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On Stability of Servo-mechanisms with Timely Varying Elements—II.

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Abstract

With an object of contributing to the study of stability of servo-mechanisms with timelyvarying elements, a disscussion is made about the stability of solution of a system of (non-linear) ordinary differential equations. The coefficients of this system of ordinary differential equations are taken to vary gradually with the time t.

I. Introduction

In the previous report[1)] of the same title as the present one, the author has reported some results of his study, for the case of linear systems. In the present report, the same problem for the case of non-linear systems is studied. Namely, here we consider a system of non-linear ordinary differential equations, whose coefficients vary gradually with the time t, and study the condition under which this system of differential equations may have regular and stable solution.

The method adopted in the present study is not a new one, but is one obtained by only a slight modification to that followed by E. Picard.[2)]

II. Statement of the problem

Our present object is the study of a system of ordinary non-linear differential equation of the form,

$$\frac{dx_i}{dt} = X_i(x_1, \dots x_n ; \xi), \qquad (1)$$

$$(i=1, 2, \dots n)$$

wherein $x_1, \dots, x_n$ are unknown variables, t (the time) is the indepentent variable, while ξ is a function of t defined by

$$\xi = e^{\alpha t}. \quad (\alpha < 0) \qquad (2)$$

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1) F. KITO, This Proceedings, Vol. 17 No. 65.

2) E. Picard, Traité d'Analyse, Tome III, Chap. VIII, 1908.

The right-hand side of this equation (1) are regular functions of $x_1, \dots, x_n$, and can be expressed as power series in $x_1, \dots x_n$, whose coefficients are linear functions of ξ. Thus, the given equation (1) can be written in the following form (after a linear transformation of dependent variables $x_1, \dots, x_n$, if necessary): —

$$\frac{dx_i}{dt} = \lambda_i x_i + \sum_j A_j^{(i)} \xi x_j + \sum_j \sum_k [B_{jk}^{(i)} + \xi C_{jk}^{(i)}] x_j x_k + \text{etc., etc.,} \tag{3}$$

where $i=1, 2, \dots n$, and the indices j and k are to be taken for $1, 2, \dots, n$. In this equation, only terms up to second order are shown explicitly, on the right-hand side. Also, A's, B's and C's are constants, which may have complex values.

The initial values are, in usual problems of technology, taken to be the values at initial instant $t=0$, thus: —

$$\text{at} \quad t=0,$$

$$x_i = H_i \quad (i=1, 2, \dots, n). \tag{4}$$

However, during the course of construction of solution of the equation (1) or (3), we shall mean, by the term "initial values" the values at $t \to \infty$ (namely, $\xi=0$), thus,

$$x_i = 0, \quad \frac{\partial x_i}{\partial y_i} = K_i, \quad (i=1, 2, \dots, n)$$

$$\text{at} \quad y_1 = 0, \dots, y_n = 0,$$

where y_i are specified functions of t, which are defined in the next section.

In short, our aim in the present paper is the discussion of a non-linear servo-mechanisms, every coefficient or some coefficients of which vary gradually with the time t, in such manner as shown graphically in Fig. 1. The constants $\lambda_1, \dots, \lambda_n$ appearing in this equation (3) are assumed, for simplicity of treatment, to have different values each other.

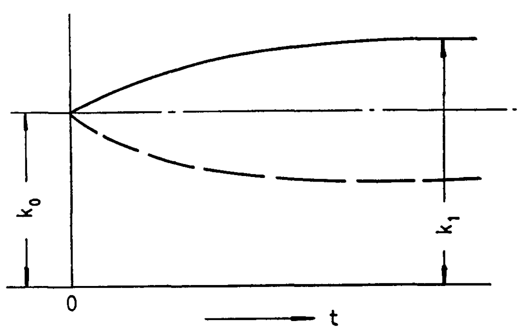

Fig. 1. gradual variation of coefficients.

