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Effects of Collision on the Propagation of Electromagnetic Wave in a Plasma

— II. Propagation along Homogeneous Static Magnetic Field —

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Abstract

In the lossy plasma electromagnetic wave with any value of frequency is shown to be able to propagate, but the wave is necessarily accompanied by attenuation.

Group velocity differs from velocity of energy transfer and even becomes negative and infinite in the stop band of the case of absence of collision. The velocity of energy transfer varies as the wave propagates through the lossy plasma.

I. Introduction

In the previous paper[1)] (hereafter refered to as the paper I) propagation of electromagnetic wave through a homogeneous isotropic plasma is investigated with consideration of the effects of collision. In the present paper we will treat the case of propagation along a homogeneous static magnetic field externally applied to a plasma.

Originally linearly polarized electromagnetic wave is decomposed into two counter-rotating modes in a magnetically biased plasma. Propagation and attenuation constants etc. for the two modes are calculated and discussed in sections III and IV. In the lossless plasma there is a range of frequencies where the electromagnetic wave cannot propagate (stop band). In the lossy plasma, however, the wave can propagate with any value of the frequency but the wave is always accompanied by attenuation.

In the last chapter V, velocity of energy transfer is shown to vary as the wave

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1) M. Ogasawara, T. Kajiura and Y. Kubo; This Proceedings **19** (1966) 1.

travels through the lossy plasma. Group velocity differs from the velocity of energy transfer in the presence of collision and becomes even negative and infinite in the region of stop band of the case of $\nu=0$, ν being the collision frequency.

Section II is devoted, as preliminaries, to the evaluations of the electrical conductivity and the dielectric constant of the plasma in a homogeneous static magnetic field.

II. Constitutive Parameters of a Plasma in the Presence of Magnetic Field

In order to investigate electromagnetic wave propagation in a lossy plasma, we must know the constitutive parameters of a plasma, i.e., electrical conductivity and dielectric constant. In this section we will calculate these parameters.

Equation of motion for an electron in an electric field $\boldsymbol{E}$ and a static magnetic field $\boldsymbol{H}_0$ is given by

$$m\frac{d\boldsymbol{v}}{dt}+\nu m\boldsymbol{v}=-e(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{H}_0)\ , \tag{2.1}$$

where m, $-e$, and $\boldsymbol{v}$ are mass, charge and velocity of an electron and ν is the collision frequency between an electron and other heavy particles in the plasma. As we confine ourselves to the special case of propagation along the static magnetic field $\boldsymbol{H}_0$ which is parallel to the z direction, the wave electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{H}$ are expected to be in the x-y plane. For an oscillating electric field varying with time as $e^{i\omega t}$, the equation (2.1) is explicitly given by

$$\left.\begin{aligned}(i\omega+\nu)v_x+\omega_c v_y&=-\frac{e}{m}E_x\ ,\\ -\omega_c v_x+(i\omega+\nu)v_y&=-\frac{e}{m}E_y\ ,\end{aligned}\right\} \tag{2.2}$$

where

$$\omega_c=\frac{eH_0}{m}\ .$$

In these expressions subscripts x and y mean x and y components respectively, and ω_c is the cyclotron frequency. From the x-y symmetry of (2.2), it will be convenient to use the rotating coordinate rather than the cartesian one,

$$\boldsymbol{e}_+=\boldsymbol{e}_x+i\boldsymbol{e}_y,\quad \boldsymbol{e}_-=\boldsymbol{e}_x-i\boldsymbol{e}_y, \tag{2.3}$$

where $\boldsymbol{e}_+$ and $\boldsymbol{e}_-$ are the rotating unit vectors and $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$ are the unit vectors in the cartesian coordinate. In the +component, the y-component is equal in magnitude to the x-component and proceeds it by $\pi/2$. Thus as time goes on the +vector rotates in a left-handed sense about the z axis and the −vector in the opposite sense.

With use of the rotating coordinate we can write $\boldsymbol{v}$ and $\boldsymbol{E}$ as

$$\boldsymbol{v}=v_+\boldsymbol{e}_+ + v_-\boldsymbol{e}_-, \quad \boldsymbol{E}=E_+\boldsymbol{e}_+ + E_-\boldsymbol{e}_-, \tag{2.4}$$

and

$$\boldsymbol{v}_\pm \times \boldsymbol{H}_0 = v_\pm H_0(\boldsymbol{e}_x \pm i\boldsymbol{e}_y)\times \boldsymbol{e}_z = iH_0\boldsymbol{v}_\pm \,, \tag{2.5}$$

that is, formally $\boldsymbol{v}_\pm$ and $\boldsymbol{H}_0$ are parallel each other. With the aid of (2.4) and (2.5), we can solve the equation of motion (2.1)

$$v_\pm = -\frac{e}{m}\frac{1}{\nu+i(\omega\pm\omega_c)}E_\pm \,. \tag{2.6}$$

Using this, we have the total current $\boldsymbol{J}_t$

$$\left.\begin{aligned} \boldsymbol{J}_t &= -en(v_+\boldsymbol{e}_+ + v_-\boldsymbol{e}_-) = J_+\boldsymbol{e}_+ + J_-\boldsymbol{e}_- \,, \\ J_\pm &= \varepsilon_0\omega_p{}^2\frac{1}{\nu+i(\omega\pm\omega_c)}E_\pm \,, \\ \omega_p{}^2 &= \frac{ne^2}{m\varepsilon_0} \,, \end{aligned}\right\} \tag{2.7}$$

where n, ω_p and ε_0 are respectively number density of the electrons, electron plasma frequency and dielectric constant of the vacuum. The total current $\boldsymbol{J}_t$ is a sum of the conduction current $\boldsymbol{J}=\sigma\boldsymbol{E}$, σ being the electrical conductivity, and the current due to polarization $\partial\boldsymbol{P}/\partial t$, $\boldsymbol{P}$ being the polarization[2)]

$$\boldsymbol{J}_t = \boldsymbol{J} + \frac{\partial \boldsymbol{P}}{\partial t} = \sigma\boldsymbol{E} + i\omega(\varepsilon-\varepsilon_0)\boldsymbol{E} \,, \tag{2.8}$$

where σ and ε are both real, ε being the dielectric constant of the plasma. From (2.7) and (2.8), we have

$$\left.\begin{aligned} \sigma_\pm &= \varepsilon_0\omega_p{}^2\frac{\nu}{\nu^2+(\omega\pm\omega_c)^2} \,, \\ \frac{\varepsilon_\pm}{\varepsilon_0} &= 1 - \frac{\omega_p{}^2(\omega\pm\omega_c)}{\omega[\nu^2+(\omega\pm\omega_c)^2]} \,. \end{aligned}\right\} \tag{2.9}$$

