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Successive Approximation Technique Applied to Continuous Line Segments Curve Fitting

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Abstract

Successive approximation technique is applied to obtain the best fit continuous line segments with a finite number of vertices with predetermined abscissae to a given curve or a set of points, in the least square sense. The Lipschitz constant associated to the recurrence relation is evaluated, so as to be able to obtain the stopping rule in numerical iterations.

I. Introduction

It is required, recently, to fit line segments to a given function or a set of points, related to some computational techniques. Some of their examples are the numerical solution of convex programming problems via solving approximated linear programming problems and generation of functions on electronic analogue computer systems, employing integrators and comparators.

Algorithms as to various types of line segments curve fitting problems are given : Stone [3] by classical method, Bellman and Gluss [4]~[6] by dynamic programming technique, and Pontryagin et al [7] by his maximum principle.

In this article, it will be described how the successive approximation technique can be applied to obtain the best fit continuous line segments with a finite number of vertices with predetermined abscissae to a given curve or a set of points in the least square sense. The evaluations of the Lipschitz constant associated to the recurrence relation will be used in estimating the errors of the approximations.

After describing the convergence criterions as to the recurrence relations in II, the recurrence relation giving the successive approximations in obtaining the best fit continuous line segments to a curve will be given in III, and to a set of points in IV. It will be also noted in V, that the method described in this article may be considered as an example of a particular approach to the minimization problem.

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II. Contraction theory and Gauss-Seidel method in solving systems of linear equations

Let us first recall the theorem of contraction operator, defined on a complete metric space as the fundamental theorem of the method evolved in this article.

Theorem

If an operator [1)]

$$\varphi(x) \qquad x \in R(X, \rho) \tag{2·1}$$

defined on a complete metric space $R(X, \rho)$ mapping R into itself is associated with the Lipschitz constant

$$\rho(\varphi(x), \varphi(x')) \leqq L\rho(x, x') \quad x, x' \in R(X, \rho) \tag{2·2}$$

less than unity

$$L<1, \tag{2·3}$$

the sequence

$$\{x_0, x_1, \ldots\ldots, x_n, \ldots\ldots\} \tag{2·4}$$

beginning with an arbitrary element of $R(X, \rho)$, defined by the recurrence relation

$$x_{n+1} = \varphi(x_n) \qquad x_0 \in R(X, \rho) \tag{2·5}$$

converges to the unique fixed point of the operator

$$x = \varphi(x). \tag{2·6}$$

Moreover, it is seen that

$$\rho(x_n, x) \leqq \frac{L^n}{1-L}\rho(x_0, x_1). \tag{2·7}$$

Proof

The proof of the theorem may be seen elsewhere in ordinary text books (eg. [8]) of functional analysis and will not be repeated here.

Next, let us consider Gauss-Seidel method from the point of view of contraction operator.

One of the numerical method of solving system of linear equations

$$\begin{aligned} c_{11}x^1 + c_{12}x^2 + \ldots\ldots + c_{1M}x^M &= d^1 \\ c_{21}x^1 + c_{22}x^2 + \ldots\ldots + c_{2M}x^M &= d^2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ c_{M1}x^1 + c_{M2}x^2 + \ldots\ldots + c_{MM}x^M &= d^M \end{aligned} \tag{2·8}$$

1) The operator associated with a Lipschitz constant less than unity is sometimes call a *contraction operator*.

is to convert the system (2·8) into the form

$$\begin{aligned} x^1 &= a_{11}x^1 + a_{12}x^2 + \ldots\ldots + a_{1M}x^M + b_1 \\ x^2 &= a_{21}x^1 + a_{22}x^2 + \ldots\ldots + a_{2M}x^M + b_2 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x^M &= a_{M1}x^1 + a_{M2}x^2 + \ldots\ldots + a_{MM}x^M + b_M \end{aligned} \qquad (2\cdot 9)$$

or in vector-matrix form

$$\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b} \qquad \boldsymbol{x} \in R_M \qquad (2\cdot 10)$$

where

$$\boldsymbol{x} = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^M \end{pmatrix} \quad \boldsymbol{A} = \begin{pmatrix} a_{11}, & a_{12}, & \ldots, & a_{1M} \\ a_{21}, & a_{22}, & \ldots, & a_{2M} \\ & & & \\ a_{M1}, & a_{M2}, & \ldots, & a_{MM} \end{pmatrix} \quad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_M \end{pmatrix} \qquad (2\cdot 11)$$

