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Collisional Damping of Transverse Oscillation in a Plasma

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Abstract

High-frequency conductivity and dispersion relation for transverse oscillation in a homogeneous plasma are derived on the basis of Fokker-Planck equation. Damping coefficient ω_i is given by

$$\omega_i = -\frac{\nu_c}{3\sqrt{2\pi}}\frac{\omega_p^2}{\omega_p^2+c^2k^2}\left[1+\frac{\omega_p^2}{\omega_p^2+c^2k^2}\left(\frac{k}{k_D}\right)^2\left(2+\frac{3\sqrt{2}}{5}-\frac{2\omega_p^2}{\omega_p^2+c^2k^2}\right)\right],$$

ω_p, k_D and ν_c being the plasma frequency, Debye's characteristic constant and the electron collision frequency. The term containing $\frac{3\sqrt{2}}{5}$ in the bracket arises from electron-electron collision, and the remaining terms in the bracket are due to electron-ion collision.

I. Introduction

Transverse high-frequency conductivity in a plasma was calculated quantum mechanically by Gilinsky and DuBois [1]. Their result, however, contains only electron-electron collision. Berk [2] also calculated the same quantity with use of the model of Dawson and Oberman [3]. But by this model we cannot take into account of electron-ion collision. Thourson and Lewis [4] solved approximately the BBGKY equation and gave the transverse high-frequency conductivity including both the electron-electron and the electron-ion collisions.

Recently Buti and Jain [5] calculate the dispersion relation for the transverse oscillation with use of the Fokker-Planck equation. But their result disagrees with the present author's [6] which is obtained by a generalization of the method developped in obtaining the dispersion relation for plasma oscillations with use of the Boltzmann

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1) V. Gilinsky and D. G. DuBois; The RAND Corporation, RM-4109-PR June 1964.
2) H. L. Berk; Phys. Fluids **7** (1964) 257.
3) J. Dawson and C. Oberman; Phys. Fluids 5 (1962) 517, ibid. **6** (1963) 394.
4) T. L. Thourson and M. B. Lewis; Phys. Fluids 8 (1965) 1119.
5) B. Buti and R. K. Jain; Phys. Fluids 8 (1965) 2080.
6) M. Ogasawara; Not yet published.

equation [7]. Our result agrees with those obtained by Thourson and Lewis, and by Berk. It seems that Buti and Jain make mistakes in the course of calculation. So in the present paper we will recalculate the dispersion relation based on the Fokker-Planck equation in a slightly different way from Buti and Jain's.

In Section II general expressions of the high-frequency conductivity and the dispersion relation are derived. In Sections III and IV the collision integrals involved in the general expressions derived in Section II are evaluated and compared with the results of Buti and Jain. In V the high-frequency conductivity is written down and compared with other works. The dispersion relation in the long wavelength limit is presented.

II. High-Frequency Conductivity and Dispersion Relation

This section is mainly devoted to the derivation of a general expression of high-frequency conductivity. Once this is obtained dispersion relation is easily written down.

In the absence of an electric field, electrons and ions obey Maxwellian velocity distributions :

$$f_{0e} = n\left(\frac{1}{2\pi v_0^2}\right)^{\frac{3}{2}} e^{-\frac{v^2}{2v_0^2}}, \qquad f_{0i} = n\left(\frac{1}{2\pi V_0^2}\right)^{\frac{3}{2}} e^{-\frac{v^2}{2V_0^2}}, \tag{2.1}$$

where $v_0^2=T/m$, $V_0^2=T/M$, n is the number density, T the temperature in energy unit, m and M masses of an electron and an ion respectively.

When an electric field $\boldsymbol{E}$ is applied, linearized equation which the electrons must obey is given by

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f}{\partial \boldsymbol{x}} - \frac{e}{m}\boldsymbol{E}\cdot\frac{\partial f_{0e}}{\partial \boldsymbol{v}} = \sum_{j=1}^{3}\left(\frac{\partial f}{\partial t}\right)_{cj}, \tag{2.2}$$

where $f(\boldsymbol{x}, \boldsymbol{v}, t)$ is the perturbed electron velocity distribution. The right side of (2. 2) represents the collision term that are given by [8]

$$\left(\frac{\partial f}{\partial t}\right)_{c1} = -\frac{\partial}{\partial \boldsymbol{v}}\cdot(\langle\boldsymbol{\Delta}\rangle_{0i} f) + \frac{1}{2}\frac{\partial^2}{\partial \boldsymbol{v}\,\partial \boldsymbol{v}} : (\langle\boldsymbol{\Delta\Delta}\rangle_{0i} f), \tag{2.3}$$

$$\left(\frac{\partial f}{\partial t}\right)_{c2} = -\frac{\partial}{\partial \boldsymbol{v}}\cdot(\langle\boldsymbol{\Delta}\rangle_{0e} f) + \frac{1}{2}\frac{\partial^2}{\partial \boldsymbol{v}\,\partial \boldsymbol{v}} : (\langle\boldsymbol{\Delta\Delta}\rangle_{0e} f), \tag{2.4}$$

$$\left(\frac{\partial f}{\partial t}\right)_{c3} = -\frac{\partial}{\partial \boldsymbol{v}}\cdot(\langle\boldsymbol{\Delta}\rangle f_{0e}) + \frac{1}{2}\frac{\partial^2}{\partial \boldsymbol{v}\,\partial \boldsymbol{v}} : (\langle\boldsymbol{\Delta\Delta}\rangle f_{0e}), \tag{2.5}$$

7) M. Ogasawara; Proc. Fac. Eng. Keio Univ. **17** (1964) 1, J. Phys. Soc. Japan **18** (1963) 1066.
8) M. N. Rosenbluth, W. M. MacDonald, and D. L. Judd; Phyg. Rev. **107** (1957) 1.

with

$$<\Delta>_{0i}=\left(1+\frac{m}{M}\right)n\Gamma\frac{\partial}{\partial \boldsymbol{v}}\int d\boldsymbol{v}'\frac{f_{0i}(\boldsymbol{v}')}{|\boldsymbol{v}-\boldsymbol{v}'|}, \tag{2.6}$$

$$<\Delta\Delta>_{0i,e}=n\Gamma\frac{\partial^2}{\partial \boldsymbol{v}\,\partial \boldsymbol{v}}\int d\boldsymbol{v}'f_{0i,e}(\boldsymbol{v}')\,|\boldsymbol{v}-\boldsymbol{v}'|, \tag{2.7}$$

$$<\Delta>_{0e}=2n\Gamma\frac{\partial}{\partial \boldsymbol{v}}\int d\boldsymbol{v}'\frac{f_{0e}(\boldsymbol{v}')}{|\boldsymbol{v}-\boldsymbol{v}'|}, \tag{2.8}$$

$$<\Delta>=2n\Gamma\frac{\partial}{\partial \boldsymbol{v}}\int d\boldsymbol{v}'\frac{f(\boldsymbol{v}')}{|\boldsymbol{v}-\boldsymbol{v}'|}, \tag{2.9}$$

$$<\Delta\Delta>=n\Gamma\frac{\partial^2}{\partial \boldsymbol{v}\,\partial \boldsymbol{v}}\int d\boldsymbol{v}'f(\boldsymbol{v}')\,|\boldsymbol{v}-\boldsymbol{v}'|. \tag{2.10}$$

