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Author	小笠原, 正忠(Ogasawara, Masatada)
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Collisional Damping of Plasma Oscillation
—Application of the Unified Theory—
I. Collision Theory

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Masatada OGASAWARA*

Abstract

Damping rate of plasma oscillation is calculated with use of the Boltzmann equation for the screened coulomb potential $\exp(-k_s r)/r$, r and k_s^{-1} being the distance between particles and the screening distance. The result is required for applying the unified theory.

I. Introduction

Effect of collisions on the plasma oscillation has been investigated by the present author [1] (hereafter refered to as the paper 1) by using Boltzmann's collision term. As is well known collision integrals, however, diverge logarithmically at large impact parameter due to the long-range nature of coulomb force. Usually cut off with the Debye length is employed to obtain finite collision integral. In this way we have a logarithmic factor which is called coulomb logarithm, with physically introduced cut off length. Thus our rate of damping of plasma oscillation contains ambiguity through the argument of coulomb logarithm.

Recently Kihara and Aono [2] developed a new theory which enables us to obtain transport coefficients or relaxation constants with exact arguments in the coulomb logarithms. The new theory is called the unified theory.

The aim of the present series of papers is to obtain the damping coefficient of plasma oscillation with the exact coulomb logarithms in the long wavelength limit by applying the unified theory.

Let us briefly survey the unified theory. The interactions between particles much closer than the Debye length are binary and can be treated with the usual Boltzmann equation. Those between particles much remoter than the close impact radius are collective and can be treated with the "wave theory", in which the system of particles is regarded as a continuous medium with a dielectric constant $\varepsilon(\boldsymbol{k}, \omega)$ of the wave number $\boldsymbol{k}$ and frequency ω. For high temperature plasmas

$$\textit{close impact radius} \ll \textit{Debye radius}\,. \tag{1・1}$$

*小笠原正忠 School of Engineering Keio University, Koganei Tokyo Japan.

Therefore the regions of validity of the "collision theory", which is based on the Boltzmann equation, and the wave theory greatly overlap each other. By taking this fact as a clue to the problem, Kihara [3] proved the following theorem.

$$X = X_\kappa + \int_0^\infty [X(k) - X_\kappa(k)]\, dk,$$
$$X_\kappa(k) = \frac{k^3}{(k^2+\kappa^2)^2} \lim_{k\to\infty} kX(k). \qquad (1 \cdot 2)$$

Here X_κ is the expression of the transport coefficient or the relaxation constant obtained by the collision theory with use of the screened coulomb potential $\exp(-\kappa r)/r$, r being the distance between the particles. And $\int_0^\infty X(k)\, dk \equiv X_{wave}$ is that obtained by the wave theory in the integral form with respect to the wave number k. An auxiliary parameter κ must be chosen in the intermediate region

$$\textit{close impact radius} \ll \kappa^{-1} \ll \textit{Debye radius}. \qquad (1 \cdot 3)$$

The final result is independent of κ. The name "unified theory" comes from the fact that the exact expression of X is obtained by unifying X_κ and X_{wave} by the theorem $(1 \cdot 2)$.

The expression X_κ for the damping rate of the plasma oscillation is obtained in the first paper of our series. The expressions X_{wave} and X are to be found in the next papers of the series. The plasma under consideration is composed of electrons and ions of one type, and in thermal equilbrium at temperature T_0. In II we write the Boltzmann equation in the relation between velocity moments of the electron velocity distribution function. In III the high-frequency conductivity is evaluated with use of the equation which is derived in II. Then the dispersion relation for the plasma oscillation is obtained. In Appendix B the algebraic error contained in the paper 1 is corrected.

II. Integrated moment equation

In this section we write the Boltzmann equation in the velocity integrated form. This equation enables us to obtain macroscopic quantities only by simple algebraic calculation.

Let $f(\boldsymbol{v})$ be a velocity distribution function of electrons, $\boldsymbol{v}$ being the velocity. The Boltzmann equation for the electrons in an external electric field $\boldsymbol{E}$ is given as

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{x}} - \frac{e\boldsymbol{E}}{m} \cdot \frac{\partial f}{\partial \boldsymbol{v}} = \left(\frac{\partial f}{\partial t}\right)_{coll} \qquad (2 \cdot 1)$$

where $\boldsymbol{x}$, $-e$, m, $\boldsymbol{E}$ and $\left(\frac{\partial f}{\partial t}\right)_{coll}$ mean position, charge, mass of the electron, the external electric field and the effect of collisions, respectively. The ions are smeared out as a positive charge background for retaining charge neutrality.

Regarding $\boldsymbol{E}$ as a perturbation, we write

$$f(\boldsymbol{v}, \boldsymbol{x}, t)=f_0(v)+f^{(1)}(\boldsymbol{v}, \boldsymbol{x}, t),$$
$$f^{(1)}(\boldsymbol{v}, \boldsymbol{x}, t)=f_0(v)\phi(\boldsymbol{v}, \boldsymbol{x}, t), \quad |\phi|\ll 1. \tag{2・2}$$

f_0 being Maxwellian velocity distribution ;

$$f_0(v)=n_0\left(\frac{m}{2\pi\kappa T_0}\right)^{\frac{3}{2}}\exp\left(-\frac{mv^2}{2\kappa T_0}\right), \tag{2・3}$$

where n_0, T_0, and κ express number density, temperature and the Boltzmann constant, respectively. Linearizing the Boltzmann equation (2・1), we have

$$\frac{\partial\phi}{\partial t}+\boldsymbol{v}\cdot\frac{\partial\phi}{\partial\boldsymbol{x}}-\frac{e}{m}\boldsymbol{E}\cdot\frac{1}{f_0}\frac{\partial f_0}{\partial\boldsymbol{v}}=-n_0J\phi, \tag{2・4}$$

where J is the Boltzmann collision operator which is defined as

$$J\phi(\boldsymbol{v}_1)=\frac{1}{n_0}\int\int\int f_0(v_2)[\phi(\boldsymbol{v}_1)+\phi(\boldsymbol{v}_2)-\phi(\boldsymbol{v}_1')$$
$$-\phi(\boldsymbol{v}_2')]\,gI(g,\chi)\sin\chi\,d\chi\,d\varepsilon\,d\boldsymbol{v}_2. \tag{2・5}$$

Here prime means the velocity after the collision and $I(g, \chi)\sin\chi\,d\chi\,d\varepsilon$ indicates the differential cross section for scattering into the solid angle $\sin\chi\,d\chi\,d\varepsilon$ with the speed $g=|\boldsymbol{v}_1-\boldsymbol{v}_2|=|\boldsymbol{v}_1'-\boldsymbol{v}_2'|$.