III. Dispersion Relation

In this section we will solve Maxwell's equations and give the expression of dispersion relation.

As we consider the electromagnetic wave which propagates along the z-direction, the direction of $\boldsymbol{H}_0$, the oscillating electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{H}$ is assumed to vary with time and space as

2) V. L. Ginzburg; *The Propagation of Electromagnetic Waves in Plasmas.* Translated by J. B. Sykes and R. J. Tayler (1964), Pergamon Press, New York.

$$\boldsymbol{E}_{\pm},\ \boldsymbol{H}_{\pm} \propto \exp(i\omega t-\gamma_{\pm}z). \tag{3.1}$$

Here $\gamma_{\pm}$ is the complex propagation constant which is written as

$$\gamma_{\pm}=\alpha_{\pm}+i\beta_{\pm}, \quad \alpha_{\pm},\ \beta_{\pm}=\text{real}, \tag{3.2}$$

here $\alpha_{\pm}$ and $\beta_{\pm}$ mean the attenuation constant and the propagation constant, respectively. The real and imaginary parts of refractive index of the plasma which are written as $n_{r\pm}$ and $n_{i\pm}$, are connected with $\beta_{\pm}$ and $\alpha_{\pm}$ by

$$n_{r\pm}=\frac{c\beta_{\pm}}{\omega}, \qquad n_{i\pm}=\frac{c\alpha_{\pm}}{\omega}, \tag{3.3}$$

c is the velocity of light.

Maxwell's equations for the magnetically biased homogeneous plasma are given by

$$\text{rot}\,\boldsymbol{H}_{\pm}=\sigma_{\pm}\boldsymbol{E}_{\pm}+i\omega\varepsilon_{\pm}\boldsymbol{E}_{\pm}\ , \tag{3.4}$$

$$\text{rot}\,\boldsymbol{E}_{\pm}=-i\omega\mu_0\boldsymbol{H}_{\pm}\ , \tag{3.5}$$

$$\text{div}\,\boldsymbol{E}_{\pm}=0\ , \tag{3.6}$$

$$\text{div}\,\boldsymbol{H}_{\pm}=0\ , \tag{3.7}$$

where we have taken the magnetic peameability of the plasma to be equal to that of the vacuum μ_0. For the circularly polarized fields, rot $\boldsymbol{E}_{\pm}$ is evaluated as

$$\text{rot}\,\boldsymbol{E}_{\pm}=-\gamma_{\pm}\boldsymbol{e}_z\times(\boldsymbol{e}_x\pm i\boldsymbol{e}_y)E_{\pm}=\pm i\gamma_{\pm}\boldsymbol{E}_{\pm}\ . \tag{3.8}$$

Using (3.4), (3.5) and (3.8), we have

$$\pm i\gamma_{\pm}\boldsymbol{H}_{\pm}=(\sigma_{\pm}+i\omega\varepsilon_{\pm})\boldsymbol{E}_{\pm}\ , \tag{3.9}$$

$$\pm\gamma_{\pm}\boldsymbol{E}_{\pm}=-\omega\mu_0\boldsymbol{H}_{\pm}\ . \tag{3.10}$$

Multiplying $\pm\gamma_{\pm}$ on both sides of (3.9), and using (3.10), we have the dispersion relation for circularly polarized waves;

$$\gamma_{\pm}{}^2=-\omega^2\mu_0\varepsilon_{\pm}+i\omega\mu_0\sigma_{\pm}\ . \tag{3.11}$$

From (3.2), (3.3) and (3.11) we have

$$\left.\begin{aligned} n_{r\pm}&=\sqrt{\frac{1}{2}\left[\frac{\varepsilon_{\pm}}{\varepsilon_0}+\sqrt{\left(\frac{\varepsilon_{\pm}}{\varepsilon_0}\right)^2+\left(\frac{\sigma_{\pm}}{\varepsilon_0\omega}\right)^2}\right]}\ , \\ n_{i\pm}&=\sqrt{\frac{1}{2}\left[-\frac{\varepsilon_{\pm}}{\varepsilon_0}+\sqrt{\left(\frac{\varepsilon_{\pm}}{\varepsilon_0}\right)^2+\left(\frac{\sigma_{\pm}}{\varepsilon_0\omega}\right)^2}\right]}\ , \end{aligned}\right\} \tag{3.12}$$

$$\beta_{\pm}=\frac{\omega}{c}n_{r\pm}\ , \qquad \alpha_{\pm}=\frac{\omega}{c}n_{i\pm}\ . \tag{3.13}$$

These expressions are formally similar to those of a homogeneous isotropic plasma[1]. The difference between them lies only in the constitutive parameters of plasmas.

IV. Results of Numerical Evaluation of the Dispersion Relation

In this section we will investigate the dispersion relation based on numerical evaluation.