and to evaluate the vector sequence $\{\boldsymbol{x}_n\}$ given by the recurrence relation

$$\begin{aligned} x^1_{n+1} &= a_{11}x^1_n + a_{12}x^2_n + \ldots a_{1M}x^M_n + b_1 \\ x^2_{n+1} &= a_{21}x^1_{n+1} + a_{22}x^2_n + \ldots + a_{2M}x^M_n + b_2 \\ x^M_{n+1} &= a_{M1}x^1_{n+1} + a_{M2}x^2_{n+1} + \ldots a_{M,\,M-1}a^{M-1}_n + a_{MM}x^M_n + b_M \\ &\boldsymbol{x}_0 : \text{arbitrary} \end{aligned} \qquad (2\cdot 12)$$

until the distance between $\boldsymbol{x}_n$ and $\boldsymbol{x}_{n+1}$ becomes negligible. This method is called the Gauss-Seidel iteration method in solving systems of linear equations.

It may be readily remarked that this method is applicable only when the sequence given by (2·12) is convergent. Referring to the theorem above, we may examine the convergence of the sequence by evaluating the Lipschitz constant of (2·12) considered as an operator. The Lipschitz constant of the operator can be evaluated as follows, corresponding to the metric defined on the M-dimensional space R_M. It is desirable not only to have the evaluation of the Lipschitz constant less than unity but also to evaluate it as small as possible in order to assure faster convergence as it was indicated in (2·7). Since the derivations of the evaluations of the Lipschitz constant given below may be refered to [1] and partly to [2], they will not be given here.

It may be well known that the M-dimensional space R_M is complete as to the metrics defined in (2·13a), (2·13b) and (2·13c).

It may be also referred to [1] that if $a_{11}=0$, then we may evaluate the Lipschitz constant of the operator by (2·13a), (2·13b) and (2·13c) by the parametres

Metric	Lipschitz constant
$\rho(\boldsymbol{x}, \boldsymbol{y}) = \max_{i} \lvert x^i - y^i \rvert$	$L = \max(\{c^i\}_{i=1}^{M})$ where $c^i = \sum_{i=1}^{i-1} \lvert a_{ij} \rvert c^j + \sum_{j=1}^{M} \lvert a_{ij} \rvert$, $c^1 = \sum_{j=1}^{M} \lvert a_{1j} \rvert$ (2·13a)
$\rho(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{M} \lvert x^i - y^i \rvert$	$L = \sum_{i=1}^{M} c^i$ where $c^i = \sum_{j=1}^{i-1} \lvert a_{ij} \rvert\, c^j + \max\{\lvert a_{ij} \rvert\}_{j=i}^{M}$ $c^1 = \max\{\lvert a_{1j} \rvert\}_{j=1}^{M}$ (2·13*b*)
$\rho(\boldsymbol{x}, \boldsymbol{y}) = (\sum_{i=1}^{M} (x^2 - y^2)^2)^{1/2}$	$L = (\sum_{i=1}^{M} (c^i)^2)^{1/2}$ where $c^i = \sum_{j-1}^{i-1} \lvert a_{ij} \rvert c^j + (\sum_{j=i}^{M} (a_{ij})^2)^{1/2}$, $c^1 = (\sum_{j=1}^{M} (a_{1j})^2)^{1/2}$ (2·13c)

$$\lvert a'_{ij} \rvert = \lvert a_{ij} \rvert + \lvert a_{i1}\, a_{1j} \rvert \qquad i, j = 2, 3, \dots, M \tag{2·14}$$

as to $M-1$ variables $(x^2, x^3, \dots, x^M)$, instead of the parametres

$$\lvert a_{ij} \rvert \qquad i, j = 1, 2, \dots, M$$

as to M variables $(x^1, x^2, \dots, x^M)$.

III. Fomulation of the problem and the applicability of Gauss-Seidel method

Given a function

$$f(x) \qquad \alpha = u_0 \leqq x \leqq u_M = \beta, \tag{3·1}$$

it is desired to obtain an approximation of the form

$$\left.\begin{aligned} y(x) &= y^1(x) && u_0 \leqq x \leqq u_1 \\ &= y^2(x) && u_1 \leqq x \leqq u_2 \\ &\cdots\cdots\cdots \end{aligned}\right\} \tag{3·2}$$

$$\begin{aligned} &= y^i(x) && u_{i-1} \leqq x \leqq u_i \\ &\cdots\cdots\cdots \\ &= y^M(x) && u_{M-1} \leqq x \leqq u_M \end{aligned}$$

with

$$y^i(x) = \frac{\theta^i - \theta^{i-1}}{u_i - u_{i-1}} (x - u_{i-1}) + \theta^{i-1}, \qquad u_{i-1} \leqq x \leqq u_i \tag{3·3}$$

which is the best fit in the least squares sense, where the points $u_1, \dots, u_{M-1}$ are specified in advance. Thus, we want to fit a continuous line segments to the given curve. (Fig. 1.)