Now we introduce a complete set of orthogonal functions defined as [1]

$$\Psi_{rl}(\boldsymbol{v})=\frac{\varphi_{rl}(\boldsymbol{v})}{N_{rl}}=\frac{1}{N_{rl}}\xi^lP_l\left(\frac{\xi_z}{\xi}\right)S^{(r)}_{l+\frac{1}{2}}(\xi^2), \tag{2・6}$$

$$\xi=\frac{v}{v_0}, \quad v_0{}^2=2\kappa T_0/m,$$

$$N^2_{rl}=\frac{2}{2l+1}\frac{1}{\sqrt{\pi}\,r!}\Gamma\left(l+r+\frac{3}{2}\right), \tag{2・7}$$

where P_l and $S^{(r)}_n$ are the Legendre and Sonine polynomials, respectively. ξ and N_{rl} mean the dimensionless velocity and the normalization constant.

When we define an inner product of two arbitrary functions of $\boldsymbol{v}$, $\phi(\boldsymbol{v})$ and $\psi(\boldsymbol{v})$, by

$$(\phi, \psi)=\frac{1}{n_0}\int f_0\,\phi\psi\,d\boldsymbol{v}, \tag{2・8}$$

the relation of orthogonality of the set $\{\Psi_{rl}\}$ is given

$$(\Psi_{rl}, \Psi_{r'l'})=\delta_{rr'}\delta_{ll'}. \tag{2・9}$$

We expand $\phi(\boldsymbol{v}, \boldsymbol{x}, t)$ in terms of Ψ_{rl} with time- and space-varying coefficients as follows

$$\phi(\boldsymbol{v},\ \boldsymbol{x},\ t)=\sum_{rl} a_{rl}(\boldsymbol{x},\ t)\Psi_{rl}(\boldsymbol{v})\ , \tag{2·10}$$

$$a_{re}(\boldsymbol{x},\ t)=(\Psi_{rl},\ \phi)=\frac{1}{n_0}\int f^{(1)}\Psi_{rl}\, d\boldsymbol{v}\ . \tag{2·11}$$

The coefficients a_{rl}'s that mean the moments of $f^{(1)}$ with respect to Ψ_{rl}'s are used in place of $f^{(1)}$ throughout this paper. These coefficients are connected with the deviations of several macroscopic quantities such as number density, macroscopic velocity, temperature etc. from their equilibrium values.

Substituting the expansion (2·10) into (2 · 5), we have

$$-n_0 J\phi=-n_0 \sum_{rl} a_{rl} J\Psi_{rl}=-\sum \tilde{\lambda}_{rs}^{(l)}\, a_{rl}\, \Psi_{sl}\ . \tag{2·12}$$

where

$$\tilde{\lambda}_{rs}^{(l)} = n_0(\Psi_{rl},\ J\Psi_{sl})=\tilde{\lambda}_{sr}^{(l)}\ . \tag{2·13}$$

Here $\tilde{\lambda}_{rs}^{(l)}$'s correspond to the collision frequencies and can be shown to be positive definite, which shows the relaxation feature. We express the electron-electron and electron-ion collision frequencies by λ and Λ, respectively, then we can write

$$\tilde{\lambda}_{rs}^{(l)} = \lambda_{rs}^{(l)} + \Lambda_{rs}^{(l)}\ . \tag{2·14}$$

From the conservation laws it follows that

$$\begin{aligned} &\lambda_{00}^{(0)}=\Lambda_{00}^{(0)}=0\ ,\\ &\lambda_{01}^{(0)}=\lambda_{10}^{(0)}=\Lambda_{01}^{(0)}=\Lambda_{10}^{(0)}=0\ ,\\ &\lambda_{00}^{(1)}=0\ . \end{aligned} \tag{2·15}$$

First, second and third lines express conservations of number density, energy and momentum, respectively. The values of $\lambda_{00}^{(2)}$ and $\Lambda_{rs}^{(l)}$ are given in Appendix A.

Substitution of (2·10) into the Boltzmann equation (2 · 4) yields

$$\sum_{ij} \dot{a}_{ij}\, \Psi_{ij}+\sum \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{x}}\, a_{ij}\, \Psi_{ij}-\frac{e\boldsymbol{E}}{m}\cdot\frac{1}{f_0}\frac{\partial f_0}{\partial \boldsymbol{v}}$$

$$=-\sum_{ijs} a_{ij}\, \tilde{\lambda}_{is}^{(j)}\, \Psi_{sj}\ , \tag{2·16}$$

where dot implies time derivative. Operating $\frac{1}{n_0}\int d\boldsymbol{v}\, f_0\, \Psi_{rl}$ on both sides of this equation, we obtain, with use of the orthogonality of Ψ_{rl},

$$\dot{a}_{rl}+\sum_{ij}(\Psi_{rl},\ \boldsymbol{v}\Psi_{ij})\cdot\frac{\partial}{\partial \boldsymbol{x}}\, a_{ij}+\frac{e\boldsymbol{E}}{\kappa T_0}\cdot(\Psi_{rl},\ \boldsymbol{v})$$

$$=-\sum_{i}\tilde{\lambda}_{ri}^{(l)}\, a_{il}\ . \tag{2·17}$$

Assuming the directions of $\boldsymbol{E}$ and the space variation be in the direction of z-axis, we can write the equation (2·17) as follows

$$\dot{a}_{rl}+v_0\sum_{ij}(\Psi_{rl},\ \xi_z\Psi_{ij})\frac{\partial}{\partial z}a_{ij}+v_0\frac{eE}{\kappa T_0}(\Psi_{rl},\ \xi_z)$$

$$=-\sum_i\tilde{\lambda}_{ri}^{(l)}a_{il}\,. \tag{2·18}$$

Making use of a recurrence formula which is given by [1]