If the collisions are absent, i.e., $\nu=0$, we have, from (3.13) and (2.9)

$$\left.\begin{aligned} \beta_{\pm}&=\frac{\omega}{c}\sqrt{\frac{1}{2}\left[\frac{\varepsilon_{\pm}}{\varepsilon_0}+\left|\frac{\varepsilon_{\pm}}{\varepsilon_0}\right|\right]}\,, \\ \alpha_{\pm}&=\frac{\omega}{c}\sqrt{\frac{1}{2}\left[-\frac{\varepsilon_{\pm}}{\varepsilon_0}+\left|\frac{\varepsilon_{\pm}}{\varepsilon_0}\right|\right]}\,, \end{aligned}\right\} \tag{4.1}$$

$$\frac{\varepsilon_{\pm}}{\varepsilon_0}=1-\frac{\omega_p^2}{\omega(\omega\pm\omega_c)}\,. \tag{4.2}$$

When $\varepsilon_{\pm}/\varepsilon_0$ is positive, the propagation constant $\beta_{\pm}$ is positive and the attenuation constant $\alpha_{\pm}$ vanishes. In this case the electromagnetic wave can propagate through plasma without attenuation. On the other hand when $\varepsilon_{\pm}/\varepsilon_0$ is negative, $\beta_{\pm}$ vanishes and $\alpha_{\pm}$ has positive value, so the wave does not propagate. Thus the regions of frequencies are called "pass band" or "stop band" depending on whether $\varepsilon_{\pm}/\varepsilon_0$ is positive or negative for the regions of frequencies. With the aid of (4.2) the pass band is explicitly represented as

$$\omega_{\rm I}<\omega \qquad \text{for } +\text{wave}, \tag{4.3}$$

$$\omega_{\rm II}<\omega,\quad 0<\omega<\omega_c \qquad \text{for } -\text{wave}, \tag{4.4}$$

where

$$\left.\begin{aligned} \omega_{\rm I}&=\frac{1}{2}\left[\sqrt{\omega_c^2+4\omega_p^2}-\omega_c\right], \\ \omega_{\rm II}&=\frac{1}{2}\left[\sqrt{\omega_c^2+4\omega_p^2}+\omega_c\right], \end{aligned}\right\} \tag{4.5}$$

$\omega_{\rm I}$ and $\omega_{\rm II}$ are called the cut off frequencies for + and −waves, respectively. For lower frequencies than the cut off both + and −waves cannot propagate as shown in (4.3) and (4.4). The propagation constants $\beta_{\pm}$ and the real parts of refractive index $n_{r\pm}$ vanish at the corresponding cut off frequencies. At the electron cyclotron frequency ω_c, the right-handed (−sign) wave which rotates in the sense of electron gyration has infinite values of the propagation and attenuation constants (electron cyclotron resonance). The left-handed (+sign) wave, which rotates in an opposite sense to electron gyration, shows no resonance at the electron cyclotron frequency as it should.

Thus the +wave can propagate in the region I, II and III of $\omega-\omega_c$ plane which is shown in Fig. 1. In the region I, III and V the −wave can propagate. Neither + nor −wave can propagate in the region IV. From the definitions of the real

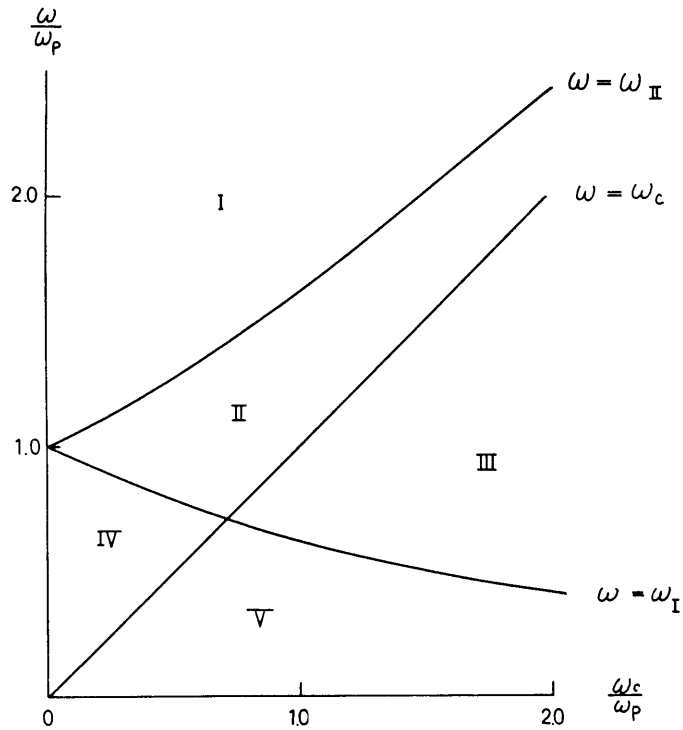

Fig. 1. Regions of propagation of the electromagnetic wave. The frequencies ω_{I} and ω_{II} are the cut off frequencies for left-handed and right-handed circularly polarized wave, ω_c is the electron cyclotron frequency. The right-handed wave can propagate in III, V and I; and the left-handed wave in I, II and III.

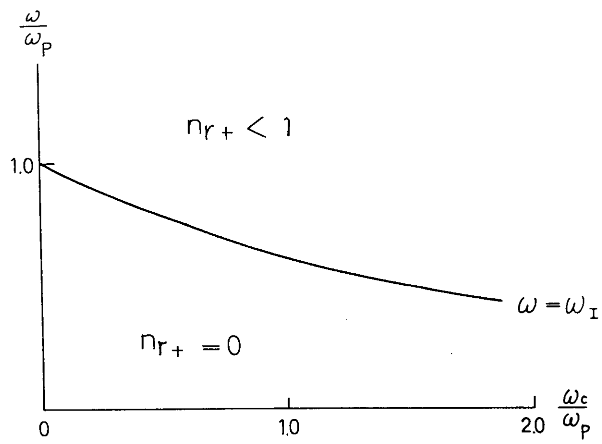

Fig. 2. Real part of refractive index n_{r+} for the left-handed wave in case of no collision.

parts of refractive indexes $n_{r\pm}$ which are given by (3.13), (4.1) and (4.2), it follows readily that

$$n_{r+}=\begin{cases}<1 \text{ for } \omega_{\mathrm{I}}<\omega; \text{ region I, II and III} \\ 0 \text{ otherwise; region IV and V,}\end{cases}$$

$$n_{r-}=\begin{cases}>1 \text{ for } \omega<\omega_c; \text{ region III and V,} \\ <1 \text{ for } \omega_{\mathrm{II}}<\omega; \text{ region I,} \\ 0 \text{ otherwise; region II and IV.}\end{cases}$$

These are shown graphycally in Figs. 2 and 3.