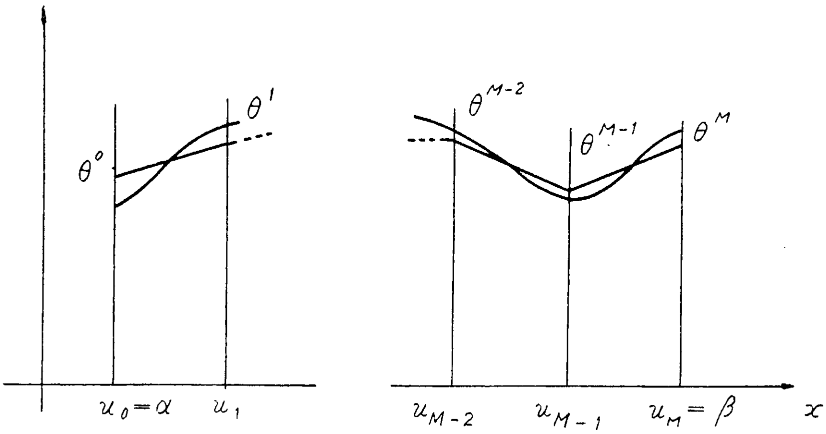

Fig. 1.

Following the least squares formulation, we wish to obtain the values of the variables $(\theta^0, \dots, \theta^M)$ which minimize

$$\begin{aligned} F(\theta^0, \theta^1, \dots, \theta^M) &= \sum_{i=1}^{M} \int_{u_{i-1}}^{u_i} (f(x) - y_i(x))^2 \, dx \\ &= \sum_{i=1}^{M} \int_{u_{i-1}}^{u_i} \left(f(x) - \frac{\theta^i - \theta^{i-1}}{u_i - u_{i-1}} (x - u_{i-1}) - \theta^{i-1}\right)^2 dx \end{aligned} \tag{3·4}$$

Since $F(\theta^0, \theta^1, \dots, \theta^M)$ is a non-negative definite quadratic form as to $\theta^0, \dots, \theta^M$, we may obtain the minimum by finding out the point at which the partial derivatives of the function (3·4) as to $(\theta^0, \dots, \theta^M)$ are zero. This leads to the following system of $M+1$ linear equations to be solved for the $M+1$ variables $\theta^0, \theta^1, \dots \theta^M$:

$$0 = \frac{1}{2} \frac{\partial F}{\partial \theta^0} = \frac{(u_1 - u_0)}{3} \theta^0 + \frac{(u_1 - u_0)}{6} \theta^1 + J(u_0, u_1) - I(u_0, u_1) \tag{3·5—0}$$

$$0=\frac{1}{2}\frac{\partial F}{\partial \theta^i}=\frac{(u_i-u_{i-1})}{6}\theta^{i-1}+\frac{(u_{i+1}-u_{i-1})}{3}\theta^i+\frac{(u_{i+1}-u_i)}{6}\theta^{i+1}$$

$$-J(u_{i-1},\ u_i)+J(u_i,\ u_{i+1})-I(u_i,\ u_{i+1}) \qquad (3\cdot5—i)$$

$$i=2,\ \ldots\ldots,\ M-1$$

$$0=\frac{1}{2}\frac{\partial F}{\partial \theta^M}=\frac{(u_M-u_{M-1})}{6}\theta^{M-1}+\frac{(u_M-u_{M-1})}{3}\theta^M-J(u_{M-1}.\ u_M) \qquad (3\cdot5—M)$$

where

$$I(u_i,\ u_{i+1})=\int_{u_i}^{u_{i+1}} f(x)\,dx \qquad (3\cdot6a)$$

$$J(u_i,\ u_{i+1})=\int_{u_i}^{u_{i+1}} f(x)\frac{x-u_i}{u_{i+1}-u_i}dx. \qquad (3\cdot6b)$$