III. Condition for existence of regular solution of our system

Following the method and reasoning of E. Picard, we shall put [3)]

$$y_i = \exp(\lambda_i t), \quad (i=1, 2, \dots n)$$
$$\xi = \exp(\alpha t), \quad (\alpha < 0), \tag{5}$$

and regard $[y_1, \dots, y_n, \xi]$ as new set of independent variables. Then, the equation (3) can be rewitten in the form:—

$$\lambda_1 y_1 \frac{\partial x_i}{\partial y_1} + \dots + \lambda_n y_n \frac{\partial x_i}{\partial y_n} + \alpha\xi \frac{\partial x_i}{\partial \xi} = \lambda_i x_i + \sum_j A_j^{(i)} \xi x_j$$
$$+ \sum_j \sum_k [B_{jk}^{(i)} + \xi C_{jk}^{(i)}] x_j x_k + \text{etc., etc..} \tag{6}$$

Regarding this system of equations (6) (for $i=1, 2, \dots, n$) as a partial differential equation of first order with repect to $x_1, \dots, x_n$ ($y_1, \dots y_n$, ξ being a set of independent variables), let us try to obtain a system of regular solutions, under the initial contition that,

$$\text{at} \quad y_1=0, \dots, y_n=0, \xi=0;$$
$$x_i=0, \quad \left(\frac{\partial x_i}{\partial y_i}\right) = K_i \tag{7}$$

[This implies that, we assume λ's have negative real parts.]

Now, taking the partial derivative, with respect to a variable $y_m (m \neq i)$, of both sides of the eq (6), and afterwards, putting $y_1=0, \dots y_n=0$, $\xi=0$ into it, we obtain the equation

$$\lambda_m \left(\frac{\partial x_i}{\partial y_m}\right)_0 = \lambda_i \left(\frac{\partial x_i}{\partial y_m}\right)_0 .$$

Since we assumed that $\lambda_1 \dots, \lambda_n$ have different values each other, we shall have,

$$\left(\frac{\partial x_i}{\partial y_m}\right) = 0, \quad \text{for} \quad m \neq i . \tag{8}$$

Next, taking partial derivative, with respect to y_i, of both sides of the eq. (6), and putting afterwards $y_1=0, \dots, y_n=0$, $\xi=0$, we obtain;

$$\lambda_i \left(\frac{\partial x_i}{\partial y_i}\right)_0 = \lambda_i \left(\frac{\partial x_i}{\partial y_i}\right)_0 ,$$

which means that $(\partial x_i/\partial y_i)_0$ remain indeterminate. We shall call this value K_i, as was already indicated in the eq. (7).

Next, taking partial derivative, with respect to ξ, and putting afterwards,

3) Hereafter, we shall understand that, by i, j, or similar indices, we mean a set of positive integers $(1, 2, \dots, n)$.

$y_1=0, \dots, y_n=0$, we obtain;

$$\alpha\left(\frac{\partial x_i}{\partial \xi}\right)_0=\lambda_i\left(\frac{\partial x_i}{\partial \xi}\right)_0 .$$

So that, we shall have (provided that $\alpha \neq \lambda_i$),

$$\left(\frac{\partial x_i}{\partial \xi}\right)_0=0 . \tag{9}$$

Furthermore, by taking the partial derivative of both sides of eq. (6), twice with respect to ξ, and, putting afterwards, $y_j=0$, $\xi=0$, we obtain,

$$2\alpha\left(\frac{\partial^2 x_i}{\partial \xi^2}\right)_0=\lambda_i\left(\frac{\partial^2 x_i}{\partial \xi^2}\right)_0 ,$$

Which shows us that

$$\left(\frac{\partial^2 x_i}{\partial \xi^2}\right)_0=0 , \tag{10}$$

provided that $2\alpha \neq \lambda_i$.

Next, by taking partial derivative of eq. (6), with respect to ξ and y_i, and putting afterwards, $y_j=0$, $\xi=0$, we obtain,

$$(\lambda_i-\lambda_i+\alpha)\left(\frac{\partial^2 x_i}{\partial \xi \partial y_i}\right)_0=A_i^{(i)} K_i . \tag{11}$$

Also, by taking partial derivative of eq. (6) with respect to y_j and y_k $(j \neq k)$, and putting afterwards $y_s=0$, $\xi=0$, we obtain,

$$(\lambda_k+\lambda_j-\lambda_i)\left(\frac{\partial^2 x_i}{\partial y_j \partial y_k}\right)_0=\sum_j \sum_k [B_{jk}^{(i)}] A_j A_k+\dots \tag{12}$$

