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On Vibration of a Circular Elastic Plate which is in Contact with Water

(Received March 4, 1966)

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Abstract

An elastic flat plate of circular form is assumed to be attached, to the top of a circular water tank, which is completely filled up with water. The side-wall and the bottom wall of the tank are assumed to be completely rigid. When the inside-water makes a vibratory motion, the top-plate will also vibrate, transversely. In this paper, the author give the result of his study about the free vibration of this circular plate, which is in contact with water, and which vibrates simultaneously with the water. The vibration is assumed to be of infinitesimally small amplitude. The water is assumed to be an incompressible, non-viscous fluid.

The author shows a formula for the natural frequency of system under consideration, in the form of a determinantal equation (of infinite order). The mode of dependence of the natural frequency ω to the factor U, which represents the mass-ratio, is examined. And numerical values for the case of $n=1$ are given.

It is pointed out that the author's method does not apply, without further modification, for the vibration-modes which have no nodal line ($n=0$).

I. Introduction

Let us consider a circular water tank, as sketched in Fig. 1. As for the top plate, an elastic flat plate of circular from (of uniform thickness) is assumed to be tightly attached. The tank is completely filled up with water. The side-wall and the bottom wall of the tank is assumed to be completely rigid. When the inside water makes a vibratory motion, the top-plate will also vibrate. This vibratory motion of the top plate will be a kind of transverse vibration of a flat elastic circular plate.

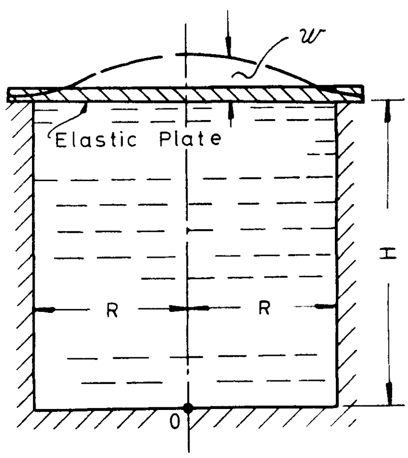

Fig. 1. A Circular Water Tank with a Top-plate.

Our problem is to find out the frequency of free vibration of the whole system composed of water and elastic plate. The amplitude of vibration will be assumed to be infinitesimally

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small. The water will be regarded as an incompressible, non-viscous fluid. The mode of vibration of the circular plate is assumed to be of n nodal lines, where n is a whole number. Numerical example is shown about the case of $n=1$. It is pointed out, that the author's method does not apply to the case of $n=0$ (that is, the case of no nodal line), unless further modification is made to the present procedure.

II. Notations used in this paper

The following notations will be used in the present paper.

(1). R=radius of the tank, H=its height, (r, θ, z)=cylindrical coodinate, the axis of the tank being taken as z-axis, as shown in Fig. 1.

(2). With regard to the circular elastic plate: ρ_m=density, γ_m=specific weight, E=young's modulus, ν=Pisson's ratio, h=thickness, D=flexural rigidity=$[Eh^3]/[12(1-\nu^2)]$, w=transverse displacement, q=normal load acting on the face of the plate.

(3). With regard to the water contained inside the tank;— ϕ=velocity potential of vibratory motion of the water, ω=angular frequency of free vibration, ρ_w=density of water, γ_w=specific weight of water, ζ=elevation at top surface $z=H$, of the water caused by the vibratory motion, p=hydraulic pressure caused by vibration, g=acceleration due to gravity of the earth.

The notation of various numerical coefficients, which is used in the discussion, will be mentioned at the place where it makes its first appearance.

III. Analysis of the vibratory motion of water contained in the tank

The author has, previously, made an analytical study of vibratory motion of water contained in a circular cylindrical tank, whose side-wall is vibrating, both end plates being kept perfectly rigid.[1] In the present paper, on the other hand, we intend to treat similar problem for which the side-wall is kept rigid, whereas the top-plate is vibrating. But some results of analysis, by the author, of previous paper [1] may be utilized here.