Γ is the coulomb logarithm and given by

$$\Gamma=\frac{4\pi e^4}{m^2}\,ln\left(\frac{\lambda_D}{b_0}\right), \tag{2.11}$$

where $\lambda_D=(T/4\pi ne^2)^{1/2}$ and $b_0=2e^2/3T$ being the Debye length and 90°-scattering impact parameter, respectively. We are concerned with the phenomena whose wavelength is much longer than the Debye length. Hence the use of the coulcmb logarithm of the form (2.11) is justified. We take the electric field which varies in space and time as

$$\boldsymbol{E}(\boldsymbol{x},t)=\boldsymbol{E}_0(\boldsymbol{k},\omega)\,e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}, \tag{2.12}$$

and the response of the electrons to the electric field as

$$f(\boldsymbol{x},\boldsymbol{v},t)=f'(\boldsymbol{k},\boldsymbol{v},\omega)\,e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}. \tag{2.13}$$

Then from eq. (2.2) we have

$$(-i\omega+i\boldsymbol{k}\cdot\boldsymbol{v})f'-\frac{e}{m}\boldsymbol{E}_0\cdot\frac{\partial f_{0e}}{\partial \boldsymbol{v}}=\sum_j\left(\frac{\partial f'}{\partial t}\right)_{cj}. \tag{2.14}$$

In order to solve this equation it is convenient to introduce the Fourier transform in velocity space defined by

$$F(\boldsymbol{k},\boldsymbol{\sigma},\omega)=\int d\boldsymbol{v}\,e^{-i\boldsymbol{\sigma}\cdot\boldsymbol{v}}f'(\boldsymbol{k},\boldsymbol{v},\omega). \tag{2.15}$$

Then we can rewrite (2.14) as

$$\left(-i\omega-\boldsymbol{k}\cdot\frac{\partial}{\partial\boldsymbol{\sigma}}\right)F(\boldsymbol{k},\boldsymbol{\sigma},\omega)-\frac{ine}{m}\boldsymbol{E}_0\cdot\boldsymbol{\sigma}e^{-\frac{(\sigma v_0)^2}{2}}=\sum_j\left(\frac{\partial F}{\partial t}\right)_{cj}. \tag{2.16}$$

The transformed collision term is represented by

$$\sum_j\left(\frac{\partial F}{\partial t}\right)_{cj}=n\Gamma\sum_j\int d\boldsymbol{\eta}\,K_j(\boldsymbol{\sigma},\boldsymbol{\eta})\,F(\boldsymbol{\eta})=n\Gamma\int d\boldsymbol{\eta}\,K(\boldsymbol{\sigma},\boldsymbol{\eta})\,F(\boldsymbol{\eta}), \tag{2.17}$$

where

$$K_1(\boldsymbol{\sigma},\boldsymbol{\eta})=\frac{1}{2\pi^2}exp\Big[-\frac{V_0^2}{2}(\boldsymbol{\sigma}-\boldsymbol{\eta})^2\Big]\Big\{\Big(1+\frac{m}{M}\Big)\frac{\boldsymbol{\sigma}\cdot(\boldsymbol{\sigma}-\boldsymbol{\eta})}{(\boldsymbol{\sigma}-\boldsymbol{\eta})^2}-\frac{[\boldsymbol{\sigma}\cdot(\boldsymbol{\sigma}-\boldsymbol{\eta})]^2}{(\boldsymbol{\sigma}-\boldsymbol{\eta})^4}\Big\}, \quad (2.18)$$

$$K_2(\boldsymbol{\sigma},\boldsymbol{\eta})=\frac{1}{2\pi^2}exp\Big[-\frac{v_0^2}{2}(\boldsymbol{\sigma}-\boldsymbol{\eta})^2\Big]\Big\{2\frac{\boldsymbol{\sigma}\cdot(\boldsymbol{\sigma}-\boldsymbol{\eta})}{(\boldsymbol{\sigma}-\boldsymbol{\eta})^2}-\frac{[\boldsymbol{\sigma}\cdot(\boldsymbol{\sigma}-\boldsymbol{\eta})]^2}{(\boldsymbol{\sigma}-\boldsymbol{\eta})^4}\Big\}, \quad (2.19)$$

$$K_3(\boldsymbol{\sigma},\boldsymbol{\eta})=\frac{1}{2\pi^2}exp\Big[-\frac{v_0^2}{2}(\boldsymbol{\sigma}-\boldsymbol{\eta})^2\Big]\Big[\frac{2\boldsymbol{\sigma}\cdot\boldsymbol{\eta}}{\eta^4}-\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\eta})^2}{\eta^4}\Big]. \quad (2.20)$$

Let us take the wave vector $\boldsymbol{k}$ to be along the z axis. Operating $\int_\infty^{\sigma_z} d\sigma_z' e^{\frac{i\omega\sigma_z'}{k}}$ on both sides of (2.16), we obtain

$$F(\boldsymbol{\sigma})=F_0(\boldsymbol{\sigma})-\frac{n\Gamma}{k}e^{-\beta\sigma_z}\int_\infty^{\sigma_z}d\sigma_z'\,e^{\beta\sigma_z'}\int d\boldsymbol{\eta}\,K(\boldsymbol{\sigma}',\boldsymbol{\eta})\,F(\boldsymbol{\eta}), \quad (2.21)$$

where

$$F_0(\boldsymbol{\sigma})=-e^{-\beta\sigma_z}\frac{ine}{km}\boldsymbol{E}_0\cdot\boldsymbol{p}(\boldsymbol{\sigma}), \quad (2.22)$$

$$\boldsymbol{p}(\boldsymbol{\sigma})=\int_\infty^{\sigma_z}d\sigma_z'\,e^{-\frac{(\sigma' v_0)^2}{2}+\beta\sigma_z'}\,\boldsymbol{\sigma}', \quad (2.23)$$

$$\boldsymbol{\sigma}'=(\boldsymbol{\sigma}_\perp,\sigma_z'),\qquad \beta=\frac{i\omega}{k}.$$

If we restrict ourselves to the case of few collisions, we can solve (2.21) by the method of iteration. The solution is given by

$$F(\boldsymbol{\sigma})=F_0(\boldsymbol{\sigma})-\frac{n\Gamma}{k}e^{-\beta\sigma_z}\int_\infty^{\sigma_z}d\sigma_z'e^{\beta\sigma_z'}\int d\boldsymbol{\eta}\,K(\boldsymbol{\sigma}',\boldsymbol{\eta})\,F_0(\boldsymbol{\eta}). \quad (2.24)$$

This is valid to the first order in the ratio of the wavelength to the mean free path.