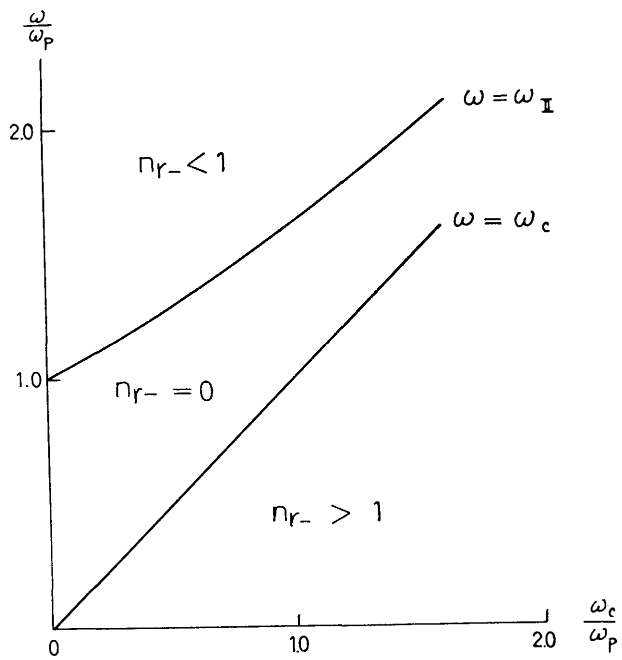

Fig. 3. Real part of refractive index n_{r-} for the right-handed wave in case of no collision.

When we take account of the collisions, the electrical conductivity $\sigma_{\pm}$ which contains the collision frequency ν does not vanish. In this case as is readily seen from (3.12) and (3.13), $n_{r\pm}$, $n_{i\pm}$, $\beta_{\pm}$ and $\alpha_{\pm}$ are positive for any values of ω and ω_c. This fact means that the electromagnetic wave can propagate in the whole region of $\omega-\omega_c$ plane, but necessarily accompanied by attenuation as it travels through the plasma.

In order to illustrate an effect of the collision, let us consider the following equation;

$$1-n_{r\pm}=(\text{positive definite function of } \omega,\ \omega_c \text{ and } \nu)A,$$

$$A=4\omega(\omega\pm\omega_c)^3+\nu^2[4\omega(\omega\pm\omega_c)-\omega_p^2] \tag{4.6}$$

$$=4\omega(\omega\pm\omega_c)\left[(\omega\pm\omega_c)^2+\nu^2\left(\frac{\varepsilon_\pm}{\varepsilon_0}\right)_{\nu=0}\right]+3\nu^2\omega_p^2\ , \tag{4.7}$$

with

$$\left(\frac{\varepsilon_\pm}{\varepsilon_0}\right)_{\nu=0}=1-\frac{\omega_p^2}{\omega(\omega\pm\omega_c)}\ ,$$

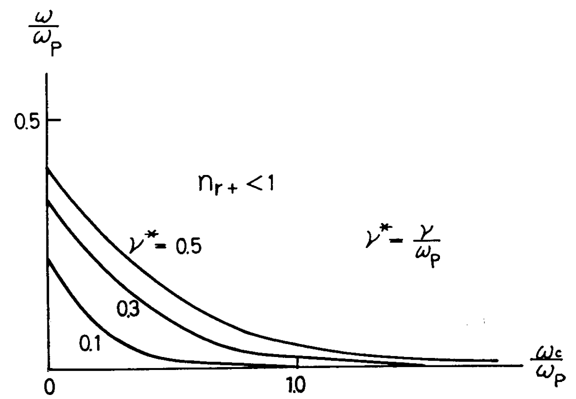

Fig. 4. Real part of refractive index n_{r+} for the left-handed wave in case of presence of collision. In the regions under the curves, $n_{r+}>1$.

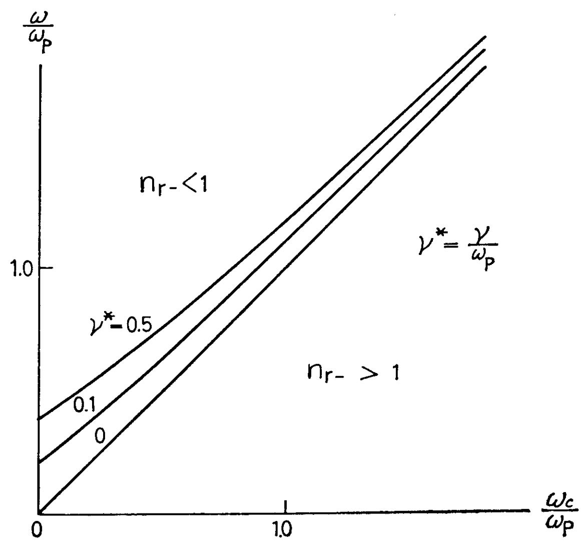

Fig. 5. Real part of refractive index n_{r-} for the right-handed wave in case of presence of collision.

where use has been made of the definition of $n_{r\pm}$ which is given by (3.12). From (4.7) it follows that in the pass band for $+$ wave $n_{r+}<1$, on account of $\varepsilon_+/\varepsilon_0>0$. On the other hand for $-$wave it follows that

$$n_{r-}=\begin{cases}<1 \text{ in the region I, } \because (\varepsilon_-/\varepsilon_0)_{\nu=0}>0,\\ >1 \text{ in the region III and V, } \because \omega<\omega_c,\end{cases}$$

from (4.6) and (4.7). These are not necessary conditions but sufficient ones. Thus we can conclude that even in the presence of collision, the inequalities $n_{r\pm}<1$ or $1<n_{r\pm}$ hold at least in the regions where the corresponding inequalities hold when $\nu=0$. The boundaries $n_{r\pm}=1$ when $\nu\neq0$ are given in Fig. 4 and 5 based on numerical computation.

