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(Received March 8, 1965)

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#### Abstract

Successive approximation technique is applied to obtain the best fit continuous line segments with a finite number of vertices with predetermined abscissae to a given curve or a set of points, in the least square sense. The Lipschitz constant associated to the recurrence relation is evaluated, so as to be able to obtain the stopping rule in numerical iterations.

### I. Introduction

It is required, recently, to fit line segments to a given function or a set of points, related to some computational techniques. Some of their examples are the numerical solution of convex programming problems via solving approximated linear programming problems and generation of functions on electronic analogue computer systems, employing integrators and comparators.

Algorithms as to various types of line segments curve fitting problems are given : Stone [3] by classical method, Bellman and Gluss  $[4]\sim[6]$  by dynamic programming technique, and Pontryagin et al [7] by his maximum principle.

In this article, it will be described how the successive approximation technique can be applied to obtain the best fit continuous line segments with a finite number of vertices with predetermined abscissae to a given curve or a set of points in the least square sense. The evaluations of the Lipschitz constant associated to the recurrence relation will be used in estimating the errors of the approximations.

After describing the convergence criterions as to the recurrence relations in II, the recurrence relation giving the successive approximations in obtaining the best fit continuous line segments to a curve will be given in II, and to a set of points in IV. It will be also noted in V, that the method described in this article may be considered as an example of a particular approach to the minimization problem.

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# II. Contraction theory and Gauss-Seidel method in solving systems of linear equations

Let us first recall the theorem of contraction operator, defined on a complete metric space as the fundamental theorem of the method evolved in this article.

### Theorem

If an operator 1)

$$\varphi(x) \qquad x \in R(X, \rho) \tag{2.1}$$

defined on a complete metric space  $R(X, \rho)$  mapping R into itself is associated with the Lipschitz constant

$$\rho(\varphi(x), \varphi(x')) \leq L\rho(x, x') \quad x, x' \in R(X, \rho)$$
(2.2)

less than unity

$$L < 1, \tag{2.3}$$

the sequence

$$\{x_0, x_1, \dots, x_n, \dots\}$$
 (2.4)

beginning with an arbitrary element of  $R(X, \rho)$ , defined by the recurrence relation

$$x_{n+1} = \varphi(x_n) \qquad x_0 \in R(X, \rho) \tag{2.5}$$

converges to the unique fixed point of the operator

$$x = \varphi(x). \tag{2.6}$$

Moreover, it is seen that

$$\rho(x_n, x) \leq \frac{L^n}{1-L} \rho(x_0, x_1). \tag{2.7}$$

# Proof

The proof of the theorem may be seen elsewhere in ordinary text books (eg. [8]) of functional analysis and will not be repeated here.

Next, let us consider Gauss-Seidel method from the point of view of contraction operator.

One of the numerical method of solving system of linear equations

$$c_{11}x^{1} + c_{12}x^{2} + \dots + c_{1M}x^{M} = d^{1}$$

$$c_{21}x^{1} + c_{22}x^{2} + \dots + c_{2M}x^{M} = d^{2}$$

$$\dots$$

$$c_{M1}x^{1} + c_{M2}x^{2} + \dots + c_{MM}x^{M} = d^{M}$$
(2.8)

<sup>1)</sup> The operator associated with a Lipschitz constant less than unity is sometimes call a *contraction operator*.

is to convert the system (2.8) into the form

$$x^{M} = a_{M1}x^{1} + a_{M2}x^{2} + \dots + a_{MM}x^{M} + b_{M}$$

or in vector-matrix form

$$\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b} \qquad \boldsymbol{x} \in \boldsymbol{R}_{\boldsymbol{M}} \tag{2.10}$$

where

$$\begin{array}{c} \mathbf{x} = \begin{pmatrix} x^{1} \\ x^{2} \\ \vdots \\ x^{M} \end{pmatrix} \qquad \begin{array}{c} \mathbf{A} = \begin{pmatrix} a_{11}, a_{12}, \dots, a_{1M} \\ a_{21}, a_{22}, \dots, a_{2M} \\ a_{M1}, a_{M2}, \dots, a_{MM} \end{pmatrix} \qquad \begin{array}{c} \mathbf{b} = \begin{pmatrix} b_{1} \\ b_{2} \\ \vdots \\ b_{M} \end{pmatrix}$$
(2.11)