Although this system of equations $(3\cdot5—0)\sim(3\cdot5—M)$ may be solved by the any of numerical methods solving system of linear equations, we wish to indicate the applicability of the Gauss-Seidel successive approximation method with the recurrence relation.

$$\theta^0_{n+1}=-\frac{1}{2}\theta^1_n+\frac{3}{(u_1-u_0)}(I(u_0,u_1)-J(u_0,u_1))$$

$$\theta^1_{n+1}=-\frac{(u_1-u_0)}{2(u_2-u_0)}\theta^0_{n+1}-\frac{u_2-u_1}{2(u_2-u_0)}\theta^2_n+$$

$$+\frac{3}{(u_2-u_0)}(I(u_1,u_2)+J(u_0,u_1)-J(u_1,u_2))$$

$$\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots$$

$$\theta^i_{n+1}=-\frac{(u_i-u_{i-1})}{2(u_{i+1}-u_{i-1})}\theta^{i-1}_{n+1}-\frac{(u_{i+1}-u_i)}{2(u_{i+1}-u_{i-1})}\theta^{i+1}_n+ \qquad (3\cdot7)$$

$$+\frac{3}{(u_{i+1}-u_{i-1})}(I(u_i,u_{i+1})+J(u_{i-1},u_i)-J(u_i,u_{i+1}))$$

$$\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots$$

$$\theta^M_{n+1}=-\frac{1}{2}\theta^{M+1}_n+\frac{3}{(u_M-u_{M-1})}J(u_{M-1},u_M)$$

We may write the recurrence relation above, also, in the form

$$\theta^i_{n+1}=\sum_{j=0}^{i}a_{ij}\theta^j_{n+1}+\sum_{j=i+1}^{M}a_{ij}\theta^j_n+b_i$$

$$\theta^0_{n+1}=\sum_{j=0}^{M}a_{0j}\theta^j_n+b_0 \qquad (3\cdot8)$$

where

$$b_0=\frac{3}{u_1-u_0}(I(u_0,\ u_1)-J(u_0,\ u_1))$$

$$\left.\begin{aligned} b_i &= \frac{3}{u_{i+1}-u_{i-1}}\left(I(u_i,\ u_{i+1})+J(u_{i-1},\ u_i)-J(u_i,\ u_{i+1})\right) \\ b_M &= \frac{3}{u_M-u_{M-1}}J(u_{M-1},\ u_M) \end{aligned}\right\} \tag{3·9}$$

and

$$a_{0,\,0}=0,\quad a_{0,\,1}=-\frac{1}{2},\quad a_{0,\,2}=a_{0,\,3}=\ldots=a_{0,\,M}=0$$

$$a_{1,\,0}=-\frac{(u_1-u_0)}{2(u_2-u_0)},\quad a_{1,\,1}=0,\quad a_{1,\,2}=-\frac{(u_2-u_1)}{2(u_2-u_0)},$$

$$a_{1,\,3}=\ldots\ldots=a_{1,\,M}=0\,,$$

....

$$a_{i,\,0}=a_{i,\,1}=\ldots=a_{i,\,i-2}=0,\quad a_{i,\,i-1}=-\frac{(u_i-u_{i-1})}{2(u_{i+1}-u_{i-1})}, \tag{3·10}$$

$$a_{i,\,i}=0,\quad a_{i,\,i+1}=-\frac{u_{i+1}-u_i}{2(u_{i+1}-u_{i-1})},$$

$$a_{i,\,i+2}=\ldots\ldots=a_{i,\,M}=0\,,$$

....

$$a_{M,\,0}=a_{M,\,1}=\ldots=a_{M,\,M-2}=0,\quad a_{M,\,M-1}=-\frac{1}{2},\quad a_{M,\,M}=0.$$

Let us now examine the convergence of the sequence $\{\boldsymbol{\theta}_n\}$ given by the recurrence relation (3·7) or (3·8). Since we may assume that

$$\frac{1}{2}\leqq\max_i\left(\frac{u_i-u_{i-1}}{u_{i+1}-u_{i-1}}\right)=2\gamma<1 \tag{3·11}$$

we see that

$$|a_{i,\,i-1}|\leqq\gamma\,,\qquad |a_{i,\,i+1}|\leqq\gamma \tag{3·12}$$

$$\frac{1}{4}\leqq\gamma<\frac{1}{2}\,. \tag{3·13}$$

Hence the Lipschitz constant L of the operator is evaluated by (2·13a) as

$$L=\max\left(\{c^i\}_{i=0}^{M}\right) \tag{3·14}$$

where

$$\begin{aligned} &c^0\leq\frac{1}{2} \\ &c^1\leq\frac{1}{2}\gamma+\gamma \\ &\ldots\ldots\ldots \\ &c^i\leq c^{i-1}\gamma+\gamma \\ &\ldots\ldots\ldots \\ &c^M\leq\frac{1}{2}c^{M-1} \end{aligned} \tag{3·15}$$

