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On Stability of Servo-mechanisms with Timely Varying Elements—I.

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Abstract

In connection with the problem of stability of servo-mechanisms, whose elements vary with time t, stability property of some ordinary differential equation is studied. Firstly, a linear ordinary differential equation, whose coefficients vary with time t, is considered. The variation with time t is regarded to be expressed by linear functions of ξ, where $\xi = e^{\alpha t}$ $(\alpha < 0)$. It is shown that the solution of the given linear differential equation, thus specified, can be expressed as power sesies with respect to ξ, by means of which, the question of stability may be examined. Secondly, it is pointed out that, the problem of similar kind, for the case of non-linear servo-mechanism, can be studied, by following the classical method of É. Picard.

I. Introduction

Several years ago, M. J. Kirby[1] has reported results of his study, about stability of sero-mechanisms with linearly varying elements. Therein, he studied an ordinary linear differential equation of the form :—

$$(a_n + b_n t)\frac{d^n\theta_0}{dt^n} + \dots + (a_i + b_i t)\frac{d^i\theta_0}{dt^i} + \dots + (a_0 + b_0 t)\theta_0 = F(t) \tag{1}$$

where $a_n, \dots, a_i, \dots, a_0$; $b_n, \dots, b_i, \dots, b_0$, are real constants. According to him, the study of servo-mechanisms whose elements vary linearly with time t, may be reduced to study of stability property of the above differential euuation (1). By the word *stability property*, we mean the behavior of magnitude of solution θ_0, as t tends to increase indefinitely (in positive sense). Looking at the equation, we observe that the coefficient $(a_i + b_i t)$ tend to infinity, unless $b_i = 0$. Thus, it may seem that equation (1) is not fitted to the study of servo-mechanisms whose elements vary with time t. But, this unfittness is only apparent, because what

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1) M. J. Kirby, Stability of Servo-mechanisms with linearly varying elements, AIEE Transactions, 1950, Vol. 69, pp. 1662~1668.

matters us is not the value of coefficients (a_i+b_it), but their ratios with respect to (a_n+b_nt).

Putting, for conventience,

$$A_i(t)=\frac{a_i+b_it}{a_n+b_nt}, \tag{2}$$

the graph of $A_i(t)$ will be as shown in Fig. 1, assuming, $a_n\neq0$, $b_n\neq0$; $a_i\neq0$, $b_i\neq0$. This graph shows that the value of the coefficient $A_i(t)$ changes gradually, fron $A_i(0)=a_i/a_n$ to b_i/b_n as t is made to vary from $t=0$ to $t\to\infty$.

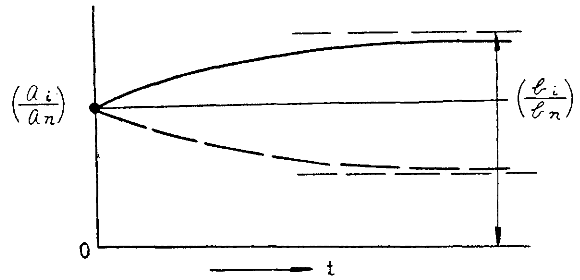

Fig. 1. Mode of variation of Coefficints.

This being so, the author wishes to propose to study the case in which the co efficients have forms as follows;

$$(A_i+a_i\xi), \quad (i=1, 2, \dots n) \qquad (3$$

in which we put $\xi=e^{\alpha t}$, α being a real constant with negative value. If we drav curves represented by (3), with time t as independent variable, the general forn will also be as shown in Fig. 1.

Thus, we are led to study an ordinary linear differential equation of the forn

$$(A_n+a_n\xi)\frac{d^n\theta}{dt^n}+\dots+(A_j+a_j\xi)\frac{d^j\theta}{dt^j}+\dots+(A_1+a_1\xi)\frac{d\theta}{dt}+(A_0+a_0\xi)\theta=0, \qquad (4$$

which may serve to study the stability-property of a servo-mechanisn whose el ments vary with the time t. In eq. (4), A_i and $a_i (i=1, 2, \dots n)$ are real constant

II. Solution of the differential equation (4).

We are to solve the linear differential equation (4), under an initial conditic that

at $t=0$ (or, $\xi=1$) ;

$$\frac{d^{n-1}\theta}{dt^{n-1}}=C_{n-1}, \dots, \frac{d\theta}{dt}=C_1, \ \theta=C_0 .$$