$$\xi_z\Psi_{rl}=A_{rl}\Psi_{r,\,l+1}-B_{rl}\Psi_{r-1,\,l+1}+C_{rl}\Psi_{r,\,l-1}-D_{rl}\Psi_{r+1,\,l-1}\,, \tag{2·19}$$

$$A_{lr}=(l+1)\sqrt{\frac{l+r+\frac{3}{2}}{(2l+1)(2l+3)}}\,,$$

$$B_{rl}=(l+1)\sqrt{\frac{r}{(2l+1)(2l+3)}}\,, \tag{2·20}$$

$$A_{rl}=C_{r,\,l+1}\,,\ \ B_{r+1,\,l}=D_{r,\,l+1}\,,$$

we obtain

$$\dot{a}_{rl}+v_0\frac{\partial}{\partial z}\Big[A_{rl}a_{r,\,l+1}-B_{rl}a_{r-1,\,l+1}+C_{rl}a_{r,\,l-1}-D_{rl}a_{r+1,\,l-1}$$

$$+v_0\frac{eE}{\kappa T}N_{01}\,\delta_{r0}\,\delta_{l1}=-\sum_i\tilde{\lambda}_{ri}^{(l)}a_{il}\,. \tag{2·21}$$

This is the integrated moment equation which manages the behavior of the moment a_{rl}.

In the next section we will use this equation to obtain the dispersion relation for the plasma oscillation.

III. Dispersion relation for plasma oscillation

The first aim of this section is to calculate high-frequency conductivity, from which we can obtain the dispersion relation for the plasma oscillation. This is the principal object of this section.

We confine ourselves to the case of long wavelength and to the case of collision frequency $\tilde{\lambda}$ much less than the frequency of plasma oscillation.

Let the space- and time-varying high-frequency electric field

$$E=E_z\exp\left[i(kz-\omega t)\right] \tag{3 · 1}$$

be applied to the plasma, then electrons are deviated from their equilblium state. Then there appears $\phi(\boldsymbol{v},\ z,\ t)$ or $a_{rl}(z,\ t)$. These are described by the equation (2·21). By putting

$$a_{re}\propto\exp\left[i(kz-\omega t)\right], \tag{3 · 2}$$

we obtain from (2·21)

$$-i\omega a_{rl}+ikv_0[A_{rl}a_{r,\,l+1}-B_{rl}a_{r-1,\,l+1}+C_{rl}a_{r,\,l-1}-D_{rl}a_{r+1,\,l-1}$$

$$+v_0\frac{eE}{\kappa T_0}N_{01}\,\delta_{r0}\,\delta_{l1}=-\sum\tilde{\lambda}_{ri}^{(l)}a_{il}\,.$$

Dividing both sides of this expression by the frequency of the plasma oscillation ω_e, which is defined by

$$\omega_e{}^2 = \frac{4\pi n_0 e^2}{m}, \tag{3・3}$$

we obtain the integrated moment equation in the dimensionless form

$$\Omega a_{rl} + \sum_i \nu_{ri}^{(l)} a_{il} - \tilde{\kappa}[A'_{rl} a_{r,\, l+1} - B'_{rl} a_{r-1,\, l+1} + C'_{rl} a_{r,\, l-1} - D'_{rl} a_{r+1,\, l-1}] = \alpha E \delta_{r0} \delta_{l1}, \tag{3・4}$$

where

$$\Omega = \frac{\omega}{\omega_e}, \quad \tilde{\kappa} = \frac{\tilde{k}}{k_D}, \quad \nu_{ri}^{(l)} = \frac{i\lambda_{ri}^{(l)}}{\omega_e}, \quad \alpha = -i\frac{e}{m}\frac{1}{\omega_e}\left(\frac{m}{\kappa T_0}\right)^{\frac{1}{2}};$$
$$A'_{rl} = \sqrt{2} A_{rl}, \quad B'_{rl} = \sqrt{2} B_{rl}, \quad C'_{rl} = \sqrt{2} C_{rl}, \quad D'_{rl} = \sqrt{2} D_{rl}, \tag{3・5}$$

and $k_D^2 = \dfrac{4\pi n_0 e^2}{\kappa T_0}$ is the Debye constant. (3・6)

From the assumptions mentioned at the beginning of this section, it follows that

$$|\tilde{\nu}_{ri}^{(l)}| \ll 1, \quad \tilde{\kappa} \ll 1. \tag{3・7}$$

Then we seek the solution of (3・4) in the form

$$a_{rl} = \sum_{\substack{0 \le m \le 2 \\ 0 \le n \le 1}} \tilde{\kappa}^m \nu^n a_{rl}^{(m,\, n)}. \tag{3・8}$$

By substituting (3・8) into (3・4), we have

$$\Omega a_{rl}^{(m,\, n)} + \sum_i \nu_{ri}^{(l)} a_{il}^{(m,\, n-1)} - [A'_{rl} a_{r,\, l+1}^{(m-1,\, n)} - B'_{rl} a_{r-1,\, l+1}^{(m-1,\, n)} + C'_{rl} a_{r,\, l-1}^{(m-1,\, n)} - D'_{rl} a_{r+1,\, l-1}^{(m-1,\, n)}] = \alpha E \delta_{r0} \delta_{l1}. \tag{3・9}$$

The electron current j is obtained as

$$\begin{aligned} j &= -e\int v_z f(\boldsymbol{v}) d\boldsymbol{v} = -e\int v_z f_0(v)\, \phi(\boldsymbol{v},\, z,\, t) d\boldsymbol{v} \\ &= -e n_0 v_0 N_{01} \sum_{rl} a_{rl} (\Psi_{01},\, \Psi_{rl}) \\ &= -n_0 e \left(\frac{\kappa T_0}{m}\right)^{\frac{1}{2}} a_{01}. \end{aligned} \tag{3・10}$$

Hence for the purpose of obtaining high-frequency conductivity the solution of (3・9) for a_{01} is required.

From (3・9) we obtain $a_{01}^{(m,\, n)}$ as follows.