It may be easily seen that

$$c^i \leq \frac{1}{2}\gamma^i + \frac{1-\gamma^{i+1}}{1-\gamma} - 1 = \left(\frac{1}{2} - \frac{\gamma}{1-\gamma}\right)\gamma^i + \frac{\gamma}{1-\gamma}$$

$$c^M \leq \frac{1}{2}c^{M-1} \qquad i=0.\ \ldots\ldots, M-1 \tag{3·16}$$

Hence, for

$$\frac{1}{4} \leqq \gamma \leqq \frac{1}{3}$$

$$L = \max(\{c^i\}_{i=0}^{M}) \leq c^0 = \frac{1}{2} \tag{3·17}$$

and for

$$\frac{1}{3} \leqq \gamma < \frac{1}{2}$$

$$L = \max(\{c_i\}_{i=0}^{M}) \leq c^{M-1} = \left(\frac{1}{2} - \frac{\gamma}{1-\gamma}\right)\gamma^{M-1} + \frac{\gamma}{1-\gamma} < \frac{\gamma}{1-\gamma} < 1 \tag{3·18}$$

which show the convergence of the sequence.

As it was remarked in the last part of II, we may have, in some cases, smaller evaluations as follows

$$L' = \max(\{c'^i\}_{i=1}^{M}) \tag{3·19}$$

where

$$c'^1 \leq \frac{3}{2}\gamma$$

$$c'^2 \leq \frac{3}{2}\gamma^2 + \gamma$$

$$\cdots\cdots\cdots\cdots\cdots \tag{3·20}$$

$$c'^{M-1} \leq c'^{i-1}\gamma + \gamma$$

$$c'^M \leq \frac{1}{2}c'^{M-1}$$

(cf. (2·14)).

It may be easily seen that

$$c'^i \leq \frac{3}{2}\gamma^i + \frac{1-\gamma^i}{1-\gamma} - 1 = \left(\frac{1}{2} - \frac{\gamma}{1-\gamma}\right)\gamma^i + \frac{\gamma}{1-\gamma}. \tag{3·21}$$

Thus

$$L' = \max(\{c'^i\}_{i=1}^{M}) \leq c'^1 = \frac{3}{2}\gamma \qquad \frac{1}{4} \leqq \gamma \leqq \frac{1}{3} \tag{3·22}$$

$$L' = \max(\{c'^i\}_{i=1}^{M}) \leq c'^{M-1} \qquad \frac{1}{3} \leqq \gamma \leqq \frac{1}{2} \tag{3·23}$$

where (3·22) is, in general, not greater than the evaluation (3·17) whereas (3·23) is equal to the evaluation (3·18).

It is also possible to evaluate the Lipschitz constant corresponding to other metrics (eg. (2・13b), (2・13c) and/or (2・14)) which can be at most in some cases, not only less than unity but also less than above evaluations.

The above evaluation can be aids in deciding when to stop numerical iterations in actual calculations (cf. 2・7). Let us take for example $\gamma=1/4$ which corresponds to the case where every subinterval, on which the line segment is given, is of the same length.

Since

$$\frac{(L)^5}{1-L} \leq \frac{\left(\frac{3}{8}\right)^5}{\frac{5}{8}} \doteqdot 0.012 ,$$

it may be almost enough with 5 iterations in practice.

Remark

We may also give the approximating vector sequence $\{(\theta_n^0,\ldots, \theta_n^M)\}$ by the (simple) recurrence relation

$$\theta_{n+1}^i = \sum_{j=0}^{M} a_{ij}\theta_n^j + b_i \tag{3・23}$$

instead of the relation (3・8). This recurrence relation is associated with the Lipschitz constant (cf. [1])

$$L=\max\left(\left\{\sum_{j=0}^{M} |a_{ij}|\right\}_{i=0}^{M}\right)=\frac{1}{2} \tag{3・24}$$

This evaluation shows that it seems to be favorable to use (3・23) if $\gamma \geqq 1/3$, *so far as* the Lipshitz constant is concerned.