In Figs. 6 and 7 normalized propagation and attenuation constants $\beta_\pm{}^*$ and $\alpha_\pm{}^*$ are given as functions of ω/ω_p for $\omega_c/\omega_p=0.5$. Here the normalized quantities are defined by

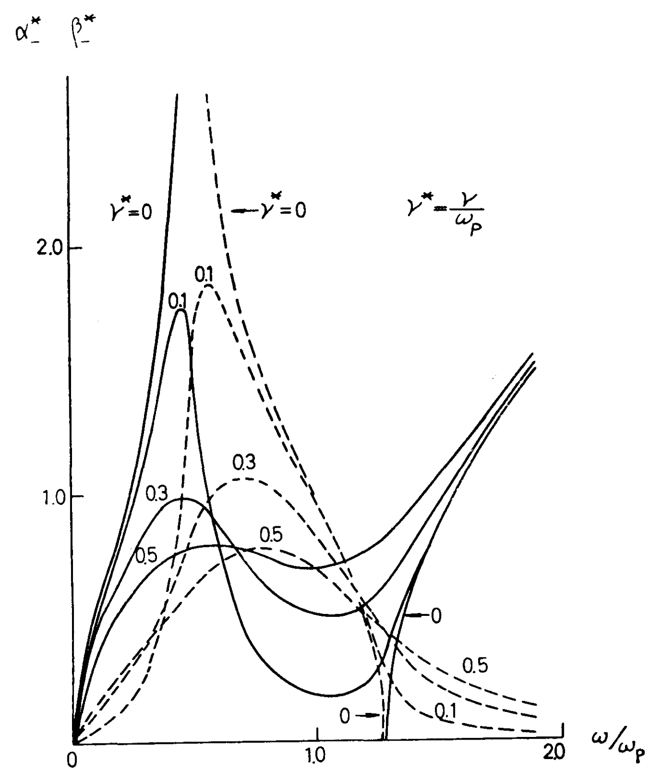

Fig. 6. Normalized attenuation constant $\alpha_-{}^*=\dfrac{c}{\omega_p}\alpha_-$ (dotted curves) and propagation constant $\beta_-{}^*=\dfrac{c}{\omega_p}\beta_-$ (solid curves) for right-handed wave as functions of ω/ω_p when $\omega_c/\omega_p=0.5$.

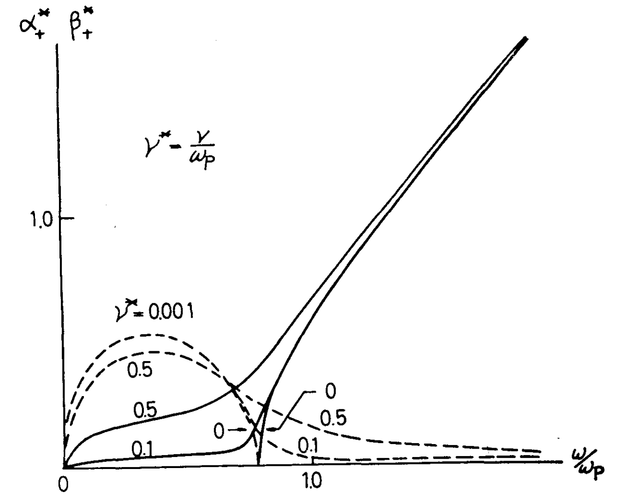

Fig. 7. Normalized attenuation constant $\alpha_+{}^* = \frac{c}{\omega_p}\alpha_+$ (dotted curves) and propagation constant $\beta_+{}^* = \frac{c}{\omega_p}\beta_+$ (solid curves) for left-handed wave as functions of ω/ω_p when $\omega_c/\omega_p = 0.5$.

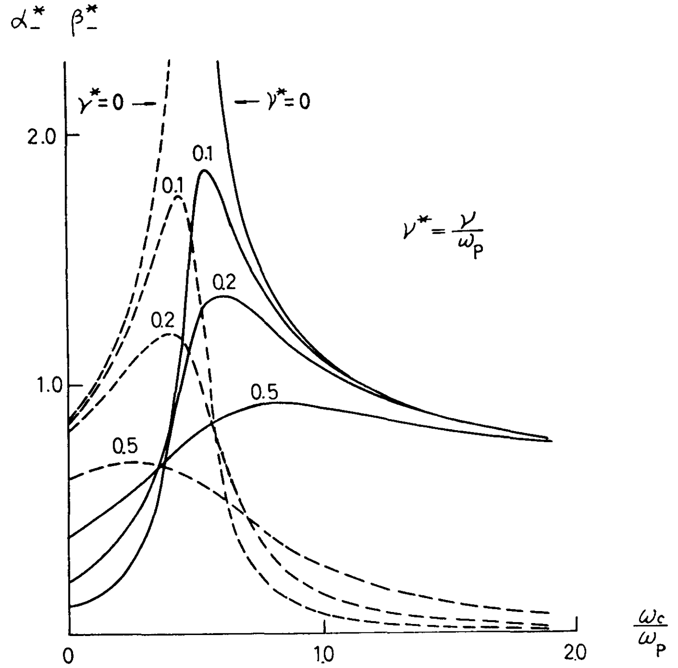

Fig. 8. Normalized attenuation constant $\alpha_-{}^*$ and propagation constant $\beta_-{}^*$ respectively by dotted curves and solid curves as functions of ω_c/ω_p when $\omega/\omega_p = 0.5$.

$$\beta_\pm{}^* = \frac{c}{\omega_p}\beta_\pm\ , \qquad \alpha_\pm{}^* = \frac{c}{\omega_p}\alpha_\pm\ ,$$

the normalizing factor ω_p/c is nearly the inverse of the distance that light travels in a period of plasma oscillation. For other choices of ω_c/ω_p, qualitative features of $\alpha_\pm{}^*$ and $\beta_\pm{}^*$ do not change from Figs. 6 and 7. The attenuation constant $\alpha_-{}^*$ and the propagation constant $\beta_-{}^*$ of the $-$wave are represented as functions of ω_c/ω_p for $\omega/\omega_p=0.5$ in Fig. 8. From the above graphs we can see the following characteristics.