[1] $(m, n)=(0, 0)$

The nonvanishing moment of this order is only $a_{01}^{(0,0)}$,

$$a_{01}^{(0,0)}=\frac{\alpha}{\Omega}E. \tag{3·11}$$

[2] $(m, n)=(1, 0)$

The nonvanishing moments are

$$a_{00}^{(1,0)}=\frac{A'_{00}}{\Omega}a_{01}^{(0,0)}, \quad a_{10}^{(1,0)}=-\frac{B'_{10}}{\Omega}a_{01}^{(0,0)}, \quad a_{02}^{(1,0)}=\frac{C'_{02}}{\Omega}a_{01}^{(0,0)}. \tag{3·12}$$

Hence

$$a_{01}^{(1,0)}=0. \tag{3·13}$$

[3] $(m, n)=(2, 0)$

$$\Omega a_{01}^{(2,0)}=A'_{01}a_{02}^{(1,0)}+C'_{01}a_{00}^{(1,0)}-D'_{01}a_{10}^{(1,0)}. \tag{3·14}$$

Substitution of (3·12) into this yields

$$\Omega^2 a_{01}^{(2,0)}=[A'^2_{01}+A'^2_{00}+B'^2_{10}]\,a_{01}^{(0,0)}=3a_{01}^{(0,0)},$$

where use has been made of (2·20). Then we have

$$a_{01}^{(2,0)}=\frac{3}{\Omega^2}a_{01}^{(0,0)}. \tag{3·15}$$

By performing similar calculations, we have the following results

$$\begin{aligned} &a_{01}^{(0,0)}=\frac{\alpha}{\Omega}E, \quad a_{01}^{(1,0)}=0, \quad a_{01}^{(2,0)}=\frac{3}{\Omega^2}a_{01}^{(0,0)}, \\ &a_{01}^{(0,1)}=-\frac{\nu_{00}^{(1)}}{\Omega}a_{01}^{(0,0)}, \qquad a_{01}^{(1,1)}=0, \\ &a_{01}^{(2,1)}=-\frac{1}{\Omega^3}\left[6\nu_{00}^{(1)}+\frac{4}{3}\nu_{00}^{(2)}-\frac{6}{5}\sqrt{10}\,\nu_{01}^{(1)}\right]a_{01}^{(0,0)}. \end{aligned} \tag{3·16}$$

By taking account of (3·5) and the law of conservation (2·15), we have

$$\begin{aligned} \frac{a_{01}}{a_{01}^{(0,0)}}=1+3\frac{\tilde{\kappa}^2}{\Omega^2}-\frac{1}{\Omega}\frac{i\Lambda_{00}^{(1)}}{\omega_e} \\ -\frac{\tilde{\kappa}^2}{\Omega^2}\left[\frac{6}{\Omega}\frac{i\Lambda_{00}^{(1)}}{\omega_e}+\frac{4}{3}\frac{1}{\Omega}\left(\frac{i\lambda_{00}^{(2)}}{\omega_e}+\frac{i\Lambda_{00}^{(2)}}{\omega_e}\right)\right. \\ \left.-\frac{6\sqrt{10}}{5}\frac{1}{\Omega}\frac{i\Lambda_{00}^{(1)}}{\omega_e}\right]\equiv\beta, \end{aligned} \tag{3·17}$$

hence

$$a_{01}=\beta\,a_{01}^{(0,0)}=\frac{\alpha\beta}{\Omega}E. \tag{3·18}$$

From (3·10) and (3·18) the conductivity $\sigma=\frac{j}{E}$ is given as

$$\sigma=-n_0 e\left(\frac{\kappa T_0}{m}\right)^{\frac{1}{2}}\frac{\alpha\beta}{\Omega}. \tag{3·19}$$

Now that σ is obtained, we can obtain the dispersion relation for the plasma oscillation by the following relations

$$\varepsilon=1-\frac{4\pi\sigma}{i\omega}, \quad \varepsilon=0. \tag{3·20}$$

Here ε is the dielectric constant of the electron plasma and $\varepsilon=0$ gives the dispersion relation for the longitudinal oscillation. Then we obtain, with use of the definition of σ and α

$$1=\left(\frac{\omega_e}{\omega}\right)^2\beta.$$

Hence the dispersion relation is obtained

$$\left(\frac{\omega}{\omega_e}\right)^2=\beta=1+3\frac{\tilde{\kappa}^2}{\Omega^2}-\frac{1}{\Omega}\frac{i\Lambda_{00}^{(1)}}{\omega_e}$$
$$-\frac{\tilde{\kappa}^2}{\Omega^2}\left[\frac{6}{\Omega}\frac{i\Lambda_{00}^{(1)}}{\omega_e}+\frac{4}{3}\frac{1}{\Omega}\left(\frac{i\lambda_{00}^{(2)}}{\omega_e}+\frac{i\Lambda_{00}^{(2)}}{\omega_e}\right)-\frac{6\sqrt{10}}{5}\frac{1}{\Omega}\frac{i\Lambda_{01}^{(1)}}{\omega_e}\right]. \tag{3·21}$$

This expression agrees with the corrected result of the paper 1. (See Appedix B.)

The dispersion relation (3·21) is solved by successive approximation when we take account of (3·7). In the first approximation we put $\Omega=1$ and obtain

$$\left(\frac{\omega}{\omega_e}\right)^2=1+3\tilde{\kappa}^2-\frac{i\Lambda_{00}^{(1)}}{\omega_e}. \tag{3·22}$$

The second approximation to the frequency is found by substituting the first approximation into the right-hand side of (3·21). Then we obtain

$$\left(\frac{\omega}{\omega_e}\right)^2=1+3\tilde{\kappa}^2-\frac{i\Lambda_{00}^{(1)}}{\omega_e}$$
$$-\tilde{\kappa}^2\left[\frac{3}{2}\frac{i\Lambda_{00}^{(1)}}{\omega_e}+\frac{4}{3}\left(\frac{i\lambda_{00}^{(2)}}{\omega_e}+\frac{i\Lambda_{00}^{(2)}}{\omega_e}\right)-\frac{6\sqrt{10}}{5}\frac{i\Lambda_{10}^{(1)}}{\omega_e}\right]. \tag{3·23}$$