Worked Example

Let us now try to approximate the function

$$f(x)=\sin x \qquad x\in[0, \pi/2] \tag{3・25}$$

by the continuous line segments

$$y^i(x)=\frac{\theta^i-\theta^{i-1}}{\frac{\pi}{6}}(x-u_i)+\theta^{i-1} \tag{3・26}$$

where

$$\begin{aligned} u_0&=0\\ u_1&=\pi/6\\ u_2&=2\pi/6\\ u_3&=\pi/2 \end{aligned} \tag{3・27}$$

Corresponding to the recurrence relation (3・7) we have

$$\theta_{n+1}^0 = \qquad -\frac{1}{2}\theta_n^1 \qquad +0.2582$$

$$\theta^1_{n+1} = -\frac{1}{4}\theta^0_{n+1} \qquad -\frac{1}{4}\theta^2_n \qquad +0.7330$$

$$\theta^2_{n+1} = \qquad -\frac{1}{4}\theta^1_{n+1} \qquad -\frac{1}{4}\theta^3_n +1.2696$$

$$\theta^3_{n+1} = \qquad -\frac{1}{2}\theta^2_{n+1} \qquad +1.4659$$

Beginning with the initial approximation

$$\theta^0_0 = \theta^1_0 = \theta^2_0 = \theta^3_0 = 0,$$

the sequence converged up the 4-th decimal place after 8 iterations.

$$\theta^0_8 = 0.0029,\ \theta^1_8 = 0.5106,\ \theta^2_8 = 0.8863,\ \theta^3_8 = 1.0227$$

(cf. Fig. 2.)

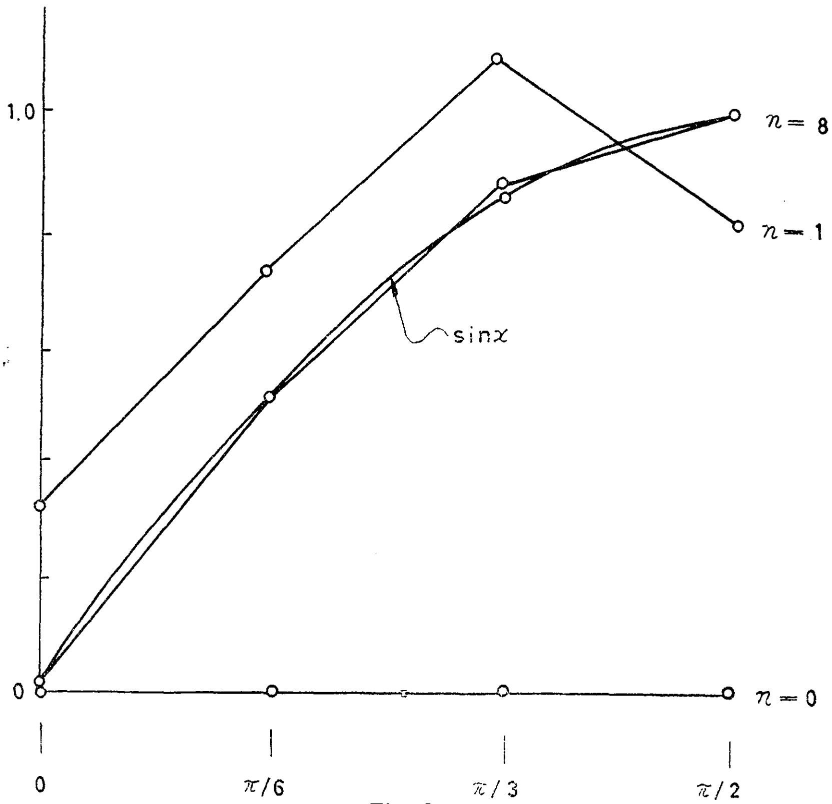

Fig. 2.

IV. Continuous line segments fitting to a set of points

We may also apply the method described in the preceeding section to obtain the best fit continuous line segments to a set of points in the least square sense.

Consider, for the sake of simplicity, that we are given the values

$$\begin{aligned} & z_{ij} \text{ at } x=(N(i-1)+j)\Delta x \\ & \Delta x>0 \\ & i=1, \dots, M \\ & j=0, \dots, N-1, (N) \qquad z_{i,N}=z_{i+1,0} \\ & N\geqq 1,\ M\geqq 1 \end{aligned} \tag{4·1}$$

to which it is desired to obtain the best fit continuous line segments

$$y_i(x)=\frac{\theta^i-\theta^{i-1}}{N\Delta x}(x-(i-1)N\Delta x)+\theta^{i-1}, \tag{4·2}$$

$$N(i-1)\Delta x\leqq x\leqq Ni\Delta x$$

in the least square sense.