Now that the velocity distribution function is obtained we can calculate the electrical current $\boldsymbol{j}(\boldsymbol{k},\omega)$ as

$$\boldsymbol{j}(\boldsymbol{k},\omega)=-e\int d\boldsymbol{v}\,\boldsymbol{v}\,f'(\boldsymbol{k},\boldsymbol{v},\omega)$$
$$=-ie\Big[\frac{\partial}{\partial\boldsymbol{\sigma}}F(\boldsymbol{k},\boldsymbol{\sigma},\omega)\Big]_{\boldsymbol{\sigma}=0}. \quad (2.25)$$

As we are concerned with the transverse conductivity, we take the direction of $\boldsymbol{E}_0$ to be along the x axis. Then we have

$$p_x=\sigma_x\int_\infty^{\sigma_z}d\sigma_z'\,e^{-\frac{(\sigma' v_0)^2}{2}+\beta\sigma_z'}=-\frac{\sqrt{2}}{v_0}\,\sigma_x\,e^{\left(\frac{\beta}{v_0}\right)^2-\frac{\sigma_\perp^2 v_0^2}{2}}\,\mathrm{erf}\,(\sigma_+), \quad (2.26)$$

where

$$\sigma_+=\frac{1}{\sqrt{2}}\Big(\sigma_z v_0-\frac{\beta}{v_0}\Big). \quad (2.27)$$

From (2.24) – (2.27) we can calculate the current

$$j_x(\boldsymbol{k},\omega)=-ie\left[\frac{\partial}{\partial\sigma_x}F(\boldsymbol{k},\boldsymbol{\sigma},\omega)\right]_{\boldsymbol{\sigma}=0}$$

$$=\frac{ne^2}{m}\frac{\sqrt{2}}{kv_0}e^{\left(\frac{\beta}{\sqrt{2}\,v_0}\right)^2}\left[\operatorname{erf}\left(\frac{-\beta}{\sqrt{2}\,v_0}\right)+\frac{n\Gamma}{k}I\right]E_{0x}, \tag{2.28}$$

where

$$I=\sum_j I_j=\sum_j\int_0^\infty d\sigma_z'\int d\boldsymbol{\eta}\,\eta_x\operatorname{erf}(\eta_+)\,e^{-\frac{\eta_\perp^2 v_0^2}{2}+\beta(\sigma_z'-\eta_z)}\left[\frac{\partial K_j(\boldsymbol{\sigma}',\boldsymbol{\eta})}{\partial\sigma_x}\right]_{\boldsymbol{\sigma}=0}, \tag{2.29}$$

and

$$\eta_+=\frac{1}{\sqrt{2}}\left(\eta_z v_0-\frac{\beta}{v_0}\right). \tag{2.30}$$

Thus we obtain the electrical conductivity $\sigma_x(\boldsymbol{k},\omega)$ defined by

$$j_x(\boldsymbol{k},\omega)=\sigma_x(\boldsymbol{k},\omega)\,E_{0x}(\boldsymbol{k},\omega), \tag{2.31}$$

where

$$\sigma_x(\boldsymbol{k},\omega)=\frac{-\omega_p^2}{4\pi i\omega}\sqrt{2}\,a\,e^{\frac{a^2}{2}}\left[\operatorname{erf}\left(\frac{a}{\sqrt{2}}\right)+\frac{\nu_c v_0^3}{k}I\right], \tag{2.32}$$

$$a=-\frac{i\omega}{kv_0}=-\frac{\beta}{v_0}, \tag{2.33}$$

In (2.32) we have introduced the electron collision frequency defined by

$$\nu_c=\frac{n\Gamma}{v_0^3}. \tag{2.34}$$

The dispersion relation for the transverse wave is given in terms of the electrical conductivity as (see Appendix)

$$\omega^2-c^2k^2=-4\pi i\omega\,\sigma_x(\boldsymbol{k},\omega), \tag{2.35}$$

where c is the velocity of light. From (2.32) and (2.35) we can rewrite the dispersion relation

$$1+\frac{\sqrt{2}\,\alpha}{v_0}e^{\frac{a^2}{2}}\left[\operatorname{erf}\left(\frac{a}{\sqrt{2}}\right)+\frac{\nu_c v_0^3}{k}I\right]=0, \tag{2.36}$$

where

$$\alpha=\frac{\omega_p^2}{\omega^2-c^2k^2}\frac{i\omega}{k} \tag{2.37}$$

has been introduced. The dispersion relation obtained here agrees with (44) of Buti and Jain that was derived by slightly different method. Our next task is to evaluate I_j's. The following two sections are devoted to this.

III. Evaluation of I_3

Differentiating $K_3(\boldsymbol{\sigma}', \boldsymbol{\eta})$ given by (2.20) with respect to σ_x and then putting $\boldsymbol{\sigma}=0$, we have

$$\left[\frac{\partial K_3(\boldsymbol{\sigma}', \boldsymbol{\eta})}{\partial \sigma_x}\right]_{\boldsymbol{\sigma}=0} = \frac{\eta_x}{\pi^2} e^{-\frac{v_0^2}{2}[\eta_\perp^2-(\sigma_z'-\eta_z)^2]}$$

$$\times\left[\frac{v_0^2\eta_z\sigma_z'}{\eta^2}\left(1-\frac{\sigma_z'\eta_z}{2\eta^2}\right)+\frac{1}{\eta^2}-\frac{\sigma_z'\eta_z}{\eta^4}\right]. \tag{3.1}$$

Substitution of this into (2.29) yields

$$I_3 = \frac{1}{\pi^2}\int_0^\infty d\sigma_z' \int d\boldsymbol{\eta}\, \eta_x^2 \operatorname{erf}(\eta_+)\, e^{-av_0(\sigma_z'-\eta_z)-v_0^2\left(\eta_\perp^2+\frac{\eta_z^2}{2}+\frac{\sigma_z'^2}{2}-\sigma_z'\eta_\perp\right)}$$

$$\times\left[\frac{v_0^2\eta_z\sigma_z'}{\eta^2}\left(1-\frac{\sigma_z'\eta_z}{2\eta^2}\right)+\frac{1}{\eta^2}-\frac{\sigma_z'\eta_z}{\eta^4}\right]. \tag{3.2}$$

The expression (47) of Buti and Jain is almost same as (3.2). The difference lies in the last term in the bracket. They give $\frac{\sigma_z'\eta_z}{2\eta^4}$ in place of $\frac{\sigma_z'\eta_z}{\eta^4}$. Perhaps they made mistake in differentiating $K_3(\boldsymbol{\sigma}', \boldsymbol{\eta})$.