We may proceed in this way, to obtain values at $y_j=0$, $\xi=0$, of higher partial derivatives of x_i. For example, by taking the derivation m times, with respect to ξ, and putting afterwards $y_j=0$, $\xi=0$, we obtain,

$$\alpha m\left(\frac{\partial^m x_i}{\partial \xi^m}\right)_0=\lambda_i\left(\frac{\partial^m x_i}{\partial \xi^m}\right)_0+\sum_j A_j^{(i)} m\left(\frac{\partial^{m-1} x_i}{\partial \xi^{m-1}}\right)_0+\sum_j \sum_k [B_{jk}^{(i)}]\left[\frac{\partial^m}{\partial \xi^m}(x_j x_k)\right]$$
$$+\sum_j \sum_k [m C_{jk}^{(i)}]\left[\frac{\partial^{m-1}}{\partial \xi^{m-1}}(x_j\, x_k)\right]+\text{etc., etc.} \tag{13}$$

From which we conclude that, if for $s=0, 1, 2, \dots, (m-1)$,

$$\left(\frac{\partial^s x_i}{\partial \xi^s}\right)_0=0,$$

then we shall have also,

$$\left(\frac{\partial^m x_i}{\partial \xi^m}\right)_0=0. \tag{14}$$

In order to obtain general conclusion, let us take partial derivative of both sides of eq. (6), p_1 times with respect to y_1; ... , p_i times with respect to y_i; ... , p_n times with respect to y_n; and s times with respect to ξ, and put afterwards, $y_1=0, \dots, y_n=0$, $\xi=0$. Then, we shall have an equation of the following form, where only terms with the highest order derivatives are written explicitly:—

$$\begin{aligned}&\lambda_1 p_1 |[p_1, p_2, \dots p_n, s][x_i]|_0 + \dots\\&+\lambda_i p_i |[p_1, p_2, \dots p_n, y][x_i]|_0 + \dots\dots\dots\\&+\lambda_n p_n |[p_1, p_2, \dots p_n, s][x_i]|_0\\&+\alpha s |[p_1, p_2, \dots p_n, s][x_i]|_0\\&=\lambda_i |[p_1, \dots p_n, s][x_i]|_0 + \text{etc., etc.}\end{aligned} \tag{15}$$

In this equation (15), the expression

$$|[p_1, p_2, \dots p_i, \dots p_n, s][x]|_0$$

is used to mean that we take the partial derivative of x, p_1 times with respect to y_1, p_2 times with respect to y_2, ... , p_n times with respet to y_n, and s times with respect to ξ, and put afterwards $y_1=0, \dots, y_n=0$, and $\xi=0$. This equation (15) can also be written,

$$(\lambda_1 p_1 + \dots + \lambda_i p_i + \dots + \lambda_n p_n + \alpha s - \lambda_i)\cdot|[p_1, \dots p_i, \dots p_n, s][x_i]|_0 = \text{etc., etc.} \tag{16}$$

From this equation (16), we conclude that, in order that we can obtain values of $|[p_1, \dots, p_n, s][x_i]|_0$ one after another, we must have

$$(\lambda_1 p_1 + \dots + \lambda_i p_i + \dots + \lambda_n p_n + \alpha s - \lambda_i) \neq 0, \tag{17}$$

for every positive integers $p_1, \dots, p_n, s$ such that $\geqq 2$.

When this inequality (17) holds, we can, at least formally, obtain the solution for x_i as power series in $y_1, \dots, y_n, \xi$, thus;

$$x_i = K_i y_i + \sum_j \left(\frac{\partial^2 x_i}{\partial \xi \partial y_j}\right)_0 \xi y_j + \frac{1}{2}\sum_j \sum_k \left(\frac{\partial^2 x_i}{\partial y_j \partial y_k}\right)_0 y_j y_k + \dots \tag{18}$$

It is to be noted that terms such as ξ, ξ^2, ... are lacking.

VI. Convergency of our solution

In order to discuss the convergency of solution of our problem, we shall use the method of majorante, which is so commonly used in the theory of partial differential equations. Writing, formally

$$x_i = \sum A_{p_1 p_2 \cdots p_n s}\, y_1^{p_1} y_2^{p_2} \dots y_n^{p_n} \xi^s \tag{19}$$

which means that x_i are power series in $y_1, y_2, \dots, y_n, \xi$, we shall examine the convergency of this power series, regarding $y_1, y_2 \dots, y_n, \xi$ as complex (independent) variables. From the discussion mentioned above, the only term of first degree in (19) is $K_i y_i$.