Now writing ω in real and imaginary parts as

$$\omega=\omega_r-i\omega_i, \qquad \left|\frac{\omega_i}{\omega_r}\right|\ll 1, \tag{3·24}$$

we have from (3·23)

$$\left(\frac{\omega_r}{\omega_e}\right)^2=1+3\tilde{\kappa}^2, \tag{3·25}$$

$$\omega_i=\frac{\omega_e}{\omega_r}\left\{\frac{1}{2}\Lambda_{00}^{(1)}+\tilde{\kappa}^2\left[\frac{3}{4}\Lambda_{00}^{(1)}+\frac{2}{3}(\lambda_{00}^{(2)}+\Lambda_{00}^{(2)})-\frac{3\sqrt{10}}{5}\Lambda_{01}^{(1)}\right]\right\}$$
$$=\frac{1}{2}\Lambda_{00}^{(1)}+\tilde{\kappa}^2\left[\frac{2}{3}(\lambda_{00}^{(2)}+\Lambda_{00}^{(2)})-\frac{3\sqrt{10}}{5}\Lambda_{01}^{(1)}\right]. \tag{3·26}$$

The imaginary part gives the rate of damping of the plasma oscillation. From Appendix A, collision integrals are given as

$$\begin{aligned}
\Lambda_{00}^{(1)} &= \frac{4\sqrt{2}}{3} n_0 K' \left[ln\,\Lambda' - \frac{1}{2} \right], \\
\Lambda_{01}^{(1)} &= \frac{4\sqrt{5}}{5} n_0 K' \left[ln\,\Lambda' - \frac{7}{6} \right], \\
\Lambda_{00}^{(2)} &= \frac{8\sqrt{2}}{5} n_0 K'\, ln\,\Lambda', \\
\lambda_{00}^{(2)} &= \frac{8}{5} n_0 K\, ln\,\Lambda,
\end{aligned} \tag{3·27}$$

where

$$\begin{aligned}
K' &= \left(\frac{\pi\kappa T_0}{m}\right)^{\frac{1}{2}} \left(\frac{Ze^2}{\kappa T_0}\right)^2, \quad K = \left(\frac{\pi\kappa T_0}{m}\right)^{\frac{1}{2}} \left(\frac{e^2}{\kappa T_0}\right)^2, \\
\Lambda' &= \frac{4\kappa T_0}{\gamma^2 Ze^2 k_s}, \quad \Lambda = \frac{4\kappa T_0}{\gamma^2 e^2 k_s},
\end{aligned} \tag{3·28}$$

here $ln\,\gamma = 0.57722$ is Euler's constant.
When $Z=1$, the damping rate is given as

$$\begin{aligned}
\omega_i = & \frac{2\sqrt{2}}{3} n_0 K \left(ln\,\Lambda - \frac{1}{2} \right) \\
& + \left(\frac{k}{k_D}\right)^2 \left[\frac{16}{15} n_0 K\, ln\,\Lambda - \frac{4\sqrt{2}}{3} n_0 K \left(ln\,\Lambda - \frac{21}{10} \right) \right].
\end{aligned} \tag{3·29}$$

The terms including $\sqrt{2}$ arise from electron-ion collision, the remaining term from electron-electron collision.

Appendix B Evaluation of $(\varphi_{pl}, J_{e-i}\, \varphi_{ql})$ and $(\varphi_{02}, J_{e-e}\, \varphi_{02})$

[1] $(\varphi_{pl}, J_{e-i}\, \varphi_{ql})$

Assuming the mass ratio of the electron mass m to the ion mass M be zero, namely

$$\frac{m}{M} = 0, \tag{A·1}$$

we have from (2·5) the collision operator J_{e-i} for electron-ion collision

$$J_{e-i}\phi(\boldsymbol{v}) = \int\int [\phi(\boldsymbol{v}) - \phi(\boldsymbol{v}')]\, v I(v, \chi) \sin\chi\, d\chi\, d\varepsilon\ . \tag{A·2}$$

The generating function of the Sonine polynomial is given by

$$(1-s)^{-(l+\frac{3}{2})} \exp(-S\,\xi^2) = \sum_{q=0}^{\infty} s^q S_{l+\frac{1}{2}}^{(q)} (\xi^2), \tag{A·3}$$

$$S = \frac{s}{1-s}.$$

Then it follows that

$$\sum s^q J_{e-i}\,\varphi_{ql}=(1-s)^{-(l+3/2)}$$
$$\times\int\int\left[\xi^l P_l\left(\frac{\xi_z}{\xi}\right)e^{-s\xi^2}-\xi'^l P_l\left(\frac{\xi_z'}{\xi'}\right)e^{-s\xi'^2}\right]vI(v,\,\chi)\,\sin\chi\,d\chi\,d\varepsilon\,. \qquad (\mathrm{A}\cdot 4)$$

On account of the assumption of infinite mass ratio (A·1), the energy of the electron is conserved in collisions with the ions. Then it follows

$$\xi=\xi'\,.$$

Hence (A·4) reduces to

$$\sum s^q J_{e-i}\,\varphi_{ql}=(1-s)^{-(l+3/2)}\,v_0\xi^{l+1}\,e^{-s\xi^2}\,A_l\,, \qquad (\mathrm{A}\cdot 5)$$

$$A_l=\int\int[P_l(\cos\theta)-P_l(\cos\theta')]\,I(v,\chi)\,\sin\chi\,d\chi\,d\varepsilon\,, \qquad (\mathrm{A}\cdot 6)$$

where

$$\cos\theta=\frac{\xi_z}{\xi}\,,\qquad \cos\theta'=\frac{\xi_z'}{\xi}\,.$$

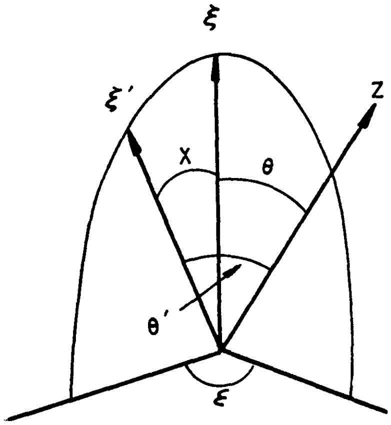

Fig. 1.