After making σ_z'-integration, we employ the cylindrical coordinates $(\eta_\perp, \theta, \eta_z)$. Performing θ-integration and putting

$$\eta_\perp^2 = y, \qquad \eta_z = z, \qquad \eta_- = \frac{a-v_0\eta_z}{\sqrt{2}}, \tag{3.3}$$

we have

$$\sqrt{2}\,\pi v_0 I_3 = 2e^{\frac{a^2}{2}}\int_{-\infty}^{\infty} dz \operatorname{erf}(\eta_+) \operatorname{erf}(\eta_-) \int_0^\infty dy\, e^{-v_0^2 y}\frac{y}{y+z^2}A$$

$$+\sqrt{2}\int_{-\infty}^{\infty} dz \operatorname{erf}(\eta_+)\, e^{av_0 z-\frac{v_0^2 z^2}{2}} \int_0^\infty dy\, e^{-v_0^2 y}\frac{y}{y+z^2}B, \tag{3.4}$$

where

$$A = \left(\frac{1}{2}-\frac{v_0\eta_{-z}}{\sqrt{2}}\right)+\left\{\frac{\eta_{-z}}{\sqrt{2}\,v_0}-\frac{z^2}{2}\left(\frac{1}{2}+\eta_-^2\right)\right\}\frac{1}{y+z^2}, \tag{3.5}$$

$$B = \frac{v_0 z}{2}+\frac{1}{2}\left(\frac{\sqrt{2}\,z^2\eta_-}{2}-\frac{z}{v_0}\right)\frac{1}{y+z^2}. \tag{3.6}$$

In order to make y-integration we define integrals I_A and I_B by

$$I_{A,B} = 4v_0^2\int_0^\infty dy\, e^{-v_0^2 y}\frac{y}{y+z^2}\{A, B\}. \tag{3.7}$$

By substituting (3.5) and (3.6) into (3.7) and using the following formulae

$$\int_0^\infty dy\, e^{-v_0^2 y} = \frac{1}{v_0^2}\,,$$

$$\int_0^\infty dy\, e^{-v_0^2 y}\frac{1}{y+z^2} = -e^{v_0^2 z^2} E_i(-v_0^2 z^2)\,, \tag{3.8}$$

$$\int_0^\infty dy\, e^{-v_0^2 y}\frac{1}{(y+z^2)^2} = \frac{1}{z^2} + v_0^2\, e^{v_0^2 z^2} E_i(-v_0^2 z^2),$$

we have

$$I_A = I_{A1} + I_{A2}\,, \qquad I_B = I_{B1} + I_{B2}\,, \tag{3.9}$$

with

$$\begin{aligned} I_{A1} &= (2+5\theta^2+\theta^4)-(4\theta+2\theta^3)\, a+\theta^2 a^2\,, \\ I_{A2} &= e^{\theta^2} E_i(-\theta^2)[(5\theta^2+6\theta^4+\theta^6)-(2\theta+6\theta^3+2\theta^5)\, a+(\theta^2+\theta^4)\, a^2], \\ I_{B1} &= (4\theta+\theta^3)-\theta^2 a\,, \\ I_{B2} &= e^{\theta^2} E_i(-\theta^2)[(2\theta+5\theta^3+\theta^5)-(\theta^2+\theta^4)\, a], \end{aligned} \tag{3.10}$$

where $\theta = v_0 z$ and $E_i(-x)$ is the exponential integral. By taking account of the asymptotic expansion of the error function, i.e.,

$$\operatorname{erf}(z) \approx \frac{e^{-z^2}}{2z}\left[1 - \frac{1}{2z^2} + \frac{1\cdot 3}{2^2 z^4} - \frac{1\cdot 3\cdot 5}{2^3 z^6} + \ldots\ldots\right], \quad |z| \gg 1, \tag{3.11}$$

we obtain

$$2\, e^{\frac{a^2}{2}} \operatorname{erf}(\eta_+) \operatorname{erf}(\eta_-) = e^{-\frac{a^2}{2}-\theta^2} P_A(\theta)\,, \tag{3.12}$$

$$\sqrt{2}\, e^{\frac{a^2}{2}} \operatorname{erf}(\eta_+) = e^{-\frac{a^2}{2}-\theta^2} P_B(\theta)\,, \tag{3.13}$$

with

$$P_A(\theta) = \frac{1}{a^2} + \frac{\theta^2-2}{a^4} + \frac{\theta^4-8\theta^2+7}{a^6}\,, \tag{3.14}$$

$$P_B(\theta) = \frac{1}{a} - \frac{\theta}{a^2} + \frac{\theta^2-1}{a^3} - \frac{\theta^3-3\theta}{a^4} + \frac{\theta^4-6\theta^2+3}{a^5}\,. \tag{3.15}$$

With use of (3.4) – (3.15) we can rewrite I_3 as

$$I_3 = \frac{e^{-\frac{a^2}{2}}}{4\sqrt{2}\,\pi v_0^4}[Q_1+Q_2]\,, \tag{3.16}$$

where

$$Q_1 = \int_{-\infty}^{\infty} dy\, e^{-\theta^2}(P_A I_{A1}+P_B I_{B1}) = \int_{-\infty}^{\infty} dy\, e^{-\theta^2}\left(\frac{2}{a^2} + \frac{8\theta^2-4}{a^4}\right) = \frac{2\sqrt{\pi}}{a^2}\,, \tag{3.17}$$

$$Q_2=\int_{-\infty}^{\infty} dy\, e^{-\theta^2}\,(P_A I_{A2}+P_B I_{B2})$$

$$=\int_{-\infty}^{\infty} d\theta\, E_i(-\theta^2)\left(\frac{2\theta^2}{a^2}+\frac{8\theta^4}{a^2}\right)=-\frac{\sqrt{\pi}}{a^2}\left(\frac{2}{3}+\frac{12}{5a^2}\right). \tag{3.18}$$

In evaluating Q_2 we have used the formula

$$E_i(-\theta^2)=-\int_1^{\infty}\frac{d\nu}{\nu}\,e^{-\nu\theta^2}. \tag{3.19}$$

Combining (3.16), (3.17) and (3.18) we have

$$I_3=\frac{e^{-\frac{a^2}{2}}}{3\sqrt{2\pi v_0{}^4 a^2}}\left(1-\frac{9}{5a^2}\right). \tag{3.20}$$

This differs from the corresponding expression of Buti and Jain as it should.

IV. Evaluations of I_1 and I_2

Differentiating $K_1(\boldsymbol{\sigma}',\boldsymbol{\eta})$ with respect to σ_x and then introducing $\boldsymbol{\xi}$ and μ which are defined by

$$\boldsymbol{\xi}=\sigma_z'\boldsymbol{e}_z-\boldsymbol{\eta}\,,\qquad \mu=\frac{m+M}{mM}\,, \tag{4.1}$$

we have, from (2.29),

$$I_1=\frac{1}{2\pi^2}\int_0^{\infty} d\sigma_z\int_{-\infty}^{\infty} d\xi_z\int_0^{\infty} d\xi_\perp\,\xi_\perp\int_0^{2\pi} d\theta\,\frac{\xi_\perp{}^2\cos^2\theta}{\xi^2}\,e^{-av_0\xi_z+\frac{V_0^2}{2}\xi_z^2-b\xi_\perp^2}$$

$$\times\,\mathrm{erf}\,(\xi_+)\left[a_0+\frac{a_2}{\xi^2}+\frac{a_4}{\xi^4}\right], \tag{4.2}$$

where

$$\begin{aligned}
a_0&=m\mu(V_0{}^2\sigma_z\xi_z-1),\\
a_2&=2(1+m\mu)\,\sigma_z\xi_z-V_0{}^2\sigma_z{}^2\xi_z{}^2,\\
a_4&=-4\sigma_z{}^2\xi_z{}^2,\\
2b&=V_0{}^2+v_0{}^2,
\end{aligned} \tag{4.3}$$

and

$$\xi_+=\frac{\varDelta_+}{\sqrt{2}}\,,\qquad \varDelta_+=a+v_0(\sigma_z-\xi_z)\,. \tag{4.4}$$

In the above expression the cylindrical coordinates have been introduced. The expression (4 2) corresponds to (63) of Buti and Jain. They give $a_4=-\sigma_z{}^2\xi_z{}^2$ in place of $-4\sigma_z{}^2\xi_z{}^2$. It seems that they made mistake in the differentiation of $K_1(\boldsymbol{\sigma}',\boldsymbol{\eta})$.

