Title	On stability of servo-mechanisms with timely varying elements-I.
Sub Title	
Author	鬼頭, 史城(Kito, Fumiki)
Publisher	慶応義塾大学藤原記念工学部
Publication year	1964
Jtitle	Proceedings of the Fujihara Memorial Faculty of Engineering Keio University (慶應義塾大学藤原記念工学部研究報告). Vol.17, No.65 (1964.),p.39(15)- 43(19)
JaLC DOI	
Abstract	In connection with the problem of stability of servo-mechanisms, whose elements vary with time t, stability property of some ordinary differential equation is studied. Firstly, a linear ordinary differential equation, whose coefficients vary with time t, is considered. The variation with time t is regarded to be expressed by linear functions of ξ , where [function]. It is shown that the solution of the given linear differential equation, thus specified, can be expressed as power sesies with respect to ξ , by means of which, the question of stability may be examined. Secondly, it is pointed out that, the problem of similar kind, for the case of non-linear servo-mechanism, can be studied, by following the classical method of É. Picard.
Notes	
Genre	Departmental Bulletin Paper
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=KO50001004-00170065- 0015

慶應義塾大学学術情報リポジトリ(KOARA)に掲載されているコンテンツの著作権は、それぞれの著作者、学会または出版社/発行者に帰属し、その権利は著作権法によって 保護されています。引用にあたっては、著作権法を遵守してご利用ください。

The copyrights of content available on the KeiO Associated Repository of Academic resources (KOARA) belong to the respective authors, academic societies, or publishers/issuers, and these rights are protected by the Japanese Copyright Act. When quoting the content, please follow the Japanese copyright act.

On Stability of Servo-mechanisms with Timely Varying Elements-I.

(Received April 21, 1965)

Fumiki KITO*

Abstract

In connection with the problem of stability of servo-mechanisms, whose elements vary with time t, stability property of some ordinary differential equation is studied. Firstly, a linear ordinary differential equation, whose coefficients vary with time t, is considered. The variation with time t is regarded to be expressed by linear functions of ξ , where $\xi = e^{\alpha t}$ ($\alpha < 0$). It is shown that the solution of the given linear differential equation, thus specified, can be expressed as power sesies with respect to ξ , by means of which, the question of stability may be examined. Secondly, it is pointed out that, the problem of similar kind, for the case of non-linear servo-mechanism, can be studied, by following the classical method of \hat{E} . Picard.

I. Introduction

Several years ago, M. J. Kirby ¹⁾ has reported results of his study, about stability of sero-mechanisms with linearly varying elements. Therein, he studied an ordinary linear differential equation of the form :—

$$(a_{n}+b_{n}t)\frac{d^{n}\theta_{0}}{dt^{n}}+\ldots+(a_{i}+b_{i}t)\frac{d^{i}\theta_{0}}{dt^{i}}+\ldots+(a_{0}+b_{0}t)\theta_{0}=F(t)$$
(1)

where $a_n, \ldots, a_i, \ldots, a_0$; $b_n, \ldots, b_i, \ldots, b_0$, are real constants. According to him, the study of servo-mechanisms whose elements vary linearly with time t, may be reduced to study of stability property of the above differential equation (1). By the word *stability property*, we mean the behavior of magnitude of solution θ_0 , as t tends to increase indefinitely (in positive sense). Looking at the equation, we observe that the coefficient (a_i+b_it) tend to infinity, unless $b_i=0$. Thus, it may seem that equation (1) is not fitted to the study of servo-mechanisms whose elements vary with time t. But, this unfittness is only apparent, because what

^{*}鬼 頭 史 城,教 授 Professor, Department of Mechanical Engineering, Faculty of Engineering, Keio University, Koganei, Tokyo, Japan.

M. J. Kirby, Stability of Servo-mechanisms with linearly varying elements, AIEE Transactions, 1950, Vol. 69, pp. 1662~1668.

matters us is not the value of coefficients (a_i+b_it) , but their ratios with respect to (a_n+b_nt) .

Putting, for conventience,

$$\Lambda_i(t) = \frac{a_i + b_i t}{a_n + b_n t},\tag{2}$$

the graph of $A_i(t)$ will be as shown in Fig. 1, assuming, $a_n \neq 0$, $b_n \neq 0$; $a_i \neq 0$, $b_i \neq 0$. This graph shows that the value of the coefficient $A_i(t)$ changes gradually, from $A_i(0) = a_i/a_n$ to b_i/b_n as t is made to vary from t=0 to $t \to \infty$.



Fig. 1. Mode of variation of Coefficients.