and to evaluate the vector sequence  $\{x_n\}$  given by the recurrence relation

$$x_{n+1}^{1} = a_{11}x_{n}^{1} + a_{12}x_{n}^{2} + \dots a_{1M}x_{n}^{M} + b_{1}$$

$$x_{n+1}^{2} = a_{21}x_{n+1}^{1} + a_{22}x_{n}^{2} + \dots + a_{2M}x_{n}^{M} + b_{2}$$

$$x_{n+1}^{M} = a_{M1}x_{n+1}^{1} + a_{M2}x_{n+1}^{2} + \dots a_{M}, \ M = a_{M1}x_{n}^{1} + a_{MM}x_{n}^{M} + b_{M}$$

$$x_{0} : \text{arbitrary}$$

$$(2 \cdot 12)$$

until the distance between  $x_n$  and  $x_{n+1}$  becomes negligible. This method is called the Gauss-Seidel iteration method in solving systems of linear equations.

It may be readily remarked that this method is applicable only when the sequence given by  $(2 \cdot 12)$  is convergent. Referring to the theorem above, we may examine the convergence of the sequence by evaluating the Lipschitz constant of  $(2 \cdot 12)$ considered as an operator. The Lipschitz constant of the operator can be evaluated as follows, corresponding to the metric defined on the *M*-dimensional space  $R_M$ . It is desirable not only to have the evaluation of the Lipschitz constant less than unity but also to evaluate it as small as possible in order to assure faster convergence as it was indicated in  $(2 \cdot 7)$ . Since the derivations of the evaluations of the Lipschitz constant given below may be referred to [1] and partly to [2], they will not be given here.

It may be well known that the *M*-dimensional space  $R_M$  is complete as to the metrics defined in (2.13a), (2.13b) and (2.13c).

It may be also referred to [1] that if  $a_{11}=0$ , then we may evaluate the Lipschitz constant of the operator by (2.13a), (2.13b) and (2.13c) by the parameters

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Metric	Lipschitz constant				
$\rho(\mathbf{x}, \mathbf{y}) = \max_{i}  x^{i} - y^{i} $	$L = \max(\{c^i\}_{i=1}^M)$				
	where				
	$c^{i} = \sum_{i=1}^{i-1}  a_{ij}  c^{j} + \sum_{j=1}^{M}  a_{ij} ,$				
	$c^{1} = \sum_{j=1}^{M}  a_{1j}  \qquad (2.13a)$				
$\rho(\boldsymbol{x},\boldsymbol{y}) = \sum_{i=1}^{M}  x^i - y^i $	$L = \sum_{i=1}^{M} c^{i}$				
	where				
	$c^{i} = \sum_{j=1}^{i-1}  a_{ij}  c^{j} + \max\{ a_{ij} \}_{j=i}^{M}$				
	$c^{1} = \max\{ a_{1j} \}_{j=1}^{M}$ (2.13b)				
$\rho(\mathbf{x}, \mathbf{y}) = (\sum_{i=1}^{M} (x^2 - y^2)^2)^{1/2}$	$L = (\sum_{i=1}^{M} (c^{i})^{2})^{1/2}$				
	where				
	$c^{i} = \sum_{j=1}^{i-1}  a_{ij}  c^{j} + (\sum_{j=i}^{M} (a_{ij})^{2})^{1/2},$				
	$c^{1} = (\sum_{j=1}^{M} (a_{1j})^{2})^{1/2}$ (2.13c)				

$$|a'_{ij}| = |a_{ij}| + |a_{i1} a_{1j}| \qquad i, j = 2, 3, \dots, M$$
(2.14)

as to M-1 variables  $(x^2, x^3, \dots, x^M)$ , instead of the parametres

 $|a_{ij}|$  *i*, *j*=1, 2, ..., *M* 

as to M variables  $(x^1, x^2, \dots, x^M)$ .