In the present case, the velocity potential ϕ of the vibratory motion of water is to satisfy the equation of Laplace $\Delta\phi=0$, together with the following boundary conditions,

$$(a) \quad \text{at } r=R, \qquad \partial\phi/\partial r=0,$$
$$(b) \quad \text{at } z=0, \qquad \partial\phi/\partial z=0.$$

Thus, we are led to the expression for ϕ, in the form;—

$$\phi=\sum_i B_i \cosh m_i z \sin n\theta \, J_n(m_i r) \sin \omega t, \qquad (i=1, 2, 3, \ldots\ldots) \tag{1}$$

1) F. Kito, On the Vibration of Cylindrical Shell, which is filled with water, This PROCEEDINGS, Vol. 4, No. 15, 1951.

where $J_n(\xi)$ denotes the Bessel function of the first kind, and of order n. $m_i=\xi_i/R$, where ξ_i's are roots of (transcendental) equation

$$J'_n(\xi_i)=0\,. \tag{2}$$

For the value of hydraulic pressure p at the top-surface $z=H$, we have,

$$\begin{aligned} p&=-\frac{\gamma_w}{g}\frac{\partial\phi}{\partial t}\\ &=-\frac{\omega\gamma_w}{g}\cos\omega t\sin n\theta\sum_i B_i\cosh m_i z\,J_n(m_iH)\,. \end{aligned} \tag{3}$$

Also we have, for the elevation ζ of water level at the top $z=H$;—

$$\begin{aligned} \zeta&=\int_0^t\left(\frac{\partial\phi}{\partial z}\right)_{z=H}dt\\ &=-\frac{1}{\omega}\cos\omega t\sin n\theta\sum_i B_i\,m_i\sinh(m_iH)J_n(m_ir) \end{aligned} \tag{4}$$

Hydro-elastic conditions to be satisfied at the top-surface, may be written;—

(*a*) w (of plate) $=\zeta$ (of fluid),
(*b*) $q(x,\,y\,;\,t)=p$ (for $z=H$).

Lastly, it may here be reminded,[1] that the set of functions $J_n(m_ir)$, $(i=1,2,3\ldots\ldots)$, where $m_i=\xi_i/R$, form a set of orthogonal (complete) functions, whose factor of normalization are given by,

$$I_i=\int_0^R[J_n(m_ir)]^2\,rdr=\frac{1}{2m_i^2}(\xi_i^2-n^2)\{J_n(\xi_i)\}^2\,.$$

We have, also

$$\int_0^R J_n(m_ir)J_n(m_jr)\,rdr=0\,,$$

if $i\neq j$.

Some numerical values of ξ_i are shown in Table 1.[1]

Table 1. Some rough estimation of the values of roots of the equation $J_n'(\xi)=0$.

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$
ξ_1	3.8317	1.822	3.05	4.3	5.3
ξ_2	7.015	5.387	6.8	8.0	9.3
ξ_3	10.174	8.509	9.9	11.4	12.7
ξ_4	13.32	11.88	13.2	14.7	16.0
ξ_5	16.47	14.81	16.2	17.8	—

IV. Analysis of vibration of the circular elastic plate

At the first instance, let us remind the case of free vibration of an elastic circular plate, with angular frequency ω. This is the case given in usual treatises on vibration. The fundamental equation of the transverse vibration of a flat plate can be written,

$$D\Delta\Delta w+\rho_m h\frac{\partial^2 w}{\partial t^2}=0, \tag{5}$$

where Δ denotes the Laplacian operator, viz., $\Delta\equiv\partial^2/\partial x^2+\partial^2/\partial y^2$. It may also be written in the simplified form,

$$\Delta\Delta w+k^4\frac{\partial^2 w}{\partial t^2}=0, \tag{6}$$

where we put

$$k^4=\rho_m h/D.$$

When the periphery of the plate where $r=R$ is in state of "clamped edge," we must have