First, we observe that the eqdation (6) can be regarded to consist of three terms, viz.,

$$(\text{Term I})=(\text{Term II})+(\text{Term III}) \tag{20}$$

where we put,

$$\text{Term I}=\lambda_1 y_1\frac{\partial x_i}{\partial y_1}+\ldots+\lambda_n y_n\frac{\partial x_i}{\partial y_n}+\alpha\xi\frac{\partial x_i}{\partial \xi}-\lambda_i x_i\,,$$

$$\text{Term II}=\sum_j A_j^{(i)}\xi x_j\,,$$

$$\text{Term III}=\sum_j\sum_k\left[B_{jk}^{(i)}+\xi C_{jk}^{(i)}\right]x_j x_k+\text{etc., etc..}$$

For the (Term I), we shall have,

$$\lambda_1 y_1\frac{\partial x_i}{\partial y_1}+\ldots+\lambda_n y_n\frac{\partial x_i}{\partial y_n}-\lambda_i x_i-\alpha\xi\frac{\partial x_i}{\partial \xi}$$
$$=\sum(\lambda_1 p_1+\ldots+\lambda_n p_n-\lambda_i-\alpha s)A_{p_1 p_2\cdots p_n s} \tag{21}$$
$$y_1^{p_1}y_2^{p_2}\ldots y_n^{p_n}\xi^s\,.$$

When the inequality (17) is satisfied, we may be justified in saying that, there exists a positive real constant ε, such that for every positive integral values of p_1, p_2, ... , p_n, s,

$$\varepsilon<|\lambda_1 p_1+\ldots+\lambda_n p_n-\lambda_i-\alpha s|\,. \tag{22}$$

For a power series of the form (19), we shall denote by X_i the power series whose each term is obtained from each term of x_i, by taking its absolute value, thus;

$$X_i=\sum|A_{p_1 p_2\cdots p_n s}|\,|y_1|^{p_1}\cdots|y_n|^{p_n}\,|\xi|^s\,. \tag{23}$$

If we treat similarly, the power series of right hand side of eq. (21), and write formally,

$$Z_i=\sum|C_{p_1 p_2\cdots p_n s}|\,|y_1|^{p_1}\cdots|y_n|^{p_n}\,|\xi|^s\,. \tag{24}$$

As mentioned above, the only term of first degree in X_i is y_i. On the other hand, the term of first degree is lacking, in Z_i, because we have

$$\lambda_1 p_1+\ldots+\lambda_n p_n-\lambda_i-\alpha s=0,$$
$$\text{for}\quad p_i=1,\ \ p_j=0\ \ (j\neq i),\ \ s=0.$$

So that, when we compare (23) and (24), we must compare

$$X_i-|K_i|y_i\qquad\text{with}\qquad Z_i.$$

and we shall have,

$$Z_i>\varepsilon(X_i-|K_i|\,|y_i|)\,. \tag{25}$$

Next, we observe that for the Term I, we may write,

$$|\text{Term II}|\leqq\sum_j|A_j^{(i)}|\,|\xi|X_j\leqq D|\xi|\sum X_j\,. \tag{26}$$

Lastly, we have

$$|\text{Term III}| \leqq [C_2+C_3|\xi|]\Big[\Big(\frac{X_1+\ldots+X_n}{a}\Big)^2+\Big(\frac{X_1+\ldots+X_n}{a}\Big)^3+\ldots\Big]M$$

$$\leqq [C_2+C_3|\xi|]\Big[\frac{M}{1-\frac{W}{a}}-M-\frac{W}{a}M\Big], \tag{27}$$

where we put

$$W=X_1+X_2+\ldots+X_n, \tag{28}$$

and a, C_2, C_3 are positive real constants of given values.