# III. Fomulation of the problem and the applicability of Gauss-Seidel method

Given a function

$$f(\mathbf{x}) \qquad \alpha = u_0 \leq \mathbf{x} \leq u_M = \beta, \tag{3.1}$$

it is desired to obtain an approximation of the form

$$y(x) = y^{1}(x) u_{0} \le x \le u_{1} \\ = y^{2}(x) u_{1} \le x \le u_{2} \\ \dots \dots \dots \qquad (3.2)$$

$$= y^{i}(x) \qquad u_{i-1} \leq x \leq u_{i}$$

$$= y^{M}(x) \qquad u_{M-1} \leq x \leq u_{M}$$

with

$$y^{i}(x) = \frac{\theta^{i} - \theta^{i-1}}{u_{i} - u_{i-1}} (x - u_{i-1}) + \theta^{i-1}, \qquad (3.3)$$
$$u_{i-1} \le x \le u_{i}$$

which is the best fit in the least squares sense, where the points  $u_1, \ldots, u_{N-1}$  are specified in advance. Thus, we want to fit a continuous line segments to the given curve. (Fig. 1.)



Fig. 1.

Following the least squares formulation, we wish to obtain the values of the variables  $(\theta^0, ..., \theta^M)$  which minimize

$$F(\theta^{0}, \theta^{1}, ..., \theta^{M}) = \sum_{i=1}^{M} \int_{u_{i-1}}^{u_{i}} (f(x) - y_{i}(x))^{2} dx$$

$$= \sum_{i=1}^{M} \int_{u_{i-1}}^{u_{i}} (f(x) - \frac{\theta^{i} - \theta^{i-1}}{u_{i} - u_{i-1}} (x - u_{i-1}) - \theta^{i-1})^{2} dx$$
(3.4)

Since  $F(\theta^0, \theta^1, ..., \theta^M)$  is a non-negative definite quadratic form as to  $\theta^0, ..., \theta^M$ , we may obtain the minimum by finding out the point at which the partial derivatives of the function (3.4) as to  $(\theta^0, ..., \theta^M)$  are zero. This leads to the following system of M+1 linear equations to be solved for the M+1 variables  $\theta^0, \theta^1, ..., \theta^M$ :

$$0 = \frac{1}{2} \frac{\partial F}{\partial \theta^0} = \frac{(u_1 - u_0)}{3} \theta^0 + \frac{(u_1 - u_0)}{6} \theta^1 + J(u_0, u_1) - I(u_0, u_1)$$
(3.5-0)

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$$0 = \frac{1}{2} \frac{\partial F}{\partial \theta^{i}} = \frac{(u_{i} - u_{i-1})}{6} \theta^{i-1} + \frac{(u_{i+1} - u_{i-1})}{3} \theta^{i} + \frac{(u_{i+1} - u_{i})}{6} \theta^{i+1} -J(u_{i-1}, u_{i}) + J(u_{i}, u_{i+1}) - I(u_{i}, u_{i+1})$$

$$i = 2, \dots, M-1$$

$$0 = \frac{1}{2} \frac{\partial F}{\partial \theta^{M}} = \frac{(u_{M} - u_{M-1})}{6} \theta^{M-1} + \frac{(u_{M} - u_{M-1})}{3} \theta^{M} - J(u_{M-1}, u_{M}) \quad (3.5-M)$$

where

$$I(u_i, u_{i+1}) = \int_{u_i}^{u_{i+1}} f(x) \, dx \tag{3.6a}$$

$$J(u_i, u_{i+1}) = \int_{u_i}^{u_{i+1}} f(x) \frac{x - u_i}{u_{i+1} - u_i} dx.$$
 (3.6b)

Although this system of equations  $(3 \cdot 5 - 0) \sim (3 \cdot 5 - M)$  may be solved by the any of numerical methods solving system of linear equations, we wish to indicate the applicability of the Gauss-Seidel successive approximation method with the recurrence relation.