Following the least square formulation, let as obtain the values of variables which minimize

$$F(\theta^0,\ \theta^1, \dots, \theta^M)=\sum_{i=1}^{M}\sum_{j=0}^{N-1}(z_{ij}-y_{ij})^2+(z_{MN}-y_{MN})^2 \tag{4·3}$$

where

$$y_{ij}=y_i((N(i-1)+j)\Delta x)=\frac{1}{N}(j\theta^i+(N-j)\theta^{i-1}). \tag{4·4}$$

Thus

$$F(\theta^0,\ \theta^1,\dots, \theta^M)=\sum_{i=1}^{M}\sum_{j=0}^{N-1}(z_{ij}-\frac{1}{N}(j\theta^i+(N-j)\theta^{i-1}))^2+(z_{MN}-\theta^M)^2 \tag{4·5}$$

Equating the partial derivatives $\partial F/\partial\theta^i$ to zero, we obtain the following $M+1$ relations

$$0=\frac{1}{2}\cdot\frac{\partial F}{\partial\theta^0}=\frac{(N+1)(2N+1)}{6N}\theta^0+\frac{N^2-1}{6N}\theta^1-\frac{1}{N}\sum_{j=0}^{N-1}(N-j)z_{1,j}$$

....

$$\begin{aligned} 0=\frac{1}{2}\cdot\frac{\partial F}{\partial\theta^i}&=\frac{N^2-1}{6N}\theta^{i-1}+\frac{2N^2+1}{3N}\theta^i+\frac{N^2-1}{6N}\theta^{i+1} \\ &-\frac{1}{N}\sum_{j=1}^{N-1}jz_{ij}-\frac{1}{N}\sum_{j=0}^{N-1}(N-j)z_{i+1,j} \end{aligned} \tag{4·6}$$

....

$$0=\frac{1}{2}\cdot\frac{\partial F}{\partial\theta^M}=\frac{N^2-1}{6N}\theta^{M-1}+\frac{(N+1)(2N+1)}{6N}\theta^M-\frac{1}{N}\sum_{j=1}^{N}jz_{ij}$$

As the corresponding recurrence relation of the Seidel method we have,

$$\theta_{n+1}^{0}=-\alpha_N'\theta_n^1+\beta_N' I_0$$
$$\cdots\cdots\cdots\cdots$$
$$\theta_{n+1}^{i}=-\alpha_N(\theta_{n+1}^{i-1}+\theta_n^{i+1})+\beta_N I_i \tag{4·7}$$
$$\cdots\cdots\cdots\cdots$$
$$\theta_{n+1}^{M}=-\alpha_N'\theta_{n+1}^{M-1}+\beta_N' I_M$$

where

$$\alpha_N'=\frac{N-1}{2N+1},\qquad \alpha_N=\frac{N^2-1}{2(2N^2+1)}$$
$$\beta_N'=\frac{6}{(N+1)(2N+1)},\qquad \beta_N=\frac{3}{2N^2+1}$$
$$I_i=\sum_{j=1}^{N-1} jz_{ij}+\sum_{j=0}^{N-1}(N-j)z_{j+1,j},\qquad i=1,\ldots,M-1 \tag{4·8}$$
$$I_0=\sum_{j=0}^{N-1}(N-j)z_{ij}$$
$$I_M=\sum_{j=1}^{N} jz_{ij}$$

Remark that the parametres α_N and α_N' are in absolute values less than the corresponding parametres in (3·7) respectively :

$$\alpha_N'<\frac{1}{2}$$
$$\alpha_N<\frac{1}{2}$$

which shows the convergence of the sequence $\{(\theta_n^0,\ldots,\theta_n^M)\}$. A check of validity of (4·7) and (4·8) is that, as $N\to\infty$, $\alpha_N'\to 1/2$ and $\alpha_N\to 1/4$, as are to be expected.

The values of the parametres α_N, α_N', β_N, β_N' are given in Table 1 for $N=1,\ldots,25$.