Changing the independent variable from t to $\xi=e^{\alpha t}$ $(\alpha<0)$, the differential equ tion (4) is reduced to the form :—

$$[B_n+b_n\xi]\xi^n\frac{d^n\theta}{d\xi^n}+[B_{n-1}+b_{n-1}\xi]\xi^{n-1}\frac{d^{n-1}\theta}{d\xi^{n-1}}$$

$$+\dots+[B_1+b_1\xi]\xi\frac{d\theta}{d\xi}+[B_0+b_0\xi]\theta=0, \tag{5}$$

where B_i, b_i $(i=1, 2, \dots\dots n)$ are real constants, and the initial condition may be written in the form,

$$\text{at } \xi=1; \quad \frac{d^{n-1}\theta}{d\xi^{n-1}}=K_{n-1}, \dots,$$

$$\frac{d\theta}{d\xi}=K_1, \ \theta=\theta_0. \tag{6}$$

It is seen that, under the condition that $(B_n+b_n\xi)$ never vanishes for an interval $0\leqq\xi\leqq1$ of the independent variable ξ, the differential equation (5) can be solved, by the method of Forbenius, in the form,

$$\theta=\theta^\lambda[1+C_1\xi+C_2\xi^2+\dots],$$

the power series contained therein having its radius of convergence no less than $|B_n/b_n|$. In what follows, we shall assume that $|B_n/b_n|>1$. This assumption may be regarded to be justified in almost all the case of servo-mechanisms. The value of the index λ will be determined by an algebraic equation,

$$[\lambda(\lambda-1)\dots(\lambda-n)]B_n+[\lambda(\lambda-1)\dots(\lambda-\overline{n-1})]B_{n-1}+\dots+\lambda B_1+B_0=0. \tag{7}$$

Denoting with λ_i $(i=1, 2, \dots n)$ the n roots of the equation (7), the system of fundamental solutions of the eq. (4) can be given in the form,

$$\Theta_i(\xi)=\xi^{\lambda_i}[1+C_{1i}\xi+C_{2i}\xi^2+\dots], \tag{8}$$

for the case in which there are n distinct roots λ_i of the eq. (7).

The general solution of the eq. (4) may be given in the form

$$\theta=\sum_i E_i\Theta_i(\xi), \tag{9}$$

where E_i are arbitrary constants. These constants E_i are to be determined by the given initial conditon (6), thus;

$$\sum_i E_i\Theta_i^{(k)}(1)=K_k, \quad (k=0, 1, \dots \overline{n-1}). \tag{10}$$

It will readily be seen that we have

$$\xi^{\lambda_i}=e^{\mu_i t}$$

where μ_i $(i=1, 2, \dots n)$ are roots of the algebraic equation

$$A_n\mu^n+A_{n-1}\mu^{n-1}+\dots+A_1\mu+A_0=0. \tag{11}$$

Thus, we see that terms ξ^{λ_i} in the solution (8) represent the behavior of solution at very large value of time t, it being the same as for the differential equation

$$A_n \frac{d^n\theta}{dt^n} + A_{n-1} \frac{d^{n-1}\theta}{dt^{n-1}} + \dots + A_1 \frac{d\theta}{dt} + A_0\theta = 0 . \tag{12}$$

On the other hand, terms in brackets, viz,

$$[1 + C_{1i}\xi + C_{2i}\xi^2 + \dots]$$

give the modification caused by the fact that the elements (coefficients) vary gradually with time t.

III. Special case of $n=2$

For the special case, in which $n=2$, the eq. (4) becomes,

$$[A_2 + a_2\xi] \frac{d^2\theta}{dt^2} + [A_1 + a_1\xi] \frac{d\theta}{dt} + [A_0 + a_0\xi]\theta = 0 \tag{4'}$$

which may be rewritten;

$$\begin{aligned} &[\alpha^2(A_2 + a_2\xi)]\xi^2 \frac{d^2\theta}{d\xi^2} \\ &+ [\alpha^2(A_2 + a_2\xi) + \alpha(A_1 + a_1\xi)]\xi \frac{d\theta}{d\xi} \\ &+ [A_0 + a_0\xi]\theta = 0 . \end{aligned} \tag{4''}$$