From Fig. 1 it follows

$$\cos\theta'=\cos\theta\,\cos\chi+\sin\theta\,\sin\chi\,\cos\varepsilon\,.$$

Then readily we have

$$\int_0^{2\pi}P_l(\cos\theta')d\varepsilon=2\pi P_l(\cos\theta)\,P_l(\cos\chi)\,.$$

Inserting this into (A·6), we obtain

$$A_l=2\pi P_l(\cos\theta)\int[1-P_l(\cos\chi)]\,I(v,\,\chi)\,\sin\chi\,d\chi$$
$$\equiv 2\pi P_l(\cos\theta)\,B_l(\xi)\,. \qquad (\mathrm{A}\cdot 7)$$

With use of (A·3), (A·5) and (A·7) we have

$$\sum_{\substack{p=0\\p=0}}^{\infty}s^q\,t^p(\varphi_{pl},\,J_{e-i}\,\varphi_{ql})$$
$$=2\pi v_0[(1-s)\,(1-t)]^{-(l+\frac{3}{2})}\,\frac{2\pi}{\pi^{\frac{3}{2}}}\int[P_l(\cos\theta)]^2\,d(\cos\theta)$$
$$\times\int_0^{\infty}e^{-(1+S+T)\xi^2}\,\xi^{2l+3}B_l(\xi)\,d\xi \qquad (\mathrm{A}\cdot 8)$$
$$=4\sqrt{\pi}\left(\frac{2\kappa T_0}{m}\right)^{\frac{1}{2}}[(1-s)(1-t)]^{-(l+\frac{3}{2})}\,\frac{2}{2l+1}\int_0^{\infty}e^{-(1+S+T)\xi^2}\,\xi^{2l+3}B_l(\xi)d\xi\,.$$

If we adot a screened coulomb potential $\frac{1}{r}\exp(-k_s r)$, where r is the distance between two interacting particles and k_s^{-1} is the screening distance, we obtain [4]

$$\int_0^\pi (1-\cos\chi) I(g,\chi)\sin\chi\, d\chi = 2\left(\frac{Ze^2}{\mu g^2}\right)^2 \left[ln\frac{2\mu g^2}{\gamma Ze^2 k_s{}^2}-\frac{1}{2}\right], \qquad (A\cdot 9)$$

$$\int_0^\pi (1-\cos^2\chi) I(g,\chi)\sin\chi\, d\chi = 4\left(\frac{Ze^2}{\mu g^2}\right)^2 \left[ln\frac{2\mu g^2}{\gamma Ze^2 k_s}-1\right], \qquad (A\cdot 10)$$

where $\mu=\dfrac{Mm}{M+m}$ is the reduced mass, Ze is the charge of an ion and $ln\,\gamma=0.57722$ is Euler's constant. By use of (A·9) and (A·10) we obtain

$$B_1(\xi)=2\left(\frac{Ze^2}{mv^2}\right)^2\left[ln\frac{2mv^2}{\gamma Ze^2 k_s}-\frac{1}{2}\right]$$

$$=\frac{1}{2}\left(\frac{Ze^2}{\kappa T_0}\right)^2\frac{1}{\xi^4}\left[ln\frac{4\kappa T_0}{\gamma Ze^2 k_s}-\frac{1}{2}+ln\,\xi^2\right], \qquad (A\cdot 11)$$

$$B_2(\xi)=\frac{3}{2}\left(\frac{Ze^2}{\kappa T_0}\right)^2\frac{1}{\xi^4}\left[ln\frac{4\kappa T_0}{\gamma Ze^2 k_s}-1+ln\,\xi^2\right]. \qquad (A\cdot 12)$$

Thus B_l is written as follows

$$B_l=\frac{1}{2}\left(\frac{Ze^2}{\kappa T_0}\right)^2\frac{C_l}{\xi^4}\,(D_l+ln\,\xi^2)\,, \qquad (A\cdot 13)$$

where

$$C_1=1,\quad C_2=3,\quad D_1=ln\frac{4\kappa T_0}{\gamma Ze^2 k_s}-\frac{1}{2},\quad D_2=ln\frac{4\kappa T_0}{\gamma Ze^2 k_s}-1. \qquad (A\cdot 14)$$

Substituting (A·13) into (A·8), we obtain

$$\sum s^q t^p(\varphi_{pl},\ J_{e-i}\varphi_{ql})$$

$$=2\sqrt{2\pi}\left(\frac{\kappa T_0}{m}\right)^{\frac{1}{2}}\left(\frac{Ze^2}{\kappa T_0}\right)^2[(1-s)(1-t)]^{-(l+\frac{3}{2})}\frac{2}{2l+1}C_l$$

$$\times\int_0^\infty e^{-(1+S+T)\xi^2}\xi^{2l-1}[D_l+ln\,\xi^2]\,d\xi\,. \qquad (A\cdot 15)$$

The integral in this expression is evaluated by changing variable as follows

$$(1+S+T)\xi^2\equiv x\,.$$

Then

$$\int_0^\infty e^{-(1+S+T)\xi^2}\xi^{2l-1}[D_l+ln\,\xi^2]\,d\xi$$

$$=\frac{1}{2(1+S+T)^l}\int_0^\infty e^{-x}x^{l-1}[D_l-ln(1+S+T)+ln\,x]\,dx$$

$$=\frac{\Gamma(l)}{2(1+S+T)^l}[D_l-ln(1+S+T)+\psi(l)], \qquad (A\cdot 16)$$

where use has been made of the formula [5]

$$\int_0^\infty e^{-x}\, x^{l-1}\, ln\, x\, dx = \Gamma'(l) = \psi(l)\, \Gamma(l)\,. \tag{A・17}$$

