathrm{ch}\, n\xi+A_n\,\mathrm{sh}\, n\xi]\cdot[B_n\cos n\eta+F_n\sin n\eta]f''(t) \tag{20}$$

The force acts upon two circular cylinders, whose amounts are estimated in the following manner,

$$\left.\begin{aligned} -F_{x1}&=\int p\cos\theta\cdot h d\theta\,, \\ -F_{y1}&=\int p\sin\theta\cdot h d\theta\,, \end{aligned}\right\} \tag{21}$$

$$\left.\begin{aligned} -F_{x2}&=\int p\cos\theta_2\cdot h_2\, d\theta_2\,, \\ -F_{y2}&=\int p\sin\theta_2\cdot h_2\, d\theta_2\,, \end{aligned}\right\} \tag{22}$$

where $(F_{x1},\ F_{y1})$ are the components of force acting upon cylinder No. 1, while $(F_{x2},\ Fy_2)$ are those acting on the cylinder No. 2. Notations with suffixes "2" are understood as quantities referring to No. 2 cylinder.

The actual values of these forces are estimated as shown below:—

(A). On the surface $\xi=\xi_1$, of No. 1 circular cylinder, we have,

$$\begin{aligned} &-\int_{-\pi}^{\pi} p\cos\theta\cdot h\,d\eta \\ &=\int_{-\pi}^{\pi}\rho f''(t)\Big[\sum_{n=1}^{\infty}(\mathrm{ch}\,n\xi_1+A_n\,\mathrm{sh}\,n\xi_1) \\ &\times\Big(B_n\cos n\eta+F_n\sin n\eta\Big)\times\frac{c\cos\theta}{(\mathrm{ch}\,\xi_1+\cos\eta)}\cdot d\eta \\ &=\rho c f''(t)\sum_{n=1}\Big(\mathrm{ch}\,n\xi_1+A_n\,\mathrm{sh}\,\xi_1 n\Big)\times\frac{\pi}{R_1}B_n b_n \\ &=\rho c f''(t)\frac{\pi}{R_1}\sum_{n=1}^{\infty}H_n B_n b_n \\ &=\rho c^2 f''(t)\frac{\pi}{R_1^2}\cos\varphi_0\sum_{n=1}^{\infty}\frac{1}{n}J_n(b_n)^2\,. \end{aligned}$$

$$\begin{aligned} &-\int_{-\pi}^{\pi} p\sin\theta\cdot h\,d\eta \\ &=\int_{-\pi}^{\pi}\rho f''(t)\Big[\sum_{n=1}^{\infty}\Big(\mathrm{ch}\,n\xi_1+A_n\,\mathrm{sh}\,n\xi_1\Big) \\ &\times\Big(B_n\cos n\eta+F_n\sin n\eta\Big)\times\frac{c\sin\theta}{(\mathrm{ch}\,\xi_1+\cos\eta)}d\eta \\ &=\rho f'(t)\frac{\pi}{R_1}\sum_{n=1}^{\infty}\Big(\mathrm{ch}\,n\xi_1+A_n\,\mathrm{sh}\,n\xi_1\Big)\Big(F_n c_n\Big) \\ &=-\rho c^2 f''(t)\frac{\pi}{R_1^2}\sum_{n=1}^{\infty}\frac{1}{n}J_n(c_n)^2\sin\varphi_0\,. \end{aligned}$$

(B). On the surface $\xi=\xi_2$, of No. 2 circular cylinder, we obtain in similar manner as (A);

$$-\int p\cos\theta\cdot h\,d\eta$$

$$=\rho f''(t)\int_{-\pi}^{\pi}\Big[\sum_{n=1}^{\infty}\Big(\operatorname{ch} n\xi_2+A_n\operatorname{sh} n\xi_2\Big)$$

$$\times\Big(B_n\cos n\eta+F_n\operatorname{sh} n\xi_2\Big)$$

$$\times\frac{c\cos\theta}{(\operatorname{ch}\xi_2+\cos\eta)}\cdot d\eta$$

$$=\rho cf''(t)\sum_{n=1}^{\infty}\frac{\pi}{R_2}G_n'\,b_n'\,B_n$$

$$=\rho cf''(t)\frac{\pi}{R_2}\sum_{n=1}^{\infty}k_n\,G_n\,B_n\,b_n$$

$$=-\rho c^2f''(t)\frac{\pi}{R_2^2}\sum_{n=1}^{\infty}\cos\varphi_0\frac{1}{n}\,J_n\,k_n\,(b_n)^2$$

where we have written G_n', b_n', for values of quantities referring to No. 2 cylinder, which correspond to G_n, b_n of No. 1 cylinder. Also we have put (for $n=1.\ 2,\ \cdots\cdots$),

$$k_n=\frac{G_n'}{G_n}\times\frac{b_n'}{b_n} \tag{23}$$

$$=\frac{\operatorname{ch} n\xi_2+A_n\operatorname{sh} n\xi_2}{\operatorname{ch} n\xi_1+A_n\operatorname{sh} n\xi_1}\cdot\frac{b_n'}{b_n}$$

$$=\frac{1}{\operatorname{ch}\,(\xi_1-\xi_2)\,n}\cdot\frac{b_n'}{b_n}. \tag{23}$$

$$-\int p\sin\theta\cdot h\,d\eta$$

$$=-\rho c^2f''(t)\frac{\pi}{R_2^2}\sum_{n=1}^{\infty}\frac{1}{n}\,k_n(c_n)^2J_n\sin\varphi_2\,.$$

