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On Stability of Servo-mechanisms with Timely Varying Elements—I.

(Received April 21, 1965)

Fumiki KITO*

Abstract

In connection with the problem of stability of servo-mechanisms, whose elements vary with time $t$, stability property of some ordinary differential equation is studied. Firstly, a linear ordinary differential equation, whose coefficients vary with time $t$, is considered. The variation with time $t$ is regarded to be expressed by linear functions of $\xi$, where $\xi = e^{\alpha t}$ ($\alpha < 0$). It is shown that the solution of the given linear differential equation, thus specified, can be expressed as power series with respect to $\xi$, by means of which, the question of stability may be examined. Secondly, it is pointed out that, the problem of similar kind, for the case of non-linear servo-mechanism, can be studied, by following the classical method of É. Picard.

I. Introduction

Several years ago, M. J. Kirby has reported results of his study, about stability of servo-mechanisms with linearly varying elements. Therein, he studied an ordinary linear differential equation of the form:

$$(a_n + b_n t) \frac{d^n \theta}{dt^n} + \cdots + (a_1 + b_1 t) \frac{d \theta}{dt} + (a_0 + b_0 t) \theta = F(t)$$

(1)

where $a_n, \ldots, a_1, a_0; b_n, \ldots, b_1, b_0$ are real constants. According to him, the study of servo-mechanisms whose elements vary linearly with time $t$, may be reduced to study of stability property of the above differential equation (1).

By the word stability property, we mean the behavior of magnitude of solution $\theta$, as $t$ tends to increase indefinitely (in positive sense). Looking at the equation, we observe that the coefficient $(a_i + b_i t)$ tend to infinity, unless $b_i = 0$. Thus, it may seem that equation (1) is not fitted to the study of servo-mechanisms whose elements vary with time $t$. But, this unfitness is only apparent, because what

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*鬼頭史雄 教授 Professor, Department of Mechanical Engineering, Faculty of Engineering, Keio University, Koganei, Tokyo, Japan.

matters us is not the value of coefficients \((a_i+b_i t)\), but their ratios with respect to \((a_n+b_n t)\).

Putting, for convenience,

\[ A_i(t) = \frac{a_i + b_i t}{a_n + b_n t}, \tag{2} \]

the graph of \(A_i(t)\) will be as shown in Fig. 1, assuming, \(a_n \neq 0\), \(b_n \neq 0\); \(a_i \neq 0\), \(b_i \neq 0\). This graph shows that the value of the coefficient \(A_i(t)\) changes gradually, from \(A_i(0) = a_i/a_n\) to \(b_i/b_n\) as \(t\) is made to vary from \(t=0\) to \(t \to \infty\).

![Graph showing mode of variation of coefficients](image)

**Fig. 1.** Mode of variation of Coefficients.

This being so, the author wishes to propose to study the case in which the coefficients have forms as follows:

\[(A_i+a_i \xi), \quad (i=1, 2, \ldots n)\] \tag{3}

in which we put \(\xi = e^{at}\), \(\alpha\) being a real constant with negative value. If we draw curves represented by (3), with time \(t\) as independent variable, the general form will also be as shown in Fig. 1.

Thus, we are led to study an ordinary linear differential equation of the form

\[(A_n+a_n \xi) \frac{d^n \theta}{dt^n} + \ldots + (A_1+a_1 \xi) \frac{d \theta}{dt} + \ldots + (A_i+a_i \xi) \frac{d \theta}{dt} + (A_0+a_0 \xi) \theta = 0, \tag{4}\]

which may serve to study the stability-property of a servo-mechanism whose elements vary with the time \(t\). In eq. (4), \(A_i\) and \(a_i\) \((i=1, 2, \ldots n)\) are real constants:

**II. Solution of the differential equation (4).**

We are to solve the linear differential equation (4), under an initial condition that

\[\text{at } t=0 \quad (\text{or, } \xi=1);\]

\[\frac{d^{n-1} \theta}{dt^{n-1}} = C_{n-1}, \ldots, \frac{d \theta}{dt} = C_1, \quad \theta = C_0.\]