Worked Example

Let us now, try to fit continuous line segments

$$y^i(x)=\frac{\theta^i-\theta^{i-1}}{\frac{\pi}{6}}(x-u_i)+\theta^{i-1}\qquad u_{i-1}\leqq x\leqq u_i \tag{4·9}$$

$(N=3,\ \Delta x=\pi/18)$

where

$$u_0=0$$
$$u_1=\pi/6$$
$$u_2=2\pi/6 \tag{4·10}$$
$$u_3=\pi/2$$
$$(M=3)$$

Table. 1.

N	α_N	α_N'	β_N	β_N'
1	0.000 000	0.000 000	1.000 000	1.000 000
2	0.166 677	0.200 000	0.333 333	0.400 000
3	0.210 526	0.285 714	0.157 895	0.214 286
4	0.227 273	0.333 333	0.090 909	0.133 333
5	0.235 294	0.363 636	0.058 824	0.090 909
6	0.239 726	0.384 615	0.041 096	0.066 592
7	0.242 424	0.400 000	0.030 303	0.050 000
8	0.244 186	0.411 765	0.023 256	0.039 216
9	0.245 399	0.421 053	0.018 405	0.031 579
10	0.246 269	0.428 571	0.014 925	0.025 974
11	0.246 914	0.434 783	0.012 346	0.021 739
12	0.247 405	0.440 000	0.010 381	0.018 462
13	0.247 788	0.444 444	0.008 850	0.015 873
14	0.248 092	0.448 276	0.007 634	0.013 245
15	0.248 337	0.451 613	0.006 652	0.012 097
16	0.248 538	0.454 545	0.005 848	0.010 695
17	0.248 705	0.457 143	0.005 181	0.009 524
18	0.248 844	0.459 459	0.004 622	0.008 535
19	0.248 963	0.461 538	0.004 149	0.007 692
20	0.249 064	0.463 415	0.003 745	0.006 969
21	0.249 151	0.465 116	0.003 398	0.006 342
22	0.249 226	0.466 667	0.003 096	0.005 797
23	0.249 292	0.468 085	0.002 833	0.005 319
24	0.249 350	0.469 388	0.002 602	0.004 898
25	0.249 400	0.470 588	0.002 398	0.004 525

to the set of points

$$z_{ij}=\sin(3(i-1)+j)\pi/18.$$

Corresponding to the recurrence relation (4·7) we have

$$\begin{aligned}
\theta^0_{n+1}&= & -0.2857\theta^1_n & & & +0.2210\\
\theta^1_{n+1}&=-0.2105\theta^0_{n+1} & & -0.2105\theta^2_n & & +0.6915\\
\theta^2_{n+1}&= & -0.2105\theta^1_{n+1} & & -0.2105\theta^3_n & +1.2059\\
\theta^3_{n+1}&= & & -0.2857\theta^2_{n+1} & & +1.2664
\end{aligned}$$

Beginning with the initial approximation

$$\theta^0_0=\theta^1_0=\theta^2_0=\theta^3_0=0,$$

the sequence converged up to the 4 th decimal place after 7 iterations.

$$\theta^0_7=0.0819,\quad \theta^1_7=0.4868,\quad \theta^2_7=0.8904,\quad \theta^3_7=1.0120$$

Compare the results with the corresponding results in the worked example in the preceeding section.

V. Some remarks on the method

Gauss-Seidel successive approximation technique applied to continuous line segments curve fitting, evolved here, may be stated in terms of optimization techniques as follows. It is desired to find the minimum of a function

$$F(\theta^0, \theta^1 \dots, \theta^M). \qquad (5\cdot1)$$

Taking a vector $\theta_0=(\theta_0^0, \theta_0^1,\dots, \theta_0^M)$ as the 0-th approximation, we modify θ^0 so as the modify function takes its minimum as to θ^0 with the remainder of the variables $\theta_0^1, \dots, \theta_0^M$ fixed. After fixing θ^0 to that value θ_1^0, modify θ^1 so as the function takes its minimum as to θ^1 with the fixed remainder of variables θ_1^0, θ_0^2, ..., θ_0^M. Continuing in this way up to θ^M, we obtain, the first approximation $\boldsymbol{\theta}_1$. Repeating this process several times, we would be able to attain the value of the variables $(\theta^0, \theta^1, \dots, \theta^M)$ which would give, in most cases, the minimum to the function F.

In other words this method consists of the process of searching *one-dimensional* minimums in the problem of searching the $M+1$ *dimensional* minimum where each *one-dimensional* minimum is searched along the straight line parallel to the axis of the corresponding argument.

In a sense, it may be also possible to recognize this method as a kind of relaxation method which is often employed in solving differential equations numerically.

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