$$\text{at}\quad r=R;\quad w=0,\quad \partial w/\partial r=0 \tag{7}$$

Denoting by $\omega_i(i=1, 2, 3, \ldots\ldots)$, the values of frequencies of free vibration, we may write

$$w=P_i(r)[A\cos n\theta+B\sin n\theta]\cos\omega_i t, \tag{8}$$

$(i=1, 2, 3, \ldots\ldots)$, where $P_i(r)$ are functions of r. Putting the expression (8) into the equation (6), we obtain,

$$\frac{d^2P_i}{dr^2}+\frac{1}{r}\frac{dP_i}{dr}+\left(\pm k^2w_i-\frac{n^2}{r^2}\right)P_i=0. \tag{9}$$

Thus, we see that we may put

$$P_i(r)=J_n(k_i r)+K_iJ_n(\boldsymbol{i}k_i r), \tag{10}$$

where $\boldsymbol{i}=\sqrt{-1}$, and $k_i=k\sqrt{\omega_i}$.

There may be raised the claim that we should write $P_{n,i}$ instead of P_i. But since we are not led to any confusion, we shall omit the suffix n, in what follows.

For the expression $P_i(r)$ of eq. (10), the boundary condition (7) becomes,

$$J_n(k_iR)+K_iJ_n(\boldsymbol{i}k_iR)=0,$$

$$J_n'(k_iR)+K_i\boldsymbol{i}J_n'(\boldsymbol{i}k_iR)=0,$$

from which we obtain an equation for k_i, where $\boldsymbol{i}=\sqrt{-1}$,

$$\frac{J_n'(kR)}{J_n(k_iR)}=\frac{\boldsymbol{i}J_n'(\boldsymbol{i}k_iR)}{J_n(\boldsymbol{i}k_iR)} \tag{11}$$

Since $\omega_i(i=1, 2, \ldots\ldots)$ gives us eigen values for free vibration of elastic plates, the functions $P_i(r)$ as defined by (10) will furnish us a set of orthogonal functions. This inference can be verified by actual calculation (with the help of formula of Bessel functions). Thus we have

$$\int_0^R P_i(r)\, P_j(r)\, r dr = \delta_{ij}\, Q_j \tag{12}$$

The value of coefficient K_i is found to be given by

$$K_i = -\frac{J_n(k_i R)}{J_n(i\, k_i R)} = -\frac{J_n'(k_i R)}{iJ_n'(i\, k_i R)} \tag{13}$$

Furthermore, if we write the roots of the transcendental equation

$$\frac{J_n(\zeta)}{J_n(i\zeta)} = \frac{J_n'(\zeta)}{iJ_n'(i\zeta)} \tag{14}$$

as $\zeta_p (p=1, 2, 3, \ldots\ldots)$, we may put

$$k_p R = \zeta_p \tag{15}$$

The values of the factor of normalization Q_j are found, by the integration formula for Bessel functions, to be given by

$$Q_s = \frac{1}{2} R^2 \left[\left\{ J_n'(\zeta_s) \right\}^2 + \left(1 - \frac{n^2}{\zeta_s^2}\right) \left\{ J_n(\zeta_s) \right\}^2 \right]$$
$$+ \frac{1}{2} R^2 K_s^2 \left[\left\{ J_n'(i\zeta_s) \right\}^2 + \left(1 + \frac{n^2}{\zeta_s^2}\right) \left\{ J_n(i\zeta_s) \right\}^2 \right].$$

From which we obtain, by virtue of eq. (13) and eq. (14), the simplified expression as follows :—

$$Q_s = R^2 \{J_n(\zeta_s)\}^2, \quad (s=1, 2, 3, \ldots\ldots). \tag{16}$$

Some numerical values of Q_s/R^2 and ζ_i are shown in Tables 2 and 3.

Table 2. Valees of numerical coefficient Q_s/R^2.

$s=$	1	2	3	4	5
$n=0$	0.0994	0.0520	0.0324	0.0245	0.0196
$n=1$	0.0650	0.0406	0.0729	0.0230	0.01935

Table 3. Values of the roots ζ_s of the equation (14).