1) For $-$wave as the collision frequency increases, heights of the resonance peaks of the propagation and attenuation constants decrease and their curves are broadened.
2) The stop band vanishes as a result of the collision. Hence the electromagnetic wave with any value of ω can propagate through the lossy plasma, but simultaneously accompanied by attenuation.
3) In the region that corresponds to the pass band of the case $\nu=0$, the larger the value of ν, the larger the attenuation.
4) In the region of the stop band of the case $\nu=0$, on the other hand, the less the value of ν, the larger the attenuation.
5) Derivative of $\beta_\pm$ with respect to ω can vanish at two points of ω within the frequency range that corresponds to stop band of the case $\nu=0$. At these points group velocity of the electromagnetic wave becomes infinite. Between the two points the group velocity becomes negative. This does not occur in case of $\nu=0$.

Physically these properties arise from the randomization effect due to the collision.

We have evaluated the values of $\alpha_\pm{}^*$, $\beta_\pm{}^*$, $n_{r\pm}$, $n_{i\pm}$ and $v_{e\pm}$ the velocity of energy transfer on the electronic computer TOSBAC 3400 for $\omega_c/\omega_p=0.1\sim2.0$, $\omega/\omega_p=0.1\sim2.0$ and $\nu/\omega_p=0.1\sim0.5$.

V. Group Velocity and Velocity of Energy Transfer

In this section we will calculate two velocities, group velocity and velocity of energy transfer, and discuss the relation between them.

(a) Group Velocity

The group velocity v_g is defined by

$$v_{g\pm} = \left(\frac{d\beta_\pm}{d\omega}\right)^{-1}. \tag{5.1}$$

With use of the relation $\beta_\pm = \frac{\omega}{c} n_{r\pm}$, we have

$$\frac{d\beta_\pm}{d\omega}=\frac{1}{4n_{r\pm}c}\left[4n_{r\pm}^2+\omega\frac{d(2n_{r\pm}^2)}{d\omega}\right]. \tag{5.2}$$

From (3.12) and (2.9) it follows that

$$2n_{r\pm}^2=\frac{\varepsilon_\pm}{\varepsilon_0}+\sqrt{\left(\frac{\varepsilon_\pm}{\varepsilon_0}\right)^2+\left(\frac{\sigma_\pm}{\varepsilon_0\omega}\right)^2}, \tag{5.3}$$

$$\left.\begin{aligned}
&\frac{\varepsilon_\pm}{\varepsilon_0}=1-\frac{\omega_p{}^2(\omega\pm\omega_c)}{\omega[\nu^2+(\omega\pm\omega_c)^2]}=n_{r\pm}^2-n_{i\pm}^2,\\
&\frac{\sigma_\pm}{\varepsilon_0\omega}=\frac{\omega_p{}^2\nu}{\omega[\nu^2+(\omega\pm\omega_c)^2]}=2n_{r\pm}n_{i\pm},\\
&\sqrt{\left(\frac{\varepsilon_\pm}{\varepsilon_0}\right)^2+\left(\frac{\sigma_\pm}{\varepsilon_0\omega}\right)^2}=n_{r\pm}^2+n_{i\pm}^2.
\end{aligned}\right. \tag{5.4}$$

Differentiating $2n_{r\pm}^2$, $\varepsilon_\pm/\varepsilon_0$ ond $\sigma_\pm/\omega\varepsilon_0$ with respect to ω and using (5.4), we have

$$\frac{d(2n_{r\pm}^2)}{d\omega}=\frac{1}{n_{r\pm}^2+n_{i\pm}^2}\left[2n_{r\pm}^2\left(\frac{\varepsilon_\pm}{\varepsilon_0}\right)'+\frac{\sigma_\pm}{\varepsilon_0\omega}\left(\frac{\sigma_\pm}{\varepsilon_0\omega}\right)'\right], \tag{5.5}$$

$$\omega\left(\frac{\varepsilon_\pm}{\varepsilon_0}\right)'=\frac{\omega_p{}^2}{\nu^2+(\omega\pm\omega_c)^2}+(1-n_{r\pm}^2+n_{i\pm}^2)-8\frac{\omega^2}{\omega_p{}^2}n_{r\pm}^2n_{i\pm}^2, \tag{5.6}$$

$$\omega\left(\frac{\sigma_\pm}{\omega\varepsilon_0}\right)\left(\frac{\sigma_\pm}{\omega\varepsilon_0}\right)'=-4n_{r\pm}^2n_{i\pm}^2\left[1+2\frac{\omega^2}{\omega_p{}^2}(1-n_{r\pm}^2+n_{i\pm}^2)\right], \tag{5.7}$$

where prime means to make differentiation with respect to ω. Substitution of (5.5), (5.6) and (5.7) into (5.2) yields

$$\frac{d\beta_\pm}{d\omega}=\frac{1}{4n_{r\pm}c}\frac{2n_{r\pm}^2}{n_{r\pm}^2+n_{i\pm}^2}\left[1+n_{r\pm}^2+n_{i\pm}^2+\frac{\omega_p{}^2}{\nu^2+(\omega\pm\omega_c)^2}-4\frac{\omega^2}{\omega_p{}^2}n_{i\pm}^2(1+n_{r\pm}^2+n_{i\pm}^2)\right].$$