Changing the independent variable from \(t\) to \(\xi = e^{at} \quad (\alpha < 0)\), the differential equation (4) is reduced to the form:

\[(16)\]

\[ \left[ B_n + b_n \xi \right] \xi^n \frac{d^n \theta}{d \xi^n} + \left[ \xi^{n-1} \frac{d^{n-1} \theta}{d \xi^{n-1}} \right] + \ldots + \left[ B_1 + b_1 \xi \right] \xi \frac{d \theta}{d \xi} + \left[ B_0 + b_0 \xi \right] \theta = 0, \tag{5} \]

where \( B_n, b_i \) are real constants, and the initial condition may be written in the form,

\[ \text{at } \xi = 1: \quad \frac{d^{n-1} \theta}{d \xi^{n-1}} = K_{n-1}, \ldots, \]

\[ \frac{d \theta}{d \xi} = K_1, \quad \theta = \theta_0. \tag{6} \]

It is seen that, under the condition that \( (B_n + b_n \xi) \) never vanishes for an interval \( 0 \leq \xi \leq 1 \) of the independent variable \( \xi \), the differential equation (5) can be solved, by the method of Frobenius, in the form,

\[ \theta = \theta^t \left[ 1 + C_1 \xi + C_2 \xi^2 + \ldots \right], \]

the power series contained therein having its radius of convergence no less than \( |B_n/b_n| \). In what follows, we shall assume that \( |B_n/b_n| > 1 \). This assumption may be regarded to be justified in almost all the case of servo-mechanisms. The value of the index \( \lambda \) will be determined by an algebraic equation,

\[ [\lambda (\lambda - 1) \ldots (\lambda - n)] B_n + [\lambda (\lambda - 1) \ldots (\lambda - n - 1)] B_{n-1} + \ldots + \lambda B_1 + B_0 = 0. \tag{7} \]

Denoting with \( \lambda_i \) the \( n \) roots of the equation (7), the system of fundamental solutions of the eq. (4) can be given in the form,

\[ \theta_i(\xi) = \xi^{\lambda_i} \left[ 1 + C_{i1} \xi + C_{i2} \xi^2 + \ldots \right], \tag{8} \]

for the case in which there are \( n \) distinct roots \( \lambda_i \) of the eq. (7).

The general solution of the eq. (4) may be given in the form

\[ \theta = \sum_i E_i \theta_i(\xi), \tag{9} \]

where \( E_i \) are arbitrary constants. These constants \( E_i \) are to be determined by the given initial condition (6), thus;

\[ \sum_i E_i \theta_i(1) = K_k, \quad (k = 0, 1, \ldots n-1). \tag{10} \]

It will readily be seen that we have

\[ \xi^{\lambda_i} = e^{\mu_i \xi} \]

where \( \mu_i \) are roots of the algebraic equation

\[ A_n \mu^n + A_{n-1} \mu^{n-1} + \ldots + A_1 \mu + A_0 = 0. \tag{11} \]

Thus, we see that terms \( \xi^{\lambda_i} \) in the solution (8) represent the behavior of solution at very large value of time \( t \), it being the same as for the differential equation

(17)
\[
A_n \frac{d^n \theta}{dt^n} + A_{n-1} \frac{d^{n-1} \theta}{dt^{n-1}} + \ldots + A_1 \frac{d \theta}{dt} + A_0 \theta = 0.
\]  \hspace{1cm} (12)

On the other hand, terms in brackets, viz,
\[ [1 + C_{1i} \xi + C_{2i} \xi^2 + \ldots] \]
give the modification caused by the fact that the elements (coefficients) vary gradually with time \( t \).