Putting $\xi_z = z$, $\xi_\perp{}^2 = y$ and making integration with respect to θ, we have

$$I_1 = \frac{1}{4\pi}\int_0^\infty d\sigma_z \int_{-\infty}^\infty dz \,\mathrm{erf}\,(\xi_+)\, e^{-av_0z - \frac{v_0{}^2z^2}{2}} J(\sigma_z, z), \tag{4.5}$$

where

$$J(\sigma_z, z) = \int_0^\infty dy\, e^{-by} \frac{y}{y+z^2}\left[a_0 + \frac{a_2}{y+z^2} + \frac{a_4}{(y+z^2)^2}\right]. \tag{4.6}$$

If we define L_n by

$$L_n = \int_0^\infty dy \frac{e^{-by}}{(y+z^2)^n}, \tag{4.7}$$

the following relations hold

$$L_n = \frac{1}{n-1}\left[\frac{1}{z^{2(n-1)}} - bL_{n-1}\right], \qquad (n > 1), \tag{4.8}$$

and

$$L_0 = \frac{1}{b}, \qquad L_1 = -e^{bz^2} E_i(-bz^2), \qquad L_2 = \frac{1}{z^2} - bL_1, \tag{4.9}$$

$$L_3 = \frac{1}{2z^4} - \frac{b}{2z^2} + \frac{b^2}{2} L_1 .$$

By making use of these relations it follows that

$$J(\sigma_z, z) = J_0 + J_1 L_1 , \tag{4.10}$$

where

$$J_0 = \frac{a_0}{b} - a_2 + \frac{a_4 b}{2} + \frac{a_4}{2z^2}, \tag{4.11}$$

$$J_1 = -z^2 a_0 + (1 + b_z{}^2)\, a_2 - \left(b + \frac{z^2b^2}{2}\right) a_4 . \tag{4.12}$$

With the help of (4.10), the expression (4.5) is rewritten as

$$I_1 = \frac{1}{4\pi}\int_0^\infty d\sigma_z\, (S_0 + S_1), \tag{4.13}$$

where

$$S_0 = \int_{-\infty}^\infty dz \,\mathrm{erf}\,(\xi_+)\, e^{-av_0z - \frac{v_0{}^2z^2}{2}} J_0 , \tag{4.14}$$

$$S_1 = \int_{-\infty}^\infty dz \,\mathrm{erf}\,(\xi_+)\, e^{-av_0z - \frac{v_0{}^2z^2}{2}} L_1 J_1 . \tag{4.15}$$

From the asymptotic expansion of the error function

$$\mathrm{erf}\left(\frac{\Delta}{\sqrt{2}}\right)=\frac{e^{-\frac{\Delta^2}{2}}}{\sqrt{2}}\sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!}{\Delta^{2n+1}}, \qquad (\Delta\gg 1), \tag{4.16}$$

we can easily obtain

$$S_0=\frac{1}{\sqrt{2}}e^{-\frac{a^2}{2}-ax-\frac{gx^2}{2}}\sum_n(-1)^n(2n-1)!!\int_{-\infty}^{\infty}dt\,\frac{e^{-bt^2}J_0(t)}{(h-v_0t)^{2n+1}}, \tag{4.17}$$

$$S_1=\frac{1}{\sqrt{2}}e^{-\frac{a^2}{2}}\int_1^{\infty}\frac{d\nu}{\nu}e^{-ax-\frac{\tilde{g}x^2}{2}}\sum_n(-1)^n(2n-1)!!\int_{-\infty}^{\infty}d\tilde{t}\,\frac{e^{-b\nu\tilde{t}^2}J_1(\tilde{t})}{(\tilde{h}-v_0\tilde{t})^{2n+1}}, \tag{4.18}$$

where

$$t=z-\frac{v_0x}{2b},\qquad g=1-\frac{v_0^2}{2b},\qquad h=a+gx,$$

$$\tilde{t}=z-\frac{v_0x}{2b\nu},\qquad \tilde{g}=1-\frac{v_0^2}{2b\nu},\qquad \tilde{h}=a+\tilde{g}x, \tag{4.19}$$

$$x=v_0\sigma_z.$$

In deriving (4.18) we have used the relation

$$L_1=-e^{bz^2}E_i(-bz^2)=\int_1^{\infty}\frac{d\nu}{\nu}e^{-(\nu-1)bz^2},$$

which follows from (3.19). Substitutions of (4.3) and (4.19) into (4.11) and (4.12) yield

$$J_0(t)=-\left(\frac{m\mu}{b}+2\sigma_z^2\right)+\left\{\frac{m\mu V_0^2}{b}-2(1+m\mu)\right\}\sigma_z\left(t+\frac{v_0x}{2b}\right)-v_0^2\sigma_z^2\left(t+\frac{v_0x}{2b}\right)^2$$

$$\equiv k_0+k_1t+k_2t^2, \tag{4.20}$$

$$J_1(\tilde{t})=2(1+m\mu)\sigma_z\left(\tilde{t}+\frac{v_0x}{2b\nu}\right)+\{(V_0^2+2v_0^2)\sigma_z^2-m\mu\}\left(\tilde{t}+\frac{v_0x}{2b\nu}\right)^2$$

$$+\{2b(1+m\mu)-m\mu V_0^2\}\sigma_z\left(\tilde{t}+\frac{v_0x}{2b\nu}\right)^3+bv_0^2\sigma_z^2\left(\tilde{t}+\frac{v_0x}{2b\nu}\right)^4$$

$$\equiv\tilde{k}_0+\tilde{k}_1\tilde{t}+\tilde{k}_2\tilde{t}^2+\tilde{k}_3\tilde{t}^3+\tilde{k}_4\tilde{t}^4. \tag{4.21}$$

Now we define H_{nm} by

$$H_{nm}=\int_{-\infty}^{\infty}dt\,e^{-bt^2}\frac{t^m}{(h-v_0t)^n}. \tag{4.22}$$

In case of $v_0/h\ll 1$, H_{nm} becomes

$$H_{nm}=\frac{1}{h^n}\sum_{i=0}^{\infty}\frac{\Gamma(n+i)}{\Gamma(n)\Gamma(i+1)}\left(\frac{v_0}{h}\right)^i\left(\frac{1}{b}\right)^{\frac{m+i+1}{2}}\Gamma\left(\frac{m+i+1}{2}\right)\frac{1+(-1)^{m+i}}{2}. \tag{4.23}$$