Thus we have, from (5.1), expression of the group velocity

$$v_{g\pm}=\frac{n_{r\pm}^2+n_{i\pm}^2}{n_{r\pm}^2}\frac{2n_{r\pm}}{A_\pm-B_\pm}c, \tag{5.8}$$

where we have introduced

$$\left.\begin{aligned}
&A_\pm=1+n_{r\pm}^2+n_{i\pm}^2+\frac{\omega_p{}^2}{\nu^2+(\omega\pm\omega_c)^2},\\
&B_\pm=4\frac{\omega^2}{\omega_p{}^2}n_{i\pm}^2(1+n_{r\pm}^2+n_{i\pm}^2).
\end{aligned}\right\} \tag{5.9}$$

If $A_\pm-B_\pm=0$, $v_{g\pm}$ becomes infinite as was noted in 5) of the preceding section. In order to see the situation a little more precisely, let us see the following expression

$$A_\pm-B_\pm=(1+n_{r\pm}^2+n_{i\pm}^2)\left(1-4\frac{\omega^2}{\omega_p{}^2}n_{i\pm}^2\right)+\frac{\omega_p{}^2}{\nu^2+(\omega\pm\omega_c)^2},$$

and

$$\frac{\omega}{\omega_p} n_{i\pm} = \frac{c}{\omega_p} \frac{\omega}{c} n_{i\pm} = \frac{c}{\omega_p} \alpha_\pm = \alpha_\pm^* ,$$

where α^* is the dimensionless attenuation constant. In the absence of collision, as can be seen in Figs. 6 and 7, $\alpha_\pm^*=0$ in the pass band, so $B_\pm=0$ and $v_{g\pm}$ is positive, while in the stop band the group velocity has no meaning because the wave cannot propagate. In the presence of collision, $1>4\alpha_\pm^{*2}$, so $v_{g\pm}$ is positive in the pass band of the case of $\nu=0$. In the region of stop band, however, $1\lesssim 4\alpha_\pm^{*2}$, so the group velocity can be negative and infinite.

(*b*) *Velocity of Energy Transfer*

From (3.1) and (3.5) it follows that

$$\boldsymbol{H}_\pm = \mp \frac{\gamma_\pm}{\omega\mu_0} \boldsymbol{E}_\pm = \mp \frac{\alpha_\pm + i\beta_\pm}{\omega\mu_0} \boldsymbol{E}_\pm . \tag{5.10}$$

By making use of this expression, time averaged pointing vector $\boldsymbol{S}$ is given by as follows

$$\boldsymbol{S} = \frac{1}{2} Re(\boldsymbol{E}\times \boldsymbol{H}^*) = \frac{1}{2} Re[(\boldsymbol{E}_+ + \boldsymbol{E}_-)\times(\boldsymbol{H}_+^* + \boldsymbol{H}_-^*)]$$

$$= \frac{1}{2} Re[\boldsymbol{E}_+ \times \boldsymbol{H}_+^* + \boldsymbol{E}_- \times \boldsymbol{H}_-^* + \boldsymbol{E}_+ \times \boldsymbol{H}_-^* + \boldsymbol{E}_- \times \boldsymbol{H}_+^*] = \boldsymbol{S}_+ + \boldsymbol{S}_- , \tag{5.11}$$

$$\boldsymbol{S}_\pm = \frac{1}{2} Re(\boldsymbol{E}_\pm \times \boldsymbol{H}_\pm^*) .$$

The 3 rd and 4 th terms in the bracket of (5.11) vanish because of vector product of the same vector. Component pointing vectors $\boldsymbol{S}_\pm$ are evaluated as

$$\boldsymbol{S}_\pm = \frac{1}{2}\, Re\left[(\boldsymbol{e}_x \pm i\boldsymbol{e}_y)\times(\boldsymbol{e}_x \mp i\boldsymbol{e}_y)\left(\mp\frac{\alpha_\pm - i\beta_\pm}{\omega\mu_0}\right)\right]|E_\pm|^2$$

$$= \frac{\beta_\pm}{\omega\mu_0}|E_\pm|^2 \boldsymbol{e}_z = 2n_{r\pm} c\frac{\varepsilon_0}{2}|E_\pm|^2 \boldsymbol{e}_z . \tag{5.11'}$$

Time averaged energy density W is represented by

$$W = W_E + W_H + W_K , \tag{5.12}$$

where W_E and W_H are the energy densities due to electric and magnetic fields respectively, and W_K kinetic energy of the electrons. They are given by

$$W_E = \frac{\varepsilon_0}{4} Re(\boldsymbol{E}\cdot \boldsymbol{E}^*) = \frac{\varepsilon_0}{4} Re[(\boldsymbol{E}_+ + \boldsymbol{E}_-)\cdot(\boldsymbol{E}_+^* + \boldsymbol{E}_-^*)] ,$$

$$W_E = W_{E+} + W_{E-} , \tag{5.13}$$

$$W_{E\pm}=\frac{\varepsilon_0}{4}Re(\boldsymbol{E}_\pm\cdot\boldsymbol{E}_\pm{}^*)=\frac{\varepsilon_0}{4}Re[(\boldsymbol{e}_x\pm i\boldsymbol{e}_y)\cdot(\boldsymbol{e}_x\mp i\boldsymbol{e}_y)]|E_\pm|^2=\frac{\varepsilon_0}{2}|E_\pm|^2\,. \tag{5.13'}$$

The expression $W_E=W_{E+}+W_{E-}$ is justified because the real parts of $\boldsymbol{E}_+\cdot\boldsymbol{E}_-{}^*$ and $\boldsymbol{E}_-\cdot\boldsymbol{E}_+{}^*$ vanish. Similarly it follows that

$$\left.\begin{aligned} W_H&=\frac{\mu_0}{4}Re(\boldsymbol{H}\cdot\boldsymbol{H}^*)=W_{H+}+W_{H-}\,,\\ W_{H\pm}&=\frac{\mu_0}{4}Re(\boldsymbol{H}_\pm\cdot\boldsymbol{H}_\pm{}^*)=\frac{\mu_0}{2}|H_\pm|^2\,. \end{aligned}\right\} \tag{5.14}$$