$$\begin{aligned} \theta_{n+1}^{0} &= -\frac{1}{2} \theta_{n}^{1} + \frac{3}{(u_{1} - u_{0})} \left( I(u_{0}, u_{1}) - J(u_{0}, u_{1}) \right) \\ \theta_{n+1}^{1} &= -\frac{(u_{1} - u_{0})}{2(u_{2} - u_{0})} \theta_{n+1}^{0} - \frac{u_{2} - u_{1}}{2(u_{2} - u_{0})} \theta_{n}^{2} + \\ &+ \frac{3}{(u_{2} - u_{0})} \left( I(u_{1}, u_{2}) + J(u_{0}, u_{1}) - J(u_{1}, u_{2}) \right) \\ & \dots \\ \theta_{n+1}^{i} &= -\frac{(u_{i} - u_{i-1})}{2(u_{i+1} - u_{i-1})} \theta_{n+1}^{i-1} - \frac{(u_{i+1} - u_{i})}{2(u_{i+1} - u_{i-1})} \theta_{n}^{i+1} + \\ &+ \frac{3}{(u_{i+1} - u_{i-1})} \left( I(u_{i}, u_{i+1}) + J(u_{i-1}, u_{i}) - J(u_{i}, u_{i+1}) \right) \\ & \dots \\ \theta_{n+1}^{M} &= -\frac{1}{2} \theta_{n}^{M+1} + \frac{3}{(u_{M} - u_{M-1})} J(u_{M-1}, u_{M}) \end{aligned}$$

$$(3.7)$$

We may write the recurrence relation above, also, in the form

$$\theta_{n+1}^{i} = \sum_{j=0}^{i} a_{ij} \theta_{n+1}^{j} + \sum_{j=i+1}^{M} a_{ij} \theta_{n}^{j} + b_{i}$$

$$\theta_{n+1}^{0} = \sum_{j=0}^{M} a_{0j} \theta_{n}^{j} + b_{0}$$
(3.8)

•

where

$$b_0 = \frac{3}{u_1 - u_0} \left( I(u_0, u_1) - J(u_0, u_1) \right)$$

$$b_{i} = \frac{3}{u_{i+1} - u_{i-1}} \left( I(u_{i}, u_{i+1}) + J(u_{i-1}, u_{i}) - J(u_{i}, u_{i+1}) \right)$$

$$b_{M} = \frac{3}{u_{M} - u_{M-1}} J(u_{M-1}, u_{M})$$
(3.9)

and

$$a_{0,0} = 0, \quad a_{0,1} = -\frac{1}{2}, \quad a_{0,2} = a_{0,3} = \dots = a_{0,M} = 0$$

$$a_{1,0} = -\frac{(u_{1} - u_{0})}{2(u_{2} - u_{0})}, \quad a_{1,1} = 0, \quad a_{1,2} = -\frac{(u_{2} - u_{1})}{2(u_{2} - u_{0})},$$

$$a_{1,3} = \dots = a_{1,M} = 0,$$

$$\dots$$

$$a_{i,0} = a_{i,1} = \dots = a_{i,i-2} = 0, \quad a_{i,i-1} = -\frac{(u_{i} - u_{i-1})}{2(u_{i+1} - u_{i-1})},$$

$$a_{i,i} = 0, \quad a_{i,i+1} = -\frac{u_{i+1} - u_{i}}{2(u_{i+1} - u_{i-1})},$$

$$a_{i,i+2} = \dots = a_{i,M} = 0,$$

$$\dots$$

$$a_{M,0} = a_{M,1} = \dots = a_{M,M-2} = 0, \quad a_{M,M-1} = -\frac{1}{2}, \quad a_{M,M} = 0.$$
(3.10)

Let us now examine the convergence of the sequence  $\{\theta_n\}$  given by the recurrence relation (3.7) or (3.8). Since we may assume that

$$\frac{1}{2} \le \max_{i} \left( \frac{u_i - u_{i-1}}{u_{i+1} - u_{i-1}} \right) = 2\gamma < 1 \tag{3.11}$$

we see that

$$|a_{i, i-1}| \leq \gamma , \qquad |a_{i, i+1}| \leq \gamma$$

$$(3 \cdot 12)$$

$$\frac{1}{4} \le \gamma < \frac{1}{2} \,. \tag{3.13}$$

Hence the Lipschitz constant L of the operator is evaluated by (2.13a) as

$$L = \max(\{c^i\}_{i=0}^{M})$$
(3.14)

where

$$c^{0} \leq \frac{1}{2}$$

$$c^{1} \leq \frac{1}{2}\gamma + \gamma$$

$$\cdots$$

$$c^{i} \leq c^{i-1}\gamma + \gamma$$

$$\cdots$$

$$c^{M} \leq \frac{1}{2}c^{M-1}$$

$$(3.15)$$

It may be easily seen that

$$c^{i} \leq \frac{1}{2} \gamma^{i} + \frac{1 - \gamma^{i+1}}{1 - \gamma} - 1 = \left(\frac{1}{2} - \frac{\gamma}{1 - \gamma}\right) \gamma^{i} + \frac{\gamma}{1 - \gamma}$$