This being so, the author wishes to propose to study the case in which the co efficients have forms as follows;

$$(A_i + a_i \xi), \quad (i = 1, 2, ..., n)$$
 (3)

in which we put $\xi = e^{\alpha t}$, α being a real constant with negative value. If we draw curves represented by (3), with time t as independent variable, the general form will also be as shown in Fig. 1.

Thus, we are led to study an ordinary linear differential equation of the form

$$(A_n + a_n\xi) \frac{d^n\theta}{dt^n} + \dots + (A_j + a_j\xi) \frac{d^j\theta}{dt^j} + \dots + (A_1 + a_1\xi) \frac{d\theta}{dt} + (A_0 + a_0\xi)\theta = 0, \quad (4)$$

which may serve to study the stability-property of a servo-mechanism whose elements vary with the time t. In eq. (4), A_i and $a_i(i=1, 2, ..., n)$ are real constant:

II. Solution of the differential equation (4).

We are to solve the linear differential equation (4), under an initial conditic that

at
$$t=0$$
 (or, $\xi=1$);

$$\frac{d^{n-1}\theta}{dt^{n-1}} = C_{n-1}, \dots, \frac{d\theta}{dt} = C_1, \ \theta = C_0.$$

Changing the independent variable from t to $\xi = e^{\alpha t}$ ($\alpha < 0$), the differential equ tion (4) is reduced to the form :—

40

$$\begin{bmatrix} B_{n} + b_{n}\xi \end{bmatrix} \xi^{n} \frac{d^{n}\theta}{d\xi^{n}} + \begin{bmatrix} B_{n-1} + b_{n-1}\xi \end{bmatrix} \xi^{n-1} \frac{d^{n-1}\theta}{d\xi^{n-1}} + \dots + \begin{bmatrix} B_{1} + b_{1}\xi \end{bmatrix} \xi \frac{d\theta}{d\xi} + \begin{bmatrix} B_{0} + b_{0}\xi \end{bmatrix} \theta = 0,$$
(5)

where B_i , b_i (i=1, 2, ..., n) are real constants, and the initial condition may be written in the form,

at
$$\xi = 1$$
; $\frac{d^{n-1}\theta}{d\xi^{n-1}} = K_{n-1}, \dots,$
 $\frac{d\theta}{d\xi} = K_1, \ \theta = \theta_0.$ (6)

It is seen that, under the condition that $(B_n+b_n\xi)$ never vanishes for an interval $0 \le \xi \le 1$ of the independent variable ξ , the differential equation (5) can be solved, by the method of Forbenius, in the form,

$$\theta = \theta^{2} [1 + C_{1} \xi + C_{2} \xi^{2} + \dots],$$

the power series contained therein having its radius of convergence no less than $|B_n/b_n|$. In what follows, we shall assume that $|B_n/b_n| > 1$. This assumption may be regarded to be justified in almost all the case of servo-mechanisms. The value of the index λ will be determined by an algebraic equation,

$$[\lambda(\lambda-1)\dots(\lambda-n)]B_n+[\lambda(\lambda-1)\dots(\lambda-\overline{n-1})]B_{n-1}+\dots+\lambda B_1+B_0=0.$$
(7)

Denoting with λ_i (i=1, 2, ..., n) the *n* roots of the equation (7), the system of fundamental solutions of the eq. (4) can be given in the form,

$$\Theta_i(\xi) = \xi^{\lambda_i} \left[1 + C_{1i}\xi + C_{2i}\xi^2 + \dots \right], \tag{8}$$

for the case in which there are *n* distinct roots λ_i of the eq. (7).

The general solution of the eq. (4) may be given in the form

$$\theta = \sum_{i} E_{i} \Theta_{i}(\xi) , \qquad (9)$$

where E_i are arbitrary constants. These constants E_i are to be determined by the given initial conditon (6), thus;

$$\sum_{i} E_{i} \Theta_{i}^{(k)}(1) = K_{k}, \quad (k = 0, 1, \dots, \overline{n-1}).$$

$$(10)$$

It will readily be seen that we have

$$\xi^{\lambda_i} = e^{\mu i t}$$

where μ_i (i=1, 2, ..., n) are roots of the algebraic equation

$$A_n \mu^n + A_{n-1} \mu^{n-1} + \dots + A_1 \mu + A_0 = 0.$$
⁽¹¹⁾

Thus, we see that terms ξ^{λ_i} in the solution (8) represent the behavior of solution at very large value of time t, it being the same as for the differential equation

$$A_n \frac{d^n \theta}{dt^n} + A_{n-1} \frac{d^{n-1} \theta}{dt^{n-1}} + \dots + A_1 \frac{d\theta}{dt} + A_0 \theta = 0.$$
 (12)

On the other hand, terms in brackets, viz,

$$[1+C_{1i}\xi+C_{2i}\xi^2+\dots]$$

give the modification caused by the fact that the elements (coefficients) vary gradually with time t.