Using (5.10) and (3.3), we have

$$\left.\begin{aligned} |H_\pm|^2&=\frac{\alpha_\pm{}^2+\beta_\pm{}^2}{(\omega\mu_0)^2}|E_\pm|^2=\frac{1}{c^2\mu_0{}^2}(n_{r\pm}^2+n_{i\pm}^2)|E_\pm|^2\,,\\ W_{H\pm}&=(n_{r\pm}^2+n_{i\pm}^2)\frac{\varepsilon_0}{2}|E_\pm|^2\,. \end{aligned}\right\} \tag{5.14'}$$

By taking account of (2.6), the solution of the equation of motion, we have

$$W_K=\frac{Nm}{4}Re(\boldsymbol{v}\cdot\boldsymbol{v}^*)=W_{K+}+W_{K-}\,, \tag{5.15}$$

$$W_{K\pm}=\frac{Nm}{4}Re(\boldsymbol{v}_\pm\cdot\boldsymbol{v}_\pm{}^*)=\frac{Nm}{2}|v_\pm|^2=\frac{\omega_p{}^2}{\nu^2+(\omega\pm\omega_c)^2}\frac{\varepsilon_0}{2}|E_\pm|^2\,, \tag{5.15'}$$

where N is the number density of the electrons.

Time averaged velocity of energy transfer $\boldsymbol{v}_e$ which is defined by

$$\boldsymbol{v}_e=\boldsymbol{S}/W\,, \tag{5.16}$$

is easily obtained as follows. Substitutions of (5.13'), (5.14') and (5.15) into (5.12) yield

$$W_\pm=W_{E\pm}+W_{H\pm}+W_{K\pm}=A_\pm\frac{\varepsilon_0}{2}|E_\pm|^2\,, \tag{5.17}$$

where

$$A_\pm=1+n_{r\pm}^2+n_{i\pm}^2+\frac{\omega_p{}^2}{\nu^2+(\omega\pm\omega_c)^2}\,, \tag{5.18}$$

which was introduced in (5.9). Substituting (5.11), (5.11') and (5.17) into (5.16), we have the velocity of energy transfer

$$\boldsymbol{v}_e=v_e\boldsymbol{e}_z\,,$$

$$v_e=\frac{S_++S_-}{W_++W_-}=\frac{W_+v_{e+}+W_-v_{e-}}{W_++W_-}=p_+v_{e+}+p_-v_{e-}\,, \tag{5.19}$$

where we have introduced component velocities of energy transfer $\boldsymbol{v}_{e\pm}$ and fractions $p_\pm$ of component energy densities which are defined by

$$v_{e\pm}=\frac{S_\pm}{W_\pm}=\frac{2n_{r\pm}}{A_\pm}c\ ,$$

$$p_\pm=\frac{W_\pm}{W_++W_-}=\frac{A_\pm e^{-2\alpha_\pm z}}{A_+e^{-2\alpha_+z}+A_-e^{-2\alpha_-z}}\ . \tag{5.20}$$

Thus the velocity of energy transfer v_e is expressed as a weighted sum of the component velocities v_{e+} and v_{e-}. The weights p_+ and p_- are functions of position as can be seen from (5.20). Therefore the velocity of energy transfer v_e varies as the wave propagates. On the other hand the component velocities v_{e+} and v_{e-} do not vary in the whole space of plasma.

As a matter of cource v_e does not exceed the velocity of light c. This is shown by proving the inequality;

$$W_++W_->\frac{S_++S_-}{c}\ . \tag{5.21}$$

This is easily shown by directly substituting (5.11′), (5.17) and (5.18) into (5.21)

$$W_++W_--\frac{S_++S_-}{c}=\left[(1-n_{r+})^2+n_{i+}^2+\frac{\omega_p{}^2}{\nu^2+(\omega+\omega_c)^2}\right]\frac{\varepsilon_0}{2}|E_+|^2$$
$$+\left[(1-n_{r-})^2+n_{i-}^2+\frac{\omega_p{}^2}{\nu^2+(\omega-\omega_c)_2}\right]\frac{\varepsilon_0}{2}|E_-|^2>0\ .$$

(*c*) *Group Velocity and Velocity of Energy Transfer*

In order to compare $v_{g\pm}$ with $v_{e\pm}$, we summalize their expressions here

$$v_{g\pm}=\frac{n_{r\pm}^2+n_{i\pm}^2}{n_{r\pm}^2}\frac{2n_{r\pm}}{A_\pm-B_\pm}c\ ,$$

$$v_{e\pm}=\frac{2n_{r\pm}}{A_\pm}c\ ,$$

$$A_\pm=1+n_{r\pm}^2+n_{i\pm}^2+\frac{\omega_p{}^2}{\nu^2+(\omega\pm\omega_c)^2}\ ,$$

$$B_\pm=4\frac{\omega^2}{\omega_p{}^2}n_{i\pm}^2(1+n_{r\pm}^2+n_{i\pm}^2)\ .$$

The ratio of $v_{g\pm}$ to $v_{e\pm}$ is given by

$$\frac{v_{g\pm}}{v_{e\pm}}=\frac{n_{r\pm}^2+n_{i\pm}^2}{n_{r\pm}^2}\frac{A_\pm}{A_\pm-B_\pm}\ . \tag{5.22}$$

From this expression we can see that $v_{g\pm}>v_{e\pm}$ in so far as $v_{g\pm}>0$, and $v_{g\pm}=v_{e\pm}$ only when $n_{i\pm}=0$. In the pass band of the case $\nu=0$, $v_{g\pm}$ is positive. Thus we can conclude that in the presence of collision the velocity of energy transfer cannot exceed the group velocity, $v_{g\pm}>v_{e\pm}$, for the frequency, which lies in the region which corresponds to the pass band in case of $\nu=0$, and only in the absence of the collision the relation $v_{g\pm}=v_{e\pm}$ holds.