$$c^{M} \leq \frac{1}{2} c^{M-1} \qquad \qquad (3 \cdot 16)$$

$$i = 0, \dots, M-1$$

Hence, for

$$\frac{1}{4} \leq \gamma \leq \frac{1}{3}$$

$$L = \max(\{c^i\}_{i=0}^{\underline{M}}) \leq c^0 = \frac{1}{2}$$
(3.17)

and for

$$L = \max(\{c_i\}_{i=0}^{M}) \le c^{M-1} = \left(\frac{1}{2} - \frac{\gamma}{1-\gamma}\right) \gamma^{M-1} + \frac{\gamma}{1-\gamma} < \frac{\gamma}{1-\gamma} < 1$$
(3.18)

which show the convergence of the sequence.

As it was remarked in the last part of  $\mathbf{I}$ , we may have, in some cases, smaller evaluations as follows

 $\frac{1}{3} \leq \gamma < \frac{1}{2}$ 

$$L' = \max(\{c'^{i}\}_{i=1}^{M}) \tag{3.19}$$

where

$$c'^{1} \leq \frac{3}{2} \gamma$$

$$c'^{2} \leq \frac{3}{2} \gamma^{2} + \gamma$$

$$\dots$$

$$c'^{M-1} \leq c'^{i-1} \gamma + \gamma$$

$$c'^{M} \leq \frac{1}{2} c'^{M-1}$$

$$(3.20)$$

(cf. (2.14)). It may be easily seen that

$$c'^{i} \leq \frac{3}{2} \gamma^{i} + \frac{1-\gamma^{i}}{1-\gamma} - 1 = \left(\frac{1}{2} - \frac{\gamma}{1-\gamma}\right) \gamma^{i} + \frac{\gamma}{1-\gamma}.$$
(3.21)

Thus

$$L' = \max(\{c'^{i}\}_{i=1}^{M}) \le c'^{1} = \frac{3}{2} \gamma \qquad \frac{1}{4} \le \gamma \le \frac{1}{3}$$
(3.22)

$$L' = \max(\{c'_i\}_{i=1}^{M}) \le c'^{M-1} \qquad \frac{1}{3} \le \gamma \le \frac{1}{2}$$
(3.23)

where  $(3\cdot22)$  is, in general, not greater than the evaluation  $(3\cdot17)$  whereas  $(3\cdot23)$  is equal to the evaluation  $(3\cdot18)$ .

It is also possible to evaluate the Lipschitz constant corresponding to other metrics (eg. (2.13b), (2.13c) and/or (2.14)) which can be at most in some cases, not only less than unity but also less than above evaluations.

The above evaluation can be aids in deciding when to stop numerical iterations in actual calculations (cf. 2.7). Let us take for example  $\gamma = 1/4$  which corresponds to the case where every subinterval, on which the line segment is given, is of the same length.

Since

$$\frac{(L)^5}{1-L} \le \frac{\left(\frac{3}{8}\right)^5}{\frac{5}{8}} \doteq 0.012 ,$$

it may be almost enough with 5 iterations in practice.

### Remark

We may also give the approximating vector sequence  $\{(\theta_n^0, ..., \theta_n^M)\}$  by the (simple) recurrence relation

$$\theta_{n+1}^{i} = \sum_{j=0}^{M} a_{ij} \theta_{n}^{j} + b_{i}$$
(3.23)

instead of the relation (3.8). This recurrence relation is associated with the Lipschitz constant (cf. [1])

$$L = \max\left(\left\{\sum_{j=0}^{M} |a_{ij}|\right\}_{i=0}^{M}\right) = \frac{1}{2}$$
(3.24)

This evaluation shows that it seems to be favorable to use (3.23) if  $\gamma \ge 1/3$ , so far as the Lipshitz constant is concerned.