It will be observed that, if $\xi_2=-\xi_1$, we have

$$b_n'/b_n=1,\quad c_n'/c_n=1\,.$$

(C). Summing up these results of (A) and (B), we conclude that, components of forces exerted on No. 1 and No. 2 cylinders, caused by the vibratory motion of No. 1 cylinder, while No. 2 cylinder stands still, can be expressed in the following manner: —

$$\left.\begin{aligned} F_{x1}&=-f''(t)\,\rho\cos\varphi_0\,\pi R_1^2\,Q_{x1}\\ F_{y1}&=-f''(t)\,\rho\sin\varphi_0\,\pi R_1^2\,Q_{y1}\end{aligned}\right\}\tag{24}$$

$$\left.\begin{aligned} F_{x2}&=-f''(t)\,\rho\cos\varphi_0\,\pi R_2^2\,Q_{x2}\\ F_{y2}&=-f''(t)\,\rho\sin\varphi_0\,\pi R_2^2\,Q_{y2}\end{aligned}\right\}\tag{25}$$

In these formulae (24) and (25), $Q_{x1}, \cdots\cdots, Q_{y2}$ are numerical constants, which depend upon the values of ξ_1 and ξ_2. It is also to be noted that formulae (24) and (25) give values of forces per unit length of each circular cylinders.

(*D*). Special Cases. The above formulae give values of hydraulic forces referred to unit length (or the length perpendicular to x, y plane), for general value of angle φ_0 (direction of vibraton of No. 1 cylinder). Therefore, we deduce the following special cases.

(*a*). If $\varphi_0=0$, the vibration is taking place in the direction of x-axis. For this case, we have

$$\left.\begin{aligned} F_{x1}&=-x_1''(t)\,\rho\,\pi\,Q_{x1}\\ F_{y1}&=0\\ F_{x2}&=-x_1''(t)\,\rho\,\pi\,Q_{x2}\\ F_{y2}&=0\end{aligned}\right\}\tag{26}$$

where we have written $x_1''(t)$ instead of $f''(t)$.

(*b*). If $\varphi_0=\pi/2$, the vibration is taking place in the direction of y-axis. For this case, we have,

$$\left.\begin{aligned} F_{x1}&=0\\ F_{y1}&=-y_1''(t)\,\rho\,\pi R_1^2 Q_{y1}\\ F_{x2}&=0\\ F_{y2}&=-y_1''(t)\,\rho\,\pi R_1^2 Q_{y2}\end{aligned}\right\}\tag{27}$$

where we have written $y_1''(t)$, instead of $f''(t)$.

VI. The case in which both No. 1 and No. 2 circular cylinders are vibrating simultaneously

We can obtain, in similar manner, expressions for the case in which No. 2 circular cylinder is vibrating, while the No. 1 cylinder is kept at stand still. Writing, for shortness, the coefficients, which correspond to Q_{x1}, Q_{x2}, Q_{y1}, Q_{y2} for the previous case, as S_{x1}, S_{x2}, S_{y1}, S_{y2}, hydraulic forces exerted upon two cylinders, respectively, may be shown to be expressed in the following form, corresponding to equations (26) and (27), thus: —

$$\left.\begin{aligned} F_{x1}&=-x_2''(t)\,\rho\pi R_1^2 S_{x1}\\ F_{y1}&=0\\ F_{x2}&=-x_2''(t)\,\rho\pi R_1^2 S_{x2}\\ F_{y2}&=0 \end{aligned}\right\} \quad (28)$$

$$\left.\begin{aligned} F_{x1}&=0\\ F_{y1}&=-y_2''(t)\,\rho\pi R_2^2 S_{y1}\\ F_{x2}&=0\\ F_{y2}&=-y_2''(t)\,\rho\pi R_2^2 S_{y2} \end{aligned}\right\} \quad (29)$$

When both cylinders are vibrating simultaneously, the resulting effect of fluid motion will be obtained by superposition of above-mentioned two cases.

What interests us is the fact that the acceleration of vibratory motion of No. 1 cylinder $x_1''(t)$ exerts a hydraulic force upon No. 2 cylinder, whose amount is proportional to $x_1''(t)$. On the other hand, acceleration of No. 2 cylinder $x_2''(t)$ will produce, upon No. 1 cylinder a hydraulic force of value proportional to $x_2''(x)$. Thus, if two circular cylinder of elastic material are placed in a water region, and made to vibrate, one cylinder will affect the vibratory motion of the other. We may say that, the effect of surrounding water is very much like that of "mutual impedance" in the theory of electric circuits.