By making use of (4.20), (4.21) and (4.22) we obtain

$$S_0=\frac{e^{-\frac{a^2}{2}-ax-\frac{gx^2}{2}}}{\sqrt{2}}\sum_{n=0}^{\infty}(-1)^n(2n-1)!!\,[k_0H_{2n+1,0}+k_1H_{2n+1,1}+k_2H_{2n+1,2}], \tag{4.24}$$

$$S_1=\frac{e^{-\frac{a^2}{2}-ax}}{\sqrt{2}}\int_1^{\infty}\frac{d\nu}{\nu}e^{-\frac{\tilde{g}x^2}{2}}\sum_n(-1)^n(2n-1)!!$$

$$\times[\tilde{k}_0\tilde{H}_{2n+1,0}+\tilde{k}_1\tilde{H}_{2n+1,1}+\tilde{k}_2\tilde{H}_{2n+1,2}+\tilde{k}_3\tilde{H}_{2n+1,3}+\tilde{k}_4\tilde{H}_{2n+1,4}], \tag{4.25}$$

where $\tilde{H}_{nm}$ is obtained by replacing b and $\tilde{h}$ by $b\nu$ and h, respectively, in the expression of H_{nm}. From (4.23) it follows that up to the order h^{-5}

$$\sum_{n=0}^{\infty}(-1)^n(2n-1)!!\,H_{2n+1,0}=\left(\frac{1}{b}\right)^{\frac{1}{2}}\Gamma\left(\frac{1}{2}\right)\left[\frac{1}{h}-\frac{g}{h^3}+\frac{1}{h^5}\left(3-\frac{3v_0^2}{b}+\frac{3v_0^4}{4b^2}\right)\right], \tag{4.26}$$

$$\sum_{n=0}^{\infty}(-1)^n(2n-1)!!\,H_{2n+1,1}=\left(\frac{1}{b}\right)^{\frac{3}{2}}\Gamma\left(\frac{3}{2}\right)\left[\frac{v_0}{h^2}-\frac{3gv_0}{h^4}\right]. \tag{4.27}$$

$$\sum_{n=0}^{\infty}(-1)^n(2n-1)!!\,H_{2n+1,2}=\left(\frac{1}{b}\right)^{\frac{3}{2}}\Gamma\left(\frac{3}{2}\right)\left[\frac{1}{h}+\frac{1}{h^3}\left(\frac{3v_0^2}{2b}-1\right)\right.$$

$$\left.+\frac{1}{h^5}\left(3-\frac{9v_0^2}{b}+\frac{15v_0^4}{4b^2}\right)\right], \tag{4.28}$$

$$\sum_{n=0}^{\infty}(-1)^n(2n-1)!!\,H_{2n+1,3}=\left(\frac{1}{b}\right)^{\frac{5}{2}}\Gamma\left(\frac{5}{2}\right)\left[\frac{v_0}{h^2}-\frac{v_0}{h^4}\left(3-\frac{5v_0^2}{2b}\right)\right], \tag{4.29}$$

$$\sum_{n=0}^{\infty}(-1)^n(2n-1)!!\,H_{2n+1,4}=\left(\frac{1}{b}\right)^{\frac{5}{2}}\Gamma\left(\frac{5}{2}\right)\left[\frac{1}{h}+\frac{1}{h^3}\left(\frac{5v_0^2}{2b}-1\right)\right.$$

$$\left.+\frac{1}{h^5}\left(3-\frac{15v_0^2}{b}+\frac{35v_0^4}{4b^2}\right)\right]. \tag{4.30}$$

By making use of (4.26) – (4.30), we can write (4.24) as

$$S_0(x)=\sqrt{\frac{\pi}{2b}}\,e^{-\frac{a^2}{2}-ax-\frac{gx^2}{2}}\left[\frac{1}{h}\left(k_0+\frac{k_2}{2b}\right)+\frac{1}{h^2}\frac{k_1v_0}{2b}\right.$$

$$+\frac{1}{h^3}\left\{-gk_0+\frac{k_2}{2b}\left(\frac{3v_0^2}{2b}-1\right)\right\}-\frac{k_1}{h^4}\frac{3gv_0}{2b}$$

$$\left.+\frac{1}{h^5}\left\{k_0\left(3-\frac{3v_0^2}{b}+\frac{3v_0^4}{4b^2}\right)+\frac{k^2}{2b}\left(3-\frac{9v_0^2}{b}+\frac{15v_0^4}{4b^2}\right)\right\}\right]. \tag{4.31}$$

For the purpose of simplifying the notation we write the expression (4.31) by taking into account of (4.20) as

$$S_0(x)=\sqrt{\frac{\pi}{2b}}\,e^{-\frac{a^2}{2}-ax-\frac{gx^2}{2}}\Big[\frac{\alpha_0+\alpha_2x^2+\alpha_4x^4}{h}+\frac{\beta_1x+\beta_3x^3}{h^2}$$

$$+\frac{\gamma_0+\gamma_2x^2+\gamma_4x^4}{h^3}+\frac{\delta_1x+\delta_3x^3}{h^4}+\frac{\varepsilon_0+\varepsilon_2x^2+\varepsilon_4x^4}{h^5}\Big]. \qquad (4.32)$$

Now let us define an integral

$$T_{mn}=\int_0^\infty dx\,\frac{x^m}{(a+gx)^n}\,e^{-ax-\frac{gx^2}{2}}. \qquad (4.33)$$

The leading term of the asymptotic expansion of this integral is easily obtained to become

$$T_{mn}\approx\frac{1}{a^n}\int_0^\infty dx\,x^m\,e^{-ax}=\frac{\Gamma(m+1)}{a^{m+n+1}}. \qquad (4.34)$$

Using (4.33), we can write

$$\frac{1}{4\pi}\int_0^\infty d\sigma_z\,S_0(x)=\frac{e^{-\frac{a^2}{2}}}{4\sqrt{2\pi b}\,v_0}\,[\alpha_0T_{01}+\alpha_2T_{21}+\alpha_4T_{41}+\beta_3T_{32}+\gamma_0T_{03}$$

$$+\gamma_2T_{23}+\gamma_4T_{43}+\delta_1T_{14}+\delta_3T_{34}+\varepsilon_0T_{05}+\varepsilon_2T_{25}+\varepsilon_4T_{45}]. \qquad (4.35)$$

When we neglect terms $0\left(\frac{1}{a^6}\right)$, it follows from (4.33) and (4.34) that

$$T_{01}=\frac{1}{a^2}-\frac{2g}{a^4},\qquad T_{21}=\frac{2}{a^4},\qquad T_{12}=T_{03}=\frac{1}{a^4}, \qquad (4.36)$$

and other T_{mn}'s which appear in (4.35) are higher order. Then we have

$$\frac{1}{4\pi}\int_0^\infty d\sigma_z\,S_0(x)=\frac{e^{-\frac{a^2}{2}}}{4\sqrt{2\pi b}\,v_0}\Big[\frac{\alpha_0}{a^2}+\frac{1}{a^4}\,\{2(\alpha_2-\alpha_0g)+\beta_1+\gamma_0\}\Big]. \qquad (4.37)$$

The coefficients α_0, α_2, β_1 and γ_0 are obtained from (4.31), (4.32) and (4.20) to be

$$\begin{aligned}\alpha_0&=-\frac{m\mu}{b},\\ \alpha_2&=-\frac{3}{2b}-\frac{2}{v_0^2}-\frac{m\mu v_0^2}{2b^2},\\ \beta_1&=-\frac{m\mu v_0^2}{2b^2}-\frac{1}{b},\\ \gamma_0&=\frac{m\mu}{b}g.\end{aligned} \qquad (4.38)$$