#### Worked Example

Let us now try to approximate the function

$$f(x) = \sin x$$
  $x \in [0, \pi/2]$  (3.25)

by the continuous line segments

$$y^{i}(x) = \frac{\theta^{i} - \theta^{i-1}}{\frac{\pi}{6}} (x - u_{i}) + \theta^{i-1}$$
(3.26)

where

$$u_{0} = 0$$

$$u_{1} = \pi/6$$

$$u_{2} = 2\pi/6$$

$$u_{3} = \pi/2$$
(3.27)

Corresponding to the recurrence relation (3.7) we have

$$\theta_{n+1}^{0} = -\frac{1}{2} \theta_{n}^{1} + 0.2582$$

$$\theta_{n+1}^{1} = -\frac{1}{4} \theta_{n+1}^{0} - \frac{1}{4} \theta_{n}^{2} + 0.7330$$
  
$$\theta_{n+1}^{2} = -\frac{1}{4} \theta_{n+1}^{1} - \frac{1}{4} \theta_{n}^{3} + 1.2696$$
  
$$\theta_{n+1}^{3} = -\frac{1}{2} \theta_{n+1}^{2} + 1.4659$$

Beginning with the initial approximation

 $\theta_0^0 = \theta_0^1 = \theta_0^2 = \theta_0^3 = 0$ ,

the sequence converged up the 4-th decimal place after 8 iterations.

$$\theta_8^0 = 0.0029, \ \theta_8^1 = 0.5106, \ \theta_8^2 = 0.8863, \ \theta_8^3 = 1.0227$$

(cf. Fig. 2.)



IV. Continuous line segments fitting to a set of points

We may also apply the method described in the preceeding section to obtain the best fit continuous line segments to a set of points in the least square sense.

Consider, for the sake of simplicity, that we are given the values

$$z_{ij} \text{ at } x = (N(i-1)+j) \Delta x$$
  

$$\Delta x > 0$$
  

$$i = 1, ..., M$$
  

$$j = 0, ..., N-1, (N)$$
  

$$N \ge 1, M \ge 1$$
  

$$(4 \cdot 1)$$
  

$$z_{i, N} = z_{i+1, 0}$$
  

$$(4 \cdot 1)$$

to which it is desired to obtain the best fit continuous line segments

$$y_{i}(x) = \frac{\theta^{i} - \theta^{i-1}}{N dx} (x - (i-1)N dx) + \theta^{i-1}, \qquad (4\cdot 2)^{n}$$
$$N(i-1)dx \le x \le N i dx$$

in the least square sense.

Following the least square formulation, let as obtain the values of variables which minimize

$$F(\theta^{0}, \theta^{1}, \dots, \theta^{M}) = \sum_{i=1}^{M} \sum_{j=0}^{N-1} (z_{ij} - y_{ij})^{2} + (z_{MN} - y_{MN})^{2}$$
(4.3)

where

$$y_{ij} = y_i((N(i-1)+j)\Delta x) = \frac{1}{N}(j\theta^i + (N-j)\theta^{i-1}).$$
(4.4)

Thus

$$F(\theta^{0}, \theta^{1}, ..., \theta^{M}) = \sum_{i=1}^{M} \sum_{j=0}^{N-1} (z_{ij} - \frac{1}{N} (j\theta^{i} + (N-j)\theta^{i-1}))^{2} + (z_{MN} - \theta^{M})^{2}$$
(4.5)

Equating the partial derivatives  $\partial F/\partial \theta^i$  to zero, we obtain the following M+1 relations

$$0 = \frac{1}{2} \cdot \frac{\partial F}{\partial \theta^{0}} = \frac{(N+1)(2N+1)}{6N} \theta^{0} + \frac{N^{2}-1}{6N} \theta^{1} - \frac{1}{N} \sum_{j=0}^{N-1} (N-j)z_{1, j}$$

$$0 = \frac{1}{2} \cdot \frac{\partial F}{\partial \theta^{i}} = \frac{N^{2}-1}{6N} \theta^{i-1} + \frac{2N^{2}+1}{3N} \theta^{i} + \frac{N^{2}-1}{6N} \theta^{i+1}$$

$$- \frac{1}{N} \sum_{j=1}^{N-1} jz_{ij} - \frac{1}{N} \sum_{j=0}^{N-1} (N-j)z_{i+1, j}$$