III. Special case of \( n = 2 \)

For the special case, in which \( n = 2 \), the eq. (4) becomes,
\[ [A_2 + a_2 \xi] \frac{d^2 \theta}{dt^2} + [A_1 + a_1 \xi] \frac{d \theta}{dt} + [A_0 + a_0 \xi] \theta = 0 \]  \hspace{1cm} (4')
which may be rewritten;
\[
[a^2(A_2 + a_2 \xi)] \xi^2 \frac{d^2 \theta}{d \xi^2} + [a^2(A_2 + a_2 \xi) + \alpha(A_1 + a_1 \xi)] \xi \frac{d \theta}{d \xi} + [A_0 + a_0 \xi] \theta = 0.
\]  \hspace{1cm} (4'')

The algebraic equation (7) becomes, in the case of \( n = 2 \),
\[ [a^2A_2] \lambda(\lambda - 1) + [a^2A_2 + \alpha A_1] \lambda + A_0 = 0 \]  \hspace{1cm} (7')

Denoting the roots of this equation by \( \lambda_1 \) and \( \lambda_2 \), and putting \( \mu_1 = \lambda_1 \alpha \), \( \mu_2 = \lambda_2 \alpha \), we see that \( \mu_1 \), \( \mu_2 \) are roots of the equation,
\[ A_2 \mu^2 + A_1 \mu + A_0 = 0. \]  \hspace{1cm} (7'')

A system of fundamental solutions of the eq. (4') is given in the form,
\[ \Theta_i = \xi^{\lambda_i} [1 + C_{1i} \xi + C_{2i} \xi^2 + \ldots], \quad (i = 1, 2) \]
where the coefficients \( C_{ki} \) can be expressed in the following form,
\[
C_{1i} = -\frac{g(\lambda_i)}{f(\lambda_i + 1)},
\]
\[
C_{2i} = \frac{g(\lambda_i) g(\lambda_i + 1)}{f(\lambda_i + 1) f(\lambda_i + 2)},
\]
\[
C_{3i} = -\frac{g(\lambda_i) g(\lambda_i + 1) g(\lambda_i + 2)}{f(\lambda_i + 1) f(\lambda_i + 2) f(\lambda_i + 3)},
\]
etc., etc.,

the functions \( f(\lambda) \) and \( g(\lambda) \) being defined as,
\[
f(\lambda) = A_2(\alpha \lambda)^2 + A_1(\alpha \lambda) + A_0,
\]
\[ g(\lambda) = a_2(\alpha \lambda)^2 + a_1(\alpha \lambda) + a_0. \]  \hspace{1cm} (18)
It is to be noted that initial values (at \( t=0 \), or \( \xi=1 \)) of \( \Theta_i \) are given by,
\[
\Theta_i = 1 + C_{1i} + C_{2i} + \ldots ,
\]
\[
\frac{d\Theta_i}{dt} = \alpha \left[ \lambda_i + (\lambda_i + 1)C_{1i} + (\lambda_i + 2)C_{2i} + \ldots \right].
\]

Especially interesting, will be the case in which the coefficient \((A_{1i} + a_i \xi)\) varies, with time \( t \), as shown in Fig. 2. This means that, the damping coefficient has at first a negative value, but it tends gradually to a positive (final) value.

**VI. Note on non-linear problem**

In connection with the discussion, mentioned above, it may be desirable to consider the case of a non-linear system.

This non-linear problem may be reduced to the study of a system of ordinary differential equations of the form:
\[
\frac{dx_i}{dt} = \sum_{j=1}^{n} A_{ij} x_j + [x_1, x_2, \ldots, x_n] (i=1, 2, \ldots n)
\]

In this equation (13), the coefficients \( A_{ij} \) of linear parts are assumed to have forms as shown below;
\[ A_{ij} = a_{ij} + b_{ij} \xi , \]
where \( a_{ij}, b_{ij} \) are numerical constants. The expression such as \([x_1, x_2, \ldots, x_n]\) are to be understood to mean power series in \( x_1, \ldots, x_n \), which begin from the second degree terms, and whose coefficients are (linear) functions of \( \xi \). It will be seen that, the study of this system of equation (13) can be made in similar manner as treatment of É. Picard. The result will be that \( x_1, \ldots, x_n \) can be given as power series with respect to
\[
\exp(\mu_1 t), \exp(\mu_2 t), \ldots, \exp(\mu_n t), \text{and } e^{\alpha t}
\]
where \( \alpha<0 \). The author intends to give fuller account about this inference, in the next part of this paper.

2) É. Picard, Traité d'Analyse, Tome III, Chap. VII.