Substituting these values into (4.37), we have

$$\frac{1}{4\pi}\int_0^\infty d\sigma_z\,S_0(x)=\frac{e^{-\frac{a^2}{2}}}{4\sqrt{2\pi b}\,v_0}\Big[-\frac{m\mu}{ba^2}+\frac{1}{a^4}\Big\{-\frac{4}{v_0^2}+\frac{3m\mu-4}{b}-\frac{3m\mu v_0^2}{b^2}\Big\}\Big]. \qquad (4.39)$$

Similar calculations as above give

$$\frac{1}{4\pi}\int_0^\infty d\sigma_z\, S_1(x) = \frac{e^{-\frac{a^2}{2}}}{4\sqrt{2\pi}\, b\, v_0}\left[\frac{m\mu}{3ba^2}+\frac{1}{a^4}\left\{\frac{8}{3v_0^2}+\frac{1}{b}\left(4\frac{2}{5}+m\mu-\frac{2V_0^2}{3v_0^2}\right)+\frac{9m\mu v_0^2}{5b^2}\right\}\right]. \quad (4.40)$$

Then we find

$$I_1 = \frac{e^{-\frac{a^2}{2}}}{4\sqrt{2\pi}\, b\, v_0}\left[-\frac{2m\mu}{3ba^2}+\frac{1}{a^4}\left\{-\frac{4}{3v_0^2}+\frac{2}{5b}+\frac{4m\mu}{b}-\frac{2V_0^2}{3v_0^2\, b}-\frac{6}{5}\frac{m\mu v_0^2}{b^2}\right\}\right]. \quad (4.41)$$

In the limiting case of $m/M=0$ and $V_0^2/v_0^2=0$,

$$m\mu = 1, \qquad 2b = v_0^2,$$

hence the expression (4.41) reduces to

$$I_1 = -\frac{e^{-\frac{a^2}{2}}}{3\sqrt{\pi}\, v_0^4 a^2}\left(1-\frac{2}{a^2}\right). \quad (4.42)$$

If we put $V_0^2=v_0^2$, $M=m$ in (4.41) we have

$$I_2 = -\frac{e^{-\frac{a^2}{2}}}{3\sqrt{2\pi}\, v_0^4 a^2}\left(1-\frac{3}{a^2}\right). \quad (4.43)$$

V. Results and Discussion

The collision integrals I_1 and I_2+I_3 are due to electron-ion and electron-electron collisions respectively. From (3.20) (4.42) and (4.43) it follows that

$$I_1 = -\frac{e^{-\frac{a^2}{2}}}{3\sqrt{\pi}\, v_0^4}\left(\frac{1}{a^2}-\frac{2}{a^4}\right), \quad (5.1)$$

$$I_2+I_3 = \frac{e^{-\frac{a^2}{2}}}{3\sqrt{\pi}\, v_0^4}\frac{3\sqrt{2}}{5a^4}. \quad (5.2)$$

Absence of a^{-2} term in the expression I_2+I_3 is due to the conservation of momentum in the electron-electron collision.

Substitutions of the asymptotic expansion of the error function, i.e.,

$$\mathrm{erf}\left(\frac{a}{\sqrt{2}}\right)\approx\frac{e^{-\frac{a^2}{2}}}{\sqrt{2}\, a}\left(1-\frac{1}{a^2}+\ldots\ldots\right),$$

and (5.1) and (5.2) into (2.32) yield the transverse high-frequency conductivity $\sigma_x(\boldsymbol{k},\omega)$;

$$\sigma_x(\boldsymbol{k},\omega) = -\frac{\omega_p^2}{4\pi i\omega}\left[1+\frac{\kappa^2}{\Omega^2}+\frac{\varLambda}{i\Omega}\left\{1+K_{OG}\frac{\kappa^2}{\Omega^2}\right\}\right], \quad (5.3)$$

with

$$K_{OG} = 2+\frac{3\sqrt{2}}{5}, \quad (5.4)$$

$$\kappa=\frac{k}{k_D},\quad \Omega=\frac{\omega}{\omega_p},\quad \Lambda=\frac{1}{\omega_p}\frac{\sqrt{2}\,\nu_c}{3\sqrt{\pi}},$$

$$k^2=\frac{4\pi ne^2}{T},\qquad \omega_p{}^2=\frac{4\pi ne^2}{m},$$

where k_D and ω_p Deby's characteristic constant and the electron plasma frequency. The same result is derived with use of the Boltzmann equation [6]. Based on the BBGKY theory Thourson and Lewis also give $\sigma_x(\boldsymbol{k}\ \omega)$ which agrees with (5.3) except for the argument of the coulomb logarithm. They simplify greatly the BBGKY equation by dropping several terms which involve the screening effect and by introducing the cut off into the coulomp potential. Berk employs Dawson and Oberman's model to calculate $\sigma_x(\boldsymbol{k},\omega)$ But the electron-electron collision cannot De taken into account in his theory. As far as the electron-ion collision concerns Berk's result agrees with (5.3).

The dispersion relation for the transverse oscillation is obtained by substituting the expression (5.3) into (2.36) :

$$\Omega^2=1+\left(\frac{c}{v_0}\right)^2\kappa^2+\frac{\kappa^2}{\Omega^2}-\frac{i\Lambda}{\Omega}-\frac{i\Lambda\kappa^2}{\Omega^3}K_{OG}. \tag{5.5}$$

We have made several assumptions for the purpose of obtaining (5.5). First we have assumed

$$\frac{\text{wavelength}}{\text{mean free path}}\ll 1, \tag{5.6}$$

to obtain (2.24). This is equivalent to

$$\frac{\Lambda}{\Omega}\ll 1, \tag{5.7}$$

since

$$\frac{\Lambda}{\Omega}\approx\frac{\nu_c}{\omega}=\frac{v_0/\omega}{v_0/\nu_c}=\frac{\text{wavelength}}{\text{mean free path}}. \tag{5.8}$$

Second we have made asymptotic expansion assuming

$$|\,a\,|\gg 1. \tag{5.9}$$

By taking into account of (2.33) we have

$$|a|^2=\frac{\omega^2}{k^2v_0{}^2}=\left(\frac{\omega}{\omega_p}\frac{\omega_p}{v_0}\frac{1}{k}\right)^2=\left(\frac{\Omega}{\kappa}\right)^2. \tag{5.10}$$

Hence (5.9) is expressed by

$$\left(\frac{\kappa}{\Omega}\right)^2\ll 1. \tag{5.11}$$

Considering (5.7), (5.11), we can deduce from (5.5) that

$$\Omega^2\gtrapprox 1. \tag{5.12}$$

Thus we will solve (5.5) in the limit

$$\Lambda \ll 1, \qquad \kappa^2 \ll 1. \tag{5.13}$$

To the zeroth order we have

$$\Omega_0{}^2 = 1 + \left(\frac{c}{v_0}\right)^2 \kappa^2. \tag{5.14}$$

The second term in this expression is taken as $O(1)$ due to $\left(\frac{c}{v_0}\right)^2 \gg 1$.