$s=$	1	2	3	4	5
$n=0$	3.19	6.306	9.425	12.56	15.71
$n=1$	4.612	7.80	10.95	14.12	17.28

V. Analysis of vibration of the circular elastic plate, which is in contact with water

Thus far, the elastic circular plate was supposed to be subject to no load upon its surface. In this section, we are to take up the case in which it is under the action of hydraulic pressure due to the fact that the plate is in contact with water (see Fig. 1).

The fundamental equation for the transverse vibration of a flat-plate can be written in the following form, wherein the effect of water pressure acting on its face is taken into consideration :—

$$D\Delta\Delta w+\rho_m h\frac{\partial^2 w}{\partial t^2}-q(x,\ y\ ;\ t)=0 \tag{17}$$

or, written in simplified form :—

$$\Delta\Delta w+k^4\frac{\partial^2 w}{\partial t^2}-\frac{1}{D}q(x,\ y\ ;\ t)=0\,. \tag{17a}$$

The periphery $r=R$, of the circular plate, being fixed in state of clamped edge, the transverse displacement w must satisfy the boundary condition that,

$$\text{at}\quad r=R\,,\quad w=0\,,\quad \partial w/\partial r=0 \tag{18}$$

Hence, it may be possible to write,

$$w=\sin n\theta\cos\omega t\sum_i W_i P_i(r)\,, \tag{19}$$

$(i=1,\ 2,\ 3,\ \ldots\ldots)$, where ω is a frequency of free vibration to be determined later. $P_i(r)$ are orthogonal functions defined by (10). W_i are numerical coefficients.

We observe that the expression (19) satisfies the boundary condition (18). On the other hand, we hope that numerical coefficients $W_i(i=1,\ 2,\ 3,\ \ldots\ldots)$ be so chosen that the expression (19) satisfies the equation of motion (17a).

The actual calculation is carried out as follows. The functions $P_i(r)$ can be expanded into a series of orthogonal functions $J_n(m_s r)$, defined in section III, in the form,

$$P_i(r)=\sum_s M_{is}J_n(m_s r)\,. \tag{20}$$

On the other hand, the functions $J_n(m_i r)$ can be expanded into a series of functions $P_i(r)$, as follows ;—

$$J_n(m_i r)=\sum_s N_{is}P_s(r)\,. \tag{21}$$

The values of numerical coefficients M_{is} and N_{is} are as follows;

$$M_{ks}=\frac{1}{I_s}\int_0^R P_k(r)J_n(m_s r)\,rdr\,,$$

$$N_{ks} = \frac{1}{Q_s} \int_0^R J_n(m_k r)\, P_s(r)\, rdr\,.$$

So that we may write

$$M_{ks} = L_{ks}/I_s\,, \quad N_{ks} = L_{sk}/Q_s\,, \tag{22}$$

where we have put,

$$L_{ps} = \int_0^R J_n(m_s r)[J_n(k_p r) + K_p J_n(i k_p r)]\, rdr$$

$$= R^2 \frac{2\zeta_p{}^3}{\xi_s{}^4 - \zeta_p{}^4} J_n(\xi_s) J_n'(\zeta_p)\,. \tag{23}$$

It should be remembered that, we have

$$L_{ps} \neq L_{sp}\,.$$

Next, the value of w, as defined by (19), must coincide with the value of ζ as given by (4), according to the condition (a) of hydro-elasticity. Comparing (19) with (4), we obtain

$$-\frac{1}{\omega} m_i \sinh(m_i H)\, B_i = \sum W_j M_{ji}\,. \tag{24}$$

$$(i = 1,\ 2,\ 3, \ldots\ldots)$$

Putting the values of coefficients B_i thus found, into (3), we obtain, for $z=H$;

$$p = \frac{\omega^2 \gamma_w}{g} \cos\omega t \sin n\theta \cdot H \sum_k W_k \sum_j \frac{\coth(m_j H)}{(m_j H)} M_{kj} \sum_i N_{ji} P_i(r) \tag{25}$$