$$0 = \frac{1}{2} \cdot \frac{\partial F}{\partial \theta^{M}} = \frac{N^{2}-1}{6N} \theta^{M-1} + \frac{(N+1)(2N+1)}{6N} \theta^{M} - \frac{1}{N} \sum_{j=1}^{N} jz_{ij}$$

$$(4 \cdot 6)$$

As the corresponding recurrence relation of the Seidel method we have,

$$\theta_{n+1}^{0} = -\alpha_{N}' \theta_{n}^{i} + \beta_{N}' I_{0}$$

$$\dots$$

$$\theta_{n+1}^{i} = -\alpha_{N} (\theta_{n+1}^{i-1} + \theta_{n}^{i+1}) + \beta_{N} I_{i}$$

$$\dots$$

$$\theta_{n+1}^{M} = -\alpha_{N}' \theta_{n+1}^{M-1} + \beta_{N}' I_{M}$$

$$(4.7)$$

.

where

$$\alpha_{N}' = \frac{N-1}{2N+1}, \qquad \alpha_{N} = \frac{N^{2}-1}{2(2N^{2}+1)}$$

$$\beta_{N}' = \frac{6}{(N+1)(2N+1)}, \qquad \beta_{N} = \frac{3}{2N^{2}+1}$$

$$I_{i} = \sum_{j=1}^{N-1} j z_{ij} + \sum_{j=0}^{N-1} (N-j) z_{j+1, j}, \qquad i=1, ..., M-1$$

$$I_{0} = \sum_{j=0}^{N-1} (N-j) z_{ij}$$

$$I_{M} = \sum_{j=1}^{N} j z_{ij}$$
(4.8)

Remark that the parametres  $\alpha_N$  and  $\alpha_{N'}$  are in absolute values less than the corresponding parametres in (3.7) respectively :

$$\alpha_N' < \frac{1}{2}$$
$$\alpha_N < \frac{1}{2}$$

which shows the convergence of the sequence  $\{(\theta_n^0, \dots, \theta_n^M)\}$ . A check of validity of (4.7) and (4.8) is that, as  $N \to \infty$ ,  $\alpha_N' \to 1/2$  and  $\alpha_N \to 1/4$ , as are to be expected.

The values of the parametres  $\alpha_N$ ,  $\alpha_{N'}$ ,  $\beta_N$ ,  $\beta_{N'}$  are given in Table 1 for N = 1, ..., 25.

# Worked Example

Let us now, try to fit continuous line segments

$$y^{i}(x) = \frac{\theta^{i} - \theta^{i-1}}{\frac{\pi}{6}} (x - u_{i}) + \theta^{i-1} \qquad u_{i-1} \le x \le u_{i}$$
(4.9)

$$(N=3, \Delta x=\pi/18)$$

where

$$u_{0} = 0$$
  

$$u_{1} = \pi/6$$
  

$$u_{2} = 2\pi/6$$
  

$$u_{3} = \pi/2$$
  
(M=3)  
(12)  
(4.10)

N	α <sub>N</sub>		$\alpha_N'$		βΝ		$\beta_N'$	
1	0.000	000	0.000	000	1.000	000	1.000	000
2	0.166	677	0.200	000	0.333	<b>3</b> 33	0.400	000
3	0.210	526	0.285	714	0.157	895	0.214	286
4	0.227	273	0.333	333	0.090	909	0.133	333
5	0.235	294	0.363	636	0.058	824	0.090	909
6	0.239	726	0.384	615	0.041	096	0.066	592
7	0.242	424	0.400	000	0.030	303	0.050	000
8	0.244	186	0.411	765	0.023	256	0.039	216
9	0.245	399	0.421	053	0.018	405	0.031	579
10	0.246	269	0.428	571	0.014	925	0.025	974
11	0.246	914	0.434	783	0.012	346	0.021	739
12	0.247	405	0.440	000	0.010	381	0.018	462
13	0.247	788	0.444	444	0.008	850	0.015	873
14	0.248	092	0.448	276	0.007	634	0.013	245
15	0.248	337	0. 451	613	0.006	652	0.012	097
16	0.248	538	0.454	545	0.005	848	0.010	695
17	0.248	705	0.457	143	0.005	181	0.009	524
18	0.248	844	0.459	459	0.004	622	0.008	535
19	0.248	963	0.461	538	0.004	149	0.007	692
20	0.249	064	0.463	415	0.003	745	0.006	969
21	0.249	151	0.465	116	0.003	398	0.006	342
22	0.249	226	0.465	667	0.003	096	0.005	797
23	0.249	292	0.468	085	0.002	833	0.005	319
24	0.249	350	0.469	388	0.002	602	0.004	898
25	0.249	400	0.470	588	0.002	398	0.004	525