From these considerations, we are led to a system of equations of following form, which is "majorant equation" for our equation (6);

$$Z_i=D|\xi|W+[C_2+C_3|\xi|]M\Big[\frac{1}{1-\frac{W}{a}}-1-\frac{W}{a}\Big]. \tag{29}$$

Putiing $\varepsilon[X_i-|K_i||y_i|]$, instead of Z_i, into this equation, we obtain,

$$\varepsilon X_i=\varepsilon|K_i||y_i|+D|\xi|W+[C_2+C_3|\xi|]\Big[\frac{1}{1-\frac{W}{a}}-1-\frac{W}{a}\Big]\cdot M. \tag{30}$$

Let us take $K_i=1$, which does not impair generality, so long as we are discussing the convergency from the view point of method of majorante. Also, let us take sum, for $i=1, 2, \ldots n$, of both sides of equation (30). Then we shall have,

$$W=Y+\frac{1}{\varepsilon}|\xi|W+\frac{M}{\varepsilon}[C_2+C_3|\xi|]\Big[\frac{1}{1-\frac{W}{a}}-1-\frac{W}{a}\Big] \tag{31}$$

where we put, for shortness $Y=\sum|y_i|$.

From this equation (31), we obtain the following algebraic equation, which gives us the value of W in terms of Y;

$$A_wW^2+B_wW+Y=0 \tag{32}$$

where we have put,

$$A_w=\frac{M}{\varepsilon a^2}[C_2+C_3|\xi|]+\frac{1}{a}-\frac{|\xi|}{\varepsilon a},$$

$$B_w=-1+\frac{|\xi|}{\varepsilon}-\frac{Y}{a}.$$

So that, we shall have,

$$W=\frac{1}{2A_w}[-B_w\pm\{(B_w)^2-4A_wY\}^{\frac{1}{2}}],$$

whereas we have,

$$(B_w)^2-4A_wY=\Big\{1-\frac{|\xi|}{\varepsilon}-\Big(\frac{Y}{a}\Big)\Big\}^2\cdot\Big[1-\frac{4M}{\varepsilon a^2}\frac{\{C_2+C_3|\xi|\}Y}{\Big\{1-\frac{|\xi|}{\varepsilon}-\Big(\frac{Y}{a}\Big)\Big\}^2}\Big]$$

It can be seen that W can be expanded into a power series (of ascending orders), which converge absolutely, at least for a sufficiently small values of Y and $|\xi|$. Thus, we are led to the conclusion, that the power-series solution, (19) of our problem converge absolutely for suffiiciently small valves of $|y_i|$ and $|\xi|$.

So far, we considered the partial differential equation (6), in which independent variables are $y_1 \dots, y_n$ and ξ. The initial values of x_i was meant to be values of x_i at $y_i=0$, $\xi=0$. But, our actual aim was the treatment of differential equation (1) in which the independent variable is t. If the values of indices λ_i were such that their real parts have negative values, then we could be justified in saying that for $t\to\infty$, we have $y_i=0$, $\xi=0$, and that the system of differential equations (1) has a set of stable solutions.

V. Determination of arbitrary constants K_i

In the above discussions, arbitrary constants K_i were left undetermined. For the case of usual problems in technology, they are to be determined from the "initial condition", that is, the values of x_i at $t=0$. Let the given initial values in this sense be,

$$\text{at} \quad t=0 \ (\text{or, } y_i=1,\ \xi=1),$$
$$x_i=H_i \ (i=1, 2, \dots n). \tag{33}$$

Putting these values of (33) into (18) [or, its equivalent, (19)], we have,

$$H_i=K_i+\sum A^{(i)}_{j1}+\tfrac{1}{2}\sum\sum A^{(i)}_{jk}+\dots \tag{34}$$

Where we put, for shortness,

$$A^{(i)}_{j1}=\left(\frac{\partial^2 x_i}{\partial\xi\partial y_i}\right)_0, \quad A^{(i)}_{jk}=\left(\frac{\partial^2 x_i}{\partial y_j\partial y_k}\right)_0, \quad \text{etc., etc.}$$

In eq. (34), the right-hand side members are functions of $K_1, \dots, K_n$. So that, formally, the equations (34) form a system of (transcendental) equations for $K_1, \dots, K_n$. Solving it, by some means, the arbitrary constants are determined (as functions of $H_1, \dots H_n$). There remains the question of convergency of right-hand side of (34) to be examined. It is hoped that this question will be discussed in the near future, by the author.