This value of p is, according to the condition (b) of hydro-elasticity, to be taken as the value $q(x, y\,;\,t)$ of normal load applied to the plate, into the equation (17a). Also, we have to put the expression (19) for w into the equation (17a). In this way, we obtain the following equation :—

$$D \sum_i (k_i{}^4 - k^4\omega^2)\, W_i P_i(r) \sin n\theta \cos\omega t$$

$$-\frac{\omega^2 \gamma_w}{g} H \cos\omega t \sin n\theta \sum_k W_k \sum_j \frac{\coth(m_j H)}{(m_j H)} M_{kj} \sum_i N_{ji} P(r) = 0\,. \tag{26}$$

This equation (26) must be satisfied for every value of independent variable r, such that $0 \leq r \leq R$. By comparison of factors of $P_i(r)$ contained in this equation (26), we obtain,

$$[k_i{}^4 - k^4\,\omega^2]\, W_i$$

$$-\frac{\omega^2 \gamma_w H}{gD} \sum_j \frac{\coth(m_j H)}{(m_j H)} N_{ji} \sum_k W_k M_{kj} = 0\,. \tag{27}$$

$(i, k, j=1, 2, 3, \ldots\ldots)$. This equation (27) is a system of homogeneous linear simultaneous equations with respect to the unknown constants $W_i (i=1, 2, 3, \ldots\ldots)$.

This system of equations (27) will have solutions W_i, which are not totally null, only if the determinantal equation formed with coefficients of W_i in the equations (27) is satisfied :—

$$\begin{vmatrix} \frac{k_1^4}{\omega^2}-k^4-AS_{11}, & -AS_{21}, \ldots\ldots & \\ -AS_{12}, & \frac{k_2^4}{\omega^2}-k-AS_{22}, \ldots\ldots & \\ -AS_{13}, \ldots\ldots & \ldots\ldots & \\ \ldots\ldots, & \ldots\ldots, & \ldots\ldots \end{vmatrix} = 0 . \tag{28}$$

In this equation, we have put, for shortness ;—

$$S_{ki} = \sum_j \frac{\coth(m_j H)}{(m_j H)} M_{kj} N_{ji} , \tag{29}$$

$$A = \frac{\gamma_w H}{gD} = \frac{\gamma_w H}{gD}\left[\left(\frac{D}{\rho_m h}\right) k^4\right] = Uk^4 , \tag{30}$$

where

$$U = \frac{\gamma_w \pi H R^2}{\gamma_m \pi h R^2} = \left(\frac{\text{weight of water}}{\text{weight of plate}}\right) . \tag{31}$$

VI. Condition of compatibility

Thus far, we obtained analytical formula concerning vibration of an elastic circular plate which is kept in contact with water. There remains, however, one more condition to be fullfilled, so far as we regard the water to be an incompressible fluid. This is condition that, for every instant t, the total amount of water displaced by the circular plate must be null. This condition is expressed by

$$\int_0^R \int_0^{2\pi} w(r, \theta ; t) r dr d\theta = 0 . \tag{A}$$

Putting the expression (19) into this equation, we observe that, this equation is satisfied by itself so long as $n \neq 0$.

For the case of $n=0$, we have from (10)

$$\int_0^R P_i(r) r dr = \frac{2R^2}{\zeta_i} J_1(\zeta_i)$$

which do not vanish. Thus, we observe that the author's procedure cannot be applied to the case of $n=0$, without further modification.

VII. Frequency of free vibration of circular elastic plate, which is in contact with water

In equation (28), the value of angular frequency of vibration ω is not yet known. If we find out such values of ω (or, ω_1, ω_2, ...) which satisfy the equation (28), they will give us values of angular frequency of free vibration of the circular elastic plate which is in contact with water, as was shown in Fig. 1.

For the estimation of roots of equation (28), we require numerical values of coefficients S_{ki}, which is defined by eq. (29). About the expression of (29), we remark the following point.