Table. 1.

to the set of points

$$z_{ij} = \sin(3(i-1)+j)\pi/18.$$

Corresponding to the recurrence relation (4.7) we have

$$\theta_{n+1}^{0} = -0.2857\theta_{n}^{1} + 0.2210$$

$$\theta_{n+1}^{1} = -0.2105\theta_{n+1}^{0} -0.2105\theta_{n}^{2} +0.6915 \theta_{n+1}^{2} = -0.2105\theta_{n+1}^{1} -0.2105\theta_{n}^{3} +1.2059$$

$$\theta_{n+1}^{s} = -0.2857\theta_{n+1}^{2} + 1.2664$$

Beginning with the initial approximation

$$\theta_0^0 = \theta_0^1 = \theta_0^2 = \theta_0^3 = 0$$

the sequence converged up to the 4 th decimal place after 7 iterations.

$$\theta_7^0 = 0.0819$$
,  $\theta_7^1 = 0.4868$ ,  $\theta_7^2 = 0.8904$ ,  $\theta_7^3 = 1.0120$ 

Compare the results with the corresponding results in the worked example in the preceeding section.

# V. Some remarks on the method

Gauss-Seidel successive approximation technique applied to continuous line segments curve fitting, evolved here, may be stated in terms of optimization techniques as follows. It is desired to find the minimum of a function

$$F(\theta^{0}, \theta^{1}..., \theta^{M}).$$

$$(5.1)$$

Taking a vector  $\theta_0 = (\theta_0^{0}, \theta_0^{1}, ..., \theta_0^{M})$  as the 0-th approximation, we modify  $\theta^0$  so as the modify function takes its minimum as to  $\theta^0$  with the remainder of the variables  $\theta_0^{1}, ..., \theta_0^{M}$  fixed. After fixing  $\theta^0$  to that value  $\theta_1^{0}$ , modify  $\theta^1$  so as the function takes its minimum as to  $\theta^1$  with the fixed remainder of variables  $\theta_1^{0}, \theta_0^{2}, ..., \theta_0^{M}$ . Continuing in this way up to  $\theta^{M}$ , we obtain, the first approximation  $\theta_1$ . Repeating this process several times, we would be able to attain the value of the variables  $(\theta^0, \theta^1, ..., \theta^{M})$  which would give, in most cases, the minimum to the function F.

In other words this method consists of the process of searching *one-dimensional* minimums in the problem of searching the M+1 dimensional minimum where each *one-dimensional* minimum is searched along the straight line parallel to the axis of the corresponding argument.

In a sense, it may be also possible to recognize this method as a kind of relaxation method which is often employed in solving differential equations numerically.

### References

[1]: Yanai, H.: Generalization of Seidel Process and some Sufficient Conditions for their Convergence.

#### to appear

- (2): Sassenfeld, H.: Ein hinreichendes Konvergenzkritrium und eine Fehlerabschäzung für die Iteration in Einzelschritten bei linearen Gleichungen, ZAMM Bd. 31, Nr. 3. 1951.
- [3]: Stone, H.: Approximation of Curves by Line Segments, Math. of Computation Vol. 15, No. 73, 1961.
- [4]: Bellman, S.: On the approximation of curves by line segments using dynamic programming, Communication of the A. C. M., Vol 4.
- (5): Gluss, B.: Further Remarks on Line Segments Curve-Fitting Using Dynamic Programming Communication of the A. C. M., Vol 15 1963.
- [6]: Gluss, B.: An Alternative Method for Continuous Line Segments Curve Fitting, Information and Control Vol. 7, 1964.
- [7]: Понтрягин, Л. С. и. лр. : Математическая Теория Оитимальных Процессов, Ф. М., Москва 1961
- [8]: Колмогоров. А. Н. и С. В. Фомин: Элементы Теории Функчий и Функуионального Анализа, Издательство Московского Университета, 1960