Here $\psi(l)$ is digamma function. Values of $\psi(l)$ are given by

$$\psi(1) = -ln\,\gamma\,, \qquad \psi(2) = 1 - ln\,\gamma\,. \tag{A・18}$$

From (A・14), (A・15), (A・16) and (A・18) we obtain

$$\sum s^q t^p (\varphi_{p1}, J_{e-i}\varphi_{q1}) = \frac{2\sqrt{2\pi}}{3}\left(\frac{\kappa T_0}{m}\right)^{\frac{1}{2}}\left(\frac{Ze^2}{\kappa T_0}\right)^2$$

$$\times \frac{[(1-s)(1-t)]^{-\frac{5}{2}}}{1+S+T}\left[ln\frac{4\kappa T_0}{\gamma^2 Ze^2 k_s} - \frac{1}{2} - ln(1+S+T)\right]. \tag{A・19}$$

$$\sum s^q t^p (\varphi_{p2}, J_{e-i}\varphi_{q2}) = \frac{6\sqrt{2\pi}}{5}\left(\frac{\kappa T_0}{m}\right)^{\frac{1}{2}}\left(\frac{Ze^2}{\kappa T_0}\right)^2$$

$$\times \frac{[(1-s)(1-t)]^{-\frac{7}{2}}}{(1+S+T)^2}\left[ln\frac{4\kappa T_0}{\gamma^2 Ze^2 k_s} - ln(1+S+T)\right]. \tag{A・20}$$

Taking account of $S = \frac{s}{1-s}$ and $T = \frac{t}{1-t}$, we can rewrite (A・19) and (A・20), namely,

$$\sum s^q t^p (\varphi_{p1}, J_{e-i}\varphi_{q1}) = \frac{2\sqrt{2\pi}}{3}\left(\frac{\kappa T_0}{m}\right)^{\frac{1}{2}}\left(\frac{Ze^2}{\kappa T_0}\right)^2$$

$$\times \frac{[(1-s)(1-t)]^{-\frac{3}{2}}}{1-st}\left[ln\frac{4\kappa T_0}{\gamma^2 Ze^2 k_s} - \frac{1}{2} - ln(1+S+T)\right], \tag{A・21}$$

$$\sum s^q t^p (\varphi_{p2}, J_{e-i}\varphi_{q2}) = \frac{6\sqrt{2\pi}}{5}\left(\frac{\kappa T_0}{m}\right)^{\frac{1}{2}}\left(\frac{Ze^2}{\kappa T_0}\right)^2$$

$$\times \frac{[(1-s)(1-t)]^{-\frac{3}{2}}}{(1-st)^2}\left[ln\frac{4\kappa T_0}{\gamma^2 Ze^2 k_s} - ln(1+S+T)\right]. \tag{A・22}$$

These expressions agree with what is given in the paper 1 (C・12) except for the factor which includes the coulomb logarithm.

[2] $(\varphi_{02}, J_{e-e}\,\varphi_{02})$

From the definitions of the inner product and the collison operator J_{e-e} for electron-electron collision, we have

$$(\varphi_{02}, J_{e-e}\varphi_{02}) = \frac{1}{n_0^2}\int\!\!\int\!\!\int\!\!\int f_0(v_1) f_0(v_2)\varphi_{02}(\boldsymbol{v}_1)[\varphi_{02}(\boldsymbol{v}_1) + \varphi_{02}(\boldsymbol{v}_2)$$

$$- \varphi_{02}(\boldsymbol{v}_1) - \varphi_{02}(\boldsymbol{v}_2')]\, gI(g,\chi)\, \sin\chi\, d\chi\, d\varepsilon\, d\boldsymbol{v}_1 d\boldsymbol{v}_2\,. \tag{A・23}$$

Substituting the expression

$$\varphi_{02} = \xi^2 P_2\left(\frac{\xi_z}{\xi}\right) S^{(0)}_{2+\frac{1}{2}}(\xi^2) = \xi^2 P_2\left(\frac{\xi_z}{\xi}\right), \tag{A・24}$$

into (A·23), we have

$$(\varphi_{02}, J_{e-e}\varphi_{02}) = \frac{1}{n_0^2}\int\int\int\int f_0(v_1) f_0(v_2)\ \xi_1^2 P_2\left(\frac{\xi_{1z}}{\xi}\right)\frac{3}{2}$$

$$\times[\xi_{1z}^2+\xi_{2z}^2-\xi_1{}'^2_z-\xi_2{}'^2_z]\ gI(g,\chi)\ \sin\chi\, d\chi\, d\varepsilon\, d\boldsymbol{v}_1\, d\boldsymbol{v}_2, \qquad (A\cdot 25)$$

where we have used the law of energy consevation;

$$\xi_1^2+\xi_2^2 = \xi'^2_1+\xi'^2_2. \qquad (A\cdot 26)$$

Let us introduce the center of mass velocity $\boldsymbol{G}$ and the relative velocity $\boldsymbol{g}$ by

$$\boldsymbol{v}_1 = \boldsymbol{G}+\frac{\boldsymbol{g}}{2}, \quad \boldsymbol{v}_2 = \boldsymbol{G}-\frac{\boldsymbol{g}}{2}, \qquad (A\cdot 27)$$

with

$$\frac{\partial(\boldsymbol{v}_1, \boldsymbol{v}_2)}{\partial(\boldsymbol{G}, \boldsymbol{g})} = 1.$$

The expression in the bracket of (A·25) is then written as

$$\frac{1}{2}\left(\frac{g}{v_0}\right)^2(\cos^2\theta-\cos^2\theta') = \frac{1}{2}\frac{2}{3}\left(\frac{g}{v_0}\right)^2\left[P_2(\cos\theta)-P_2(\cos\theta')\right] \qquad (A\cdot 28)$$

with

$$\cos\theta = \frac{g_z}{g}, \qquad \cos\theta' = \frac{g_z'}{g'} = \frac{g_z'}{g},$$

$$\cos\theta' = \cos\theta\ \cos\chi + \sin\theta\ \sin\chi\ \cos\varepsilon. \qquad (A\cdot 29)$$

Taking into account of (A·28), (A·29) and (A·10), we obtain

$$\int_0^\pi d\chi\ \sin\chi I(g,\chi)\int_0^{2\pi} d\varepsilon[\varphi_{02}(\boldsymbol{v}_1)+\varphi_{02}(\boldsymbol{v}_2)-\varphi_{02}(\boldsymbol{v}_1')-\varphi_{02}(\boldsymbol{v}_2')]$$

$$= \pi P_2(\cos\theta)\left(\frac{g}{v_0}\right)^2\int_0^\pi d\chi\ \sin\chi I(g,\chi)[1-P_2(\cos\chi)]$$

$$= 6\pi P_2(\cos\theta)\left(\frac{e^2}{\kappa T_0}\right)^2\left(\frac{v_0}{g}\right)^2\left[ln\frac{2\kappa T_0}{\gamma e^2 k_s}-1+ln\left(\frac{g}{v_0}\right)^2\right]. \qquad (A\cdot 30)$$