III. Special case of n=2

For the special case, in which n=2, the eq. (4) becomes,

$$[A_2 + a_2 \xi] \frac{d^2 \theta}{dt^2} + [A_1 + a_1 \xi] \frac{d\theta}{dt} + [A_0 + a_0 \xi] \theta = 0$$

$$(4')$$

which may be rewritten;

$$[\alpha^{2}(A_{2}+a_{2}\xi)]\xi^{2} \frac{d^{2}\theta}{d\xi^{2}}$$

$$+ [\alpha^{2}(A_{2}+a_{2}\xi)+\alpha(A_{1}+a_{1}\xi)]\xi \frac{d\theta}{d\xi}$$

$$+ [A_{0}+a_{0}\xi]\theta = 0.$$

$$(4'')$$

The algebraic equation (7) becomes, in the case of n=2,

$$\left[\alpha^2 A_2\right] \lambda \left(\lambda - 1\right) + \left[\alpha^2 A_2 + \alpha A_1\right] \lambda + A_0 = 0 \tag{7'}$$

Denoting the roots of this equation by λ_1 and λ_2 , and putting $\mu_1 = \lambda_1 \alpha$, $\mu_2 = \lambda_2 \alpha$, we see that μ_1 , μ_2 are roots of the equation,

$$A_2\mu^2 + A_1\mu + A_0 = 0. (7'')$$

A system of fundamental solutions of the eq. (4') is given in the form,

$$\Theta_i = \xi^{\lambda_i} [1 + C_{1i} \xi + C_{2i} \xi^2 + \dots], \quad (i = 1, 2)$$

where the coefficients C_{ki} can be expressed in the following form,

$$\begin{split} C_{1i} &= -\frac{g(\lambda_i)}{f(\lambda_i+1)},\\ C_{2i} &= \frac{g(\lambda_i) g(\lambda_i+1)}{f(\lambda_i+1) f(\lambda_i+2)},\\ C_{3i} &= -\frac{g(\lambda_i) g(\lambda_i+1) g(\lambda_i+2)}{f(\lambda_i+1) f(\lambda_i+2) f(\lambda_i+3)},\\ \text{etc., etc.,} \end{split}$$

the functions $f(\lambda)$ and $g(\lambda)$ being defined as,

$$f(\lambda) = A_2(\alpha \lambda)^2 + A_1(\alpha \lambda) + A_0,$$

$$g(\lambda) = a_2(\alpha \lambda)^2 + a_1(\alpha \lambda) + a_0.$$

It is to be noted that initial values (at t=0, or $\xi=1$) of Θ_i are given by,

$$\Theta_i = 1 + C_{1i} + C_{2i} + \dots,$$

$$\frac{d\Theta_i}{dt} = \alpha \left[\lambda_i + (\lambda_i + 1)C_{1i} + (\lambda_i + 2)C_{2i} + \dots \right].$$

Especially interesting, will be the case in which the coefficient $(A_1 + a_1\xi)$ varies, with time t, as shown in Fig. 2. This means that, the damping coefficient has at first a negative value, but it tends gradually to a positive (final) value.



Fig. 2. Transition from negative- to positive- damping.

VI. Note on non-linear problem

In connection with the discussion, mentioned above, it may be desirable to consider the case of a non-linear system.

This non-linear problem may be reduced to the study of a system of ordinary differential equations of the form :—

$$\frac{dx_i}{dt} = \sum_{j=1}^n A_{ij} x_j + [x_1, x_2, \dots, x_n]_2 \qquad (i=1, 2, \dots, n)$$
(13)

In this equation (13), the coefficients A_{ij} of linear parts are assumed to have forms as shown below;

$$A_{ij} = a_{ij} + b_{ij} \xi ,$$

where a_{ij} , b_{ij} are numerical constants. The expression such as $[x_1, x_2, ..., x_n]_2$ are to be understood to mean power series in $x_i, ..., x_n$, which begin from the second degree terms, and whose coefficients are (linear) functions of ξ . It will be seen that, the study of this system of equation (13) can be made in similar manner as treatment of \hat{E} . Picard.¹⁾ The result will be that $x_1, ..., x_n$ can be given as power series with respect to

$$\exp(\mu_1 t)$$
, $\exp(\mu_2 t)$, ... $\exp(\mu_n t)$, and $e^{\alpha t}$

where $\alpha < 0$. The author intends to give fuller account about this inference, in the next part of this paper.

²⁾ É. Picard, Traité d'Analyse, Tome Ⅲ, Chap. W.