VII. Numerical example

In order to illustrate the above mentioned theoretical results, some numerical calculation has been made. Here we take up values of ξ_1 as follows,

$$\xi_1=\log_e 2=0.693$$
$$\xi_1=\log_e 4=1.386$$
$$\xi_1=\log_e 6=1.792$$

For these three cases, values of λ, ε, R_1/c, E_1/c have been estimated and obtained values as shown in Table 1.

Table 1.

ξ_1	0.693	1.386	1.792
$\lambda=\mathrm{ch}\,\xi_1$	1.248	2.132	3.107
$\sqrt{\lambda^2-1}=\mathrm{sh}\,\xi_1$	0.748	1.88	2.94
R_1/c	1.34	0.530	0.338
E_1/c	1.68	1.13	1.05

For these three cases of ξ_1, values of coefficients b_n and c_n have been evaluated as shown in Table 2.

From this table 2, we observe that values of b_n and c_n decrease with increasing values of integer n, but slowly for the case of $\xi_1=0.693$. But, for the case of $\xi_1=1.386$ and 1.792, b_n and c_n decrease fairly rapidly as n increases.

Let us take up the case of $\xi_1=+1.386$ and $\xi_2=-\xi_1=-1.386$. This means that two cylinders of the same radius $R_1=0.530\,c$ are placed, at a distance of $2E_1=2.26\,c$ apart. In this case, we shall have $b_n'=b_n$, $c_n'=c_n$ and $k_n=1$. The values of inertia coefficients have been calculated by (26) and (27). The results are as follows:—

Table 2.

b_1/c	−2.18	−0.300	−0.118
b_2/c	+1.98	+0.144	+0.00066
b_3/c	−1.428	−0.0533	−0.0116
b_4/c	+0.936	+0.0178	+0.0021
b_5/c	−0.582	−0.00547	−0.0000872
c_1/c	+1.49	+0.282	+0.118
c_2/c	−1.63	−0.14	−0.038
c_3/c	+1.26	+0.0523	+0.00298
c_4/c	−0.850	−0.0173	−0.00213
c_5/c	+0.534	+0.00543	+0.000427

$$Q_{x1}=\left(\frac{c}{R_1}\right)^4\sum_{n=1}^{\infty}\frac{J_n}{n}\left(\frac{b_n}{c}\right)^2$$

$$Q_{y1}=\left(\frac{c}{R_1}\right)^4\sum_{n=1}^{\infty}\frac{J_n}{n}\left(\frac{c_n}{c}\right)^2$$

$$Q_{x2}=\left(\frac{c}{R_2}\right)^4\sum_{n=1}^{\infty}\frac{k_nJ_n}{n}\left(\frac{b_n}{c}\right)^2,$$

$$Q_{y2}=\left(\frac{c}{R_2}\right)^4\sum_{n=1}^{\infty}\frac{k_nJ_n}{n}\left(\frac{c_n}{c}\right)^2,$$

and obtained, for our case,

$$Q_{x1}=1.32,\quad Q_{y1}=1.18,$$

$$Q_{x2}=0.572,\quad Q_{y2}=0.504.$$

When a single circular cylinder is immersed in a water region of infinite extent, and is vibrating, it is known that the amount of "virtual mass" is just equal to the amount of mass of water displaced by the cylinder itself. The above numerical example shows us that, in our case of two cylinders, the "virtual mass" of vibrating cylinder itself is 1.32 or 1.18 times the mass of displaced water. On the other hand, vibrating cylinder makes an influence upon the another cylinder, as is represented by the factors Q_{x2} and Q_{y2}. In our present example in which two circular cylinders have the equal radius, we have

$$S_{x2}=Q_{x1}\,,\quad S_{x1}=Q_{x2}\,,$$

$$S_{y2}=Q_{y1}\,,\quad S_{y1}=Q_{y2}\,.$$

APPENDIX

In our preliminary discussion of section 3, one root of quadratic equation

$$z^2+2\lambda z+1=0$$

was written as $-\varepsilon$. So that we have,

$$\varepsilon=\lambda-\sqrt{\lambda^2-1}\,,\quad 1+\varepsilon^2=2\lambda\varepsilon\,.$$

Using these values, we have

$$\frac{E}{c\sqrt{\lambda^2-1}}=\frac{1}{\sqrt{\lambda^2-1}}\frac{\sqrt{\lambda^2-1}\;\lambda}{\lambda^2-1}=\frac{\lambda}{\lambda^2-1}\,,$$

$$\frac{K_0^{(2)}}{K_0^{(1)}}=\frac{2\varepsilon\,(1+\varepsilon^2)}{(1-\varepsilon^2)}=\frac{4\,\lambda\,\varepsilon^2}{4\,(\lambda^2-1)\,\varepsilon^2}=\frac{\lambda}{\lambda^2-1}\,,$$

Thus, we see that $b_0=0$, as we mentioned in the text.