The algebraic equation (7) becomes, in the case of $n=2$,

$$[\alpha^2 A_2]\lambda(\lambda-1) + [\alpha^2 A_2 + \alpha A_1]\lambda + A_0 = 0 \tag{7'}$$

Denoting the roots of this equation by λ_1 and λ_2, and putting $\mu_1 = \lambda_1\alpha$, $\mu_2 = \lambda_2\alpha$, we see that μ_1, μ_2 are roots of the equation,

$$A_2\mu^2 + A_1\mu + A_0 = 0 . \tag{7''}$$

A system of fundamental solutions of the eq. (4') is given in the form,

$$\Theta_i = \xi^{\lambda_i}[1 + C_{1i}\xi + C_{2i}\xi^2 + \dots], \quad (i = 1, 2)$$

where the coefficients C_{ki} can be expressed in the following form,

$$C_{1i} = -\frac{g(\lambda_i)}{f(\lambda_i+1)},$$

$$C_{2i} = \frac{g(\lambda_i)\,g(\lambda_i+1)}{f(\lambda_i+1)\,f(\lambda_i+2)},$$

$$C_{3i} = -\frac{g(\lambda_i)\,g(\lambda_i+1)\,g(\lambda_i+2)}{f(\lambda_i+1)\,f(\lambda_i+2)\,f(\lambda_i+3)},$$

etc., etc.,

the functions $f(\lambda)$ and $g(\lambda)$ being defined as,

$$f(\lambda) = A_2(\alpha\lambda)^2 + A_1(\alpha\lambda) + A_0 ,$$

$$g(\lambda) = a_2(\alpha\lambda)^2 + a_1(\alpha\lambda) + a_0 .$$

It is to be noted that initial values (at $t=0$, or $\xi=1$) of Θ_i are given by,

$$\Theta_i = 1 + C_{1i} + C_{2i} + \ldots,$$

$$\frac{d\Theta_i}{dt} = \alpha\left[\lambda_i + (\lambda_i+1)C_{1i} + (\lambda_i+2)C_{2i} + \ldots\right].$$

Especially interesting, will be the case in which the coefficient $(A_1 + a_1\xi)$ varies, with time t, as shown in Fig. 2. This means that, the damping coefficient has at first a negative value, but it tends gradually to a positive (final) value.

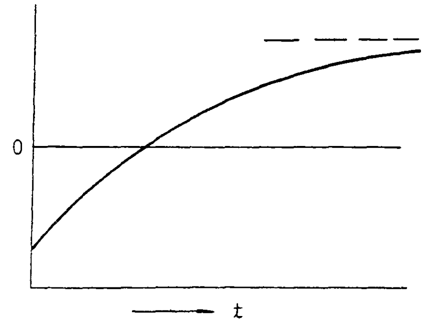

Fig. 2. Transition from negative- to positive- damping.

VI. Note on non-linear problem

In connection with the discussion, mentioned above, it may be desirable to consider the case of a non-linear system.

This non-linear problem may be reduced to the study of a system of ordinary differential equations of the form :—

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} A_{ij}x_j + [x_1, x_2, \ldots x_n]_2 \qquad (i=1, 2, \ldots n) \tag{13}$$

In this equation (13), the coefficients A_{ij} of linear parts are assumed to have forms as shown below;

$$A_{ij} = a_{ij} + b_{ij}\xi\ ,$$

where a_{ij}, b_{ij} are numerical constants. The expression such as $[x_1, x_2, \ldots x_n]_2$ are to be understood to mean power series in $x_i, \ldots\ldots x_n$, which begin from the second degree terms, and whose coefficients are (linear) functions of ξ. It will be seen that, the study of this system of equation (13) can be made in similar manner as treatment of È. Picard.[1] The result will be that $x_1, \ldots\ldots, x_n$ can be given as power series with respect to

$$\exp(\mu_1 t),\ \exp(\mu_2 t), \ldots \exp(\mu_{nt}),\ \text{and}\ e^{\alpha t}$$

where $\alpha < 0$. The author intends to give fuller account about this inference, in the next part of this paper.

2) É. Picard, Traité d'Analyse, Tome III, Chap. VIII.