The expression (A·24) is rewritten by using (A·27) as

$$\varphi_{02}(\boldsymbol{v}) = \frac{1}{v_0}\left[\frac{3}{2}\left(G_z+\frac{g_z}{2}\right)^2-\frac{1}{2}\left(\boldsymbol{G}+\frac{\boldsymbol{g}}{2}\right)^2\right].$$

The term which is proportional to $P_2(\cos\theta)$ in $\varphi_{02}(\boldsymbol{v}_1)$ is given as

$$\varphi_{02}(\boldsymbol{v}_1) \sim \frac{1}{4}\left(\frac{g}{v_0}\right)^2 P_2(\cos\theta). \qquad (A\cdot 31)$$

Only this term of all terms of $\varphi_{02}(\boldsymbol{v}_1)$ contributes to $(\varphi_{02}, J_{e-e}\ \varphi_{02})$ because of the orthogonality of $P_l(\cos\theta)$. Using (A·30), (A·31) and performing all integrations, we obtain

$$(\varphi_{02}, J_{e-e}\ \varphi_{02})=\frac{3\sqrt{2}}{2\pi^2}\left(\frac{e^2}{\kappa T_0}\right)^2\left(\frac{\kappa T_0}{m}\right)^{\frac{1}{2}}X\ Y,$$

$$X=4\pi\int_0^\infty e^{-2\left(\frac{G}{v_0}\right)^2}\left(\frac{G}{v_0}\right)^2 d\left(\frac{G}{v_0}\right)=\frac{\sqrt{2}\,\pi^{\frac{3}{2}}}{4},$$

$$Y=2\pi\int_{-1}^{1}[P_2(x)]^2dx\int_0^\infty e^{-\frac{1}{2}\left(\frac{g}{v_0}\right)^2}\left(\frac{g}{v_0}\right)^3\left[ln\frac{2\kappa T_0}{\gamma e^2 k_s}-1+ln\left(\frac{g}{v_0}\right)^2\right]$$

$$=\frac{8\pi}{5}ln\frac{4\kappa T_0}{\gamma^2 e^2 k_s},$$

thus

$$(\varphi_{02}, J_{e-e}\varphi_{02})=\frac{6\sqrt{\pi}}{5}\left(\frac{e^2}{\kappa T_0}\right)^2\left(\frac{\kappa T_0}{m}\right)^{\frac{1}{2}}ln\frac{4\kappa T_0}{\gamma^2 e^2 k_s}. \qquad (A·32)$$

Appendix B

The result of the paper 1 (5·8) contains algebraic error, so in this appendix we will show correct result.

From (1–5·7) the dispersion relation is given as

$$\left(\frac{\omega}{\omega_e}\right)^2=1-\frac{3}{2\lambda^2}+\frac{2\lambda^4}{\omega}\sum_{r,s,l}(\lambda_{rs}^{(l)}+\Lambda_{rs}^{(l)})\,J_{sl}\,J_{rl}, \qquad (B·1)$$

where

$$\lambda=\frac{i}{\sqrt{2}}\frac{\Omega}{\kappa},\quad \kappa=\frac{k}{k_D},\quad \Omega=\frac{\omega}{\omega_e},$$

and

$$J_{rl}=\left(\frac{1}{\lambda-i\xi_z},\ \Psi_{rl}\right).$$

Taking the long wavelength limit, we can write J_{rl} as

$$J_{rl}=\underline{\underline{\frac{1}{\lambda}\delta_{r0}\delta_{l0}}}+\frac{1}{\lambda^2}\frac{i}{\sqrt{2}}\delta_{r0}\delta_{l1}-\frac{1}{\lambda^3}\left[\frac{1}{\sqrt{3}}\delta_{r0}\delta_{l2}-\underline{\underline{\frac{1}{\sqrt{6}}\delta_{r1}\delta_{l0}}}+\underline{\underline{\frac{1}{2}\delta_{r0}\delta_{l0}}}\right]$$

$$-\frac{i}{\lambda^4}\left[\frac{3\sqrt{2}}{4}\delta_{r0}\delta_{l1}-\frac{3}{2\sqrt{5}}\delta_{r1}\delta_{l1}+\frac{3}{2\sqrt{30}}\delta_{r0}\delta_{l3}\right]. \qquad (B·2)$$

Terms with underline when substituted into (B·1) do not contribute to the dispersion relation due to the conservations of number density and energy. Thus we obtain

$$J_{rl}J_{sl}=\frac{1}{2\lambda^4}\delta_{r0}\delta_{s0}\delta_{l1}+\frac{1}{3\lambda^6}\delta_{r0}\delta_{s0}\delta_{l2}$$

$$+\frac{2}{\lambda^6}\left(\frac{3}{4}\delta_{r0}\delta_{s0}\delta_{l1}-\frac{3}{2\sqrt{10}}\delta_{r0}\delta_{s1}\delta_{l1}\right).$$

Then taking account of the conservation of momentum $\lambda_{00}^{(1)}=0$ in electron-electron collision, we obtain the correct result

$$\left(\frac{\omega}{\omega_e}\right)^2=1+3\frac{\kappa^2}{\Omega^2}-\frac{1}{\Omega}\frac{i\Lambda_{00}^{(1)}}{\omega_e}$$
$$-\frac{\kappa^2}{\Omega^2}\left[\frac{6}{\Omega}\frac{i\Lambda_{00}^{(1)}}{\omega_e}+\frac{4}{3}\frac{1}{\Omega}\left(\frac{i\lambda_{00}^{(2)}}{\omega_e}+\frac{i\Lambda_{00}^{(2)}}{\omega_e}\right)-\frac{6\sqrt{10}}{5}\frac{1}{\Omega}\frac{i\Lambda_{01}^{(1)}}{\omega_e}\right]. \qquad (B \cdot 3)$$

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