Thus, we have

$$\frac{\coth(m_j H)}{(m_j H)} = \frac{\coth(\xi_j H/R)}{R\xi_j(H/R)}.$$

When the ratio of height H to the radius R of the circular tank is in the neighbourhood of unity, the value of $\coth(\xi_j H/R)$ is (at least for the case of $n=0$ or 1) nearly equal to 1, except probably for $j=1$. Therefore, it will be convenient, for practical evaluation, to write the expression (29) in the form;

$$S_{kj} = \frac{R}{H}\sum_j \frac{1}{\xi_j} M_{kj} N_{ji} + \frac{R}{H}\sum_j [\coth(\xi_j H/R) - 1]\frac{1}{\xi_j} M_{kj} N_{ji}. \qquad (29a)$$

Since we have $\tanh 3.832 = 0.99905$, and $\coth(3.832) - 1 = 0.00095$, the second term of the right hand side of equation (29*a*) has a practically negligible value.

The author has carried out numerical estimation of factors M_{kj}, N_{ji}, and thence, obtained values of S_{kj}. In Tables 4 and 5, the results of estimation are given, for the case of $n=0$ and $n=1$. In these tables values of T_{ki}, as defined by

$$T_{ki} = \frac{H}{R} S_{ki}, \qquad (32)$$

are shown. It is to be noted that so long as the water tank is not too shallow, the values of T_{ki} are practically independent of the ratio R/H.

Table 4. Values of T_{ki}, for the case of $n=0$.

	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$
$k=1$	+0.124	+0.00124	+0.00127	−0.000544	+0.000207
$k=2$	+0.136	+0.0707	−0.0120	+0.00399	−0.00315
$k=3$	−0.0408	+0.0837	+0.0633	−0.0138	+0.00509
$k=4$	+0.0587	−0.0577	+0.00703	+0.0704	−0.0153
$k=5$	−0.0209	+0.0273	+0.0102	+0.0437	+0.0414

Table 5. Values of T_{ki}, for the case of $n=1$.

	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$
$k=1$	+0.179	+0.250	+0.0959	−0.0434	+0.0163
$k=2$	−0.19126	−0.03465	+0.04122	−0.10661	−0.01871
$k=3$	+0.0909	+0.0402	−0.0341	+0.0621	+0.0391
$k=4$	−0.0608	−0.0228	+0.0184	−0.0220	+0.0546
$k=5$	+0.0787	+0.00724	−0.00096	+0.00329	−0.00791

From these tables we see that the values of T_{ki} for $k<i$ are rather small in comparison with values for $k=i$.

Now, we observe that the equation (28) may, for convenience, be written in the form;

$$\begin{vmatrix} \left(\frac{k_1}{k}\right)^4 \frac{1}{\omega^2} - (1+US_{11}), & US_{12}, & \cdots\cdots \\ US_{21}, & \left(\frac{k_2}{k}\right)^4 \frac{1}{\omega^2} - (1+US_{22}), & \cdots \\ US_{31}, & US_{32}, & \cdots\cdots \\ \cdots\cdots, & \cdots\cdots, & \cdots\cdots \end{vmatrix} = 0, \qquad (28a)$$

and, we shall seek the practical meaning of this equation (28*a*).

First, if there exists no water in the tank, we have $\gamma_w=0$, and consequently, $U=0$. In this case all terms in eq. (28*a*) except the ones situated at the diagonal vanish. So that we have

$$\left[\left(\frac{k_1}{k}\right)^4 \frac{1}{\omega^2} - 1\right]\left[\left(\frac{k_2}{k}\right)^4 \frac{1}{\omega^2} - 1\right]\cdots\cdots = 0.$$

Thus we have the following values of natural frequencies;

$$\omega^2 = \left(\frac{k_i}{k}\right)^4. \qquad (i=1,\ 2,\ 3,\ldots\ldots) \qquad (33)$$

This gives us, exactly, the values of natural frequencies of circular elastic plates, as is given in usual treatises.