To the first order in small quantities we have

$$\Omega_1{}^2 = \Omega_0{}^2 + \left(\frac{\kappa}{\Omega_0{}^2}\right)^2 - \frac{i\Lambda}{\Omega_0{}^2}. \tag{5.15}$$

Next we have to the order of $\kappa^2\Lambda$

$$\Omega_2{}^2 = \Omega_0{}^2 + \left(\frac{\kappa}{\Omega_1}\right)^2 - \frac{i\Lambda}{\Omega_1} - \frac{i\Lambda\kappa^2}{\Omega_0{}^3} K_{OG}. \tag{5.16}$$

Substituting (5.15) into (5.16), we obtain

$$\Omega^2 = \Omega_0{}^2 + \left(\frac{\kappa}{\Omega_0}\right)^2 - \frac{i\Lambda}{\Omega_0} - \frac{i\Lambda\kappa^2}{\Omega_0{}^3}\left(K_{OG} - \frac{3}{2\Omega_0{}^2}\right). \tag{5.17}$$

In order to obtain the real and the imaginary parts Ω_r and Ω_i of Ω, we put

$$\Omega = \Omega_r + i\Omega_i, \qquad \left|\frac{\Omega_i}{\Omega_r}\right| \ll 1, \tag{5.18}$$

then

$$\Omega^2 = \Omega_r{}^2 + 2i\Omega_r\Omega_i.$$

From (5.17) we can obtain

$$\Omega_r{}^2 = \Omega_0{}^2 + \frac{\kappa^2}{\Omega_0{}^2}, \tag{5.19}$$

$$\begin{aligned}\Omega_i &= -\frac{\Lambda}{2\Omega_0{}^2}\left(1 - \frac{\kappa^2}{2\Omega_0{}^4}\right)\left[1 + \frac{\kappa^2}{\Omega_0{}^2}\left(K_{OG} - \frac{3}{2\Omega_0{}^2}\right)\right] \\ &= -\frac{\Lambda}{2\Omega_0{}^2}\left[1 + \frac{\kappa^2}{\Omega_0{}^2}\left(K_{OG} - \frac{2}{\Omega_0{}^2}\right)\right].\end{aligned} \tag{5.20}$$

Thus we can write down the results

$$\omega_r{}^2 = \omega_p{}^2 + c^2k^2 + \frac{\omega_p{}^4}{\omega_p{}^2 + c^2k^2}\left(\frac{k}{k_D}\right)^2, \tag{5.21}$$

$$\omega_i = -\frac{\nu_c}{3\sqrt{2\pi}}\frac{\omega_p{}^2}{\omega_p{}^2 + c^2k^2}\left[1 + \frac{\omega_p{}^2}{\omega_p{}^2 + c^2k^2}\left(\frac{k}{k_D}\right)^2\left(K_{OG} - \frac{2\omega_p{}^2}{\omega_p{}^2 + c^2k^2}\right)\right], \tag{5.22}$$

with

$$K_{OG} = 2 + \frac{3\sqrt{2}}{5}. \tag{5.23}$$

This is rearranged as

$$\omega_i = \omega_{e-e} + \omega_{e-i} , \tag{5.24}$$

$$\omega_{e-e} = -\frac{\nu_c}{3\sqrt{2\pi}} \frac{\omega_p{}^2}{\omega_p{}^2+c^2k^2} \frac{3\sqrt{2}}{5} \frac{\omega_p{}^2}{\omega_p{}^2+c^2k^2} \left(\frac{k}{k_D}\right)^2, \tag{5.25}$$

$$\omega_{e-i} = -\frac{\nu_c}{3\sqrt{2\pi}} \frac{\omega_p{}^2}{\omega_p{}^2+c^2k^2} \left[1+ \frac{\omega_p{}^2}{\omega_p{}^2+c^2k^2} \left(\frac{k}{k_D}\right)^2 \left(2-\frac{2\omega_p{}^2}{\omega_p{}^2+c^2k^2}\right)\right], \tag{5.26}$$

where ω_{e-e} and ω_{e-i} are the damping coefficients due to the electron-electron and electron-ion collisions, respectively.

From (81)—(86) of Buti and Jain we have

$$\omega_i = -\frac{\nu_c}{3\sqrt{2\pi}} \frac{\omega_p{}^2}{\omega_p{}^2+c^2k^2} \left[1+ \frac{\omega_p{}^2}{\omega_p{}^2+c^2k^2} \left(\frac{k}{k_D}\right)^2 \left(K_{BJ}-\frac{\omega_p{}^2}{\omega_p{}^2+c^2k^2}\right)\right], \tag{5.17}$$

with

$$K_{BJ} = \frac{83}{40} + \frac{41\sqrt{2}}{16} . \tag{5.25}$$

Aside from the difference of K_{BJ} from K_{OG}, Buti and Jain's result disagrees with (5.22).

Appendix — Dispersion Relation in Terms of the Electrical Conductivity

From Maxwell's equations given by

$$\text{rot}\, \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t} \tag{A.1}$$

$$\text{rot}\, \boldsymbol{H} = \frac{1}{c}\left(4\pi \boldsymbol{j}_t + \frac{\partial \boldsymbol{E}}{\partial t}\right), \tag{A.2}$$

with

$$\boldsymbol{j}_t = -e\int \boldsymbol{v} f\, d\boldsymbol{v}, \tag{A 3}$$

we have

$$\text{rot rot}\, \boldsymbol{E} = \text{grad div}\, \boldsymbol{E} - \Delta \boldsymbol{E} = -\frac{1}{c^2}\frac{\partial}{\partial t}\left(4\pi \boldsymbol{j}_t + \frac{\partial \boldsymbol{E}}{\partial t}\right). \tag{A.4}$$

If we put

$$\boldsymbol{E} = \boldsymbol{E}_0\, e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}, \qquad \boldsymbol{j}_t = \boldsymbol{j}_0\, e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}, \tag{A.5}$$

we can rewrite (A.4) as

$$\left(k^2 - \frac{\omega^2}{c^2}\right)\boldsymbol{E}_0 - \boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{E}_0) = \frac{4\pi i\omega}{c^2}\boldsymbol{j}_0 . \tag{A.6}$$

Taking the direction of the propagation vector $\boldsymbol{k}$ in z axis and then putting

$$\boldsymbol{E}_0 = \boldsymbol{E}_x + \boldsymbol{E}_z , \qquad \boldsymbol{j}_0 = \sigma_x \boldsymbol{E}_x + \sigma_z \boldsymbol{E}_z , \tag{A.7}$$

we have, from (A.6) and (A.7),

$$\left(k^2-\frac{\omega^2}{c^2}\right)\boldsymbol{E}_x-\frac{\omega^2}{c^2}\boldsymbol{E}_z=\frac{4\pi i\omega}{c^2}(\sigma_x\boldsymbol{E}_x+\sigma_z\boldsymbol{E}_z). \tag{A.8}$$

The x and y components of (A.8) give the dispersion relations of the transverse and longitudinal oscillations, respectively. Thus the dispersion relations are represented by

$$\omega^2-c^2k^2=-4\pi i\omega\sigma_x \text{ for transverse oscillation,} \tag{A.9}$$

$$\omega^2=-4\pi i\omega\sigma_z \text{ for longitudinal oscillation.} \tag{A.10}$$