Next, let us suppose that the value of the ratio U is very small in comparison with unity. In this case, we shall have approximately,

$$\left(\frac{k_1}{k}\right)^4 \frac{1}{\omega^2} - (1+US_{11}) = 0, \quad \text{etc.,} \qquad (34)$$

giving us approximate formula for the value of ω for free vibration.

Last, let us assume that the value of U is very large in comparison with unity. For this case, we rewite the equation (28*a*) into the following form :—

$$\begin{vmatrix} \zeta_1^4\Lambda - S_{11}, & -S_{21}, \ldots\ldots \\ -S_{12}, & \zeta_2^4\Lambda - S_{22}, \ldots\ldots \\ -S_{13}, & -S_{23}, \ldots\ldots \\ \ldots\ldots & \ldots\ldots \quad \ldots\ldots \end{vmatrix} = 0, \tag{35}$$

where we have put,

$$\Lambda = \frac{1}{UF\omega^2}, \quad \zeta_i = k_i R,$$

$$F = (kR)^4 = 12(1-\nu^2)\frac{\gamma_m h R^4}{gEh^3}.$$

The values of ζ_i are given in Table 3. The factor F has the dimension of (sec^2). Further, if we put,

$$S_{ij} = (R/H)T_{ij}, \quad U_a = (R/H)U,$$

$$\Lambda_a = 1/(U_a F\omega^2),$$

the above equation (35) may also be rewritten,

$$\begin{vmatrix} \zeta_1^4\Lambda_a - T_{11}, & -T_{21}, \ldots\ldots \\ -T_{12}, & \zeta_2^4\Lambda_a - T_{22}, \ldots\ldots \\ -T_{13}, & -T_{23}, \ldots\ldots \\ \ldots\ldots & \ldots\ldots \quad \ldots\ldots \end{vmatrix} = 0. \tag{36}$$

In this equation, ζ_i, T_{ij} are all numerical constants. If by solving this determinantal equation (36) with respect to Λ_a, and obtain the roots Λ_{ai} $(i=1, 2, 3, \ldots\ldots)$, these roots Λ_{ai} also will be universal constants, and will not be affected by the ratio of dimensions of the tank.

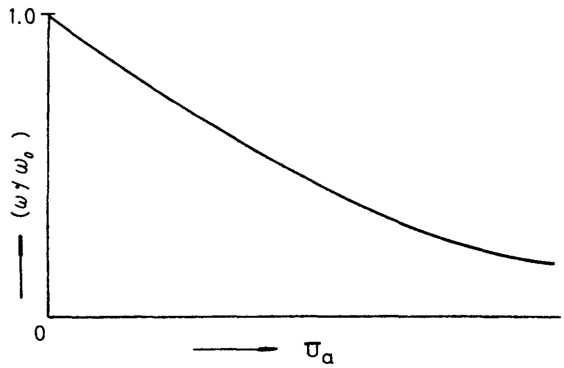

Fig. 2. Vibration of (ω/ω_0) as U_a varies from 0 to ∞.

For the case of $n=1$, the roots Λ_{ai} are found to be approximately given by;

$$\Lambda_{a1} = 1.057\times10^{-3}, \quad \Lambda_{a2} = 0.1556\times10^{-3}.$$

From the above discussion, we can infer the mode of variation of the angular frequency ω of free vibration as function of the parameter U or U_a. Especially, for the fundamental frequency ω of free vibration, we can explain the matter graphically, as shown in Fig. 2. Starting from the value $(\omega/\omega_0)=1$ for $U_a=0$, the value of (ω/ω_0) decreases gradually as U_a increases, according to the formula (34). On the other hand, for a large value of U_a, (ω/ω_0) decreases as shown by the formula

$$\frac{\omega}{\omega_0} \doteqdot \frac{1}{\omega_0\sqrt{FU_a\Lambda_{a1}}} \tag{37}$$

and, finally, we have $\omega/\omega_0=0$ for $U_a \to \infty$.