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Optimal Parametric Excitation in a Resonant Circuit

(Received December 9, 1964)

Chikara SATO*

Abstract

An $L-C$ resonant circuit with parametric excitation is dealt with when the circuit corresponds to the differential equation of the form:

$$\ddot{x}+[\omega^2+u(t)]x=0,$$

where function $u(t)$ is controllable under the condition $u_{min} \leq u(t) \leq u_{max}$. The objective is to find the optimal trajectories so that an initial phase point moves toward a terminal in a minimum time. The terminal which is used here is a circle with its center at the origin. Using both maximum principle of Pontryagin and transversality condition, synthesis is made on the phase plane for outside and inside of the circle. According to the results obtained herein the control function $u(t)$ takes either minimum value or maximum value. From the synthesis on the phase plane it is known that the function $u(t)$ changes its value four times when the argument on the phase plane increases by 2π. In other words it is most effective to induce oscillation of 1/2 subharmonic type in order to make the phase point move toward a circle in a minimum time.

I. Introduction

One of the recent topics in control engineering is the optimal problem, that was established by Pontryagin(1) and Bellman(2) beyond the classical calculus of variation. Since these mathematicians and their coworkers gave mathematical formulations and several fundamental theorems in 1956, many papers have been published in the field of control engineering. On the other hand in the field of circuit theory not so many papers have been published about optimal problems.

In this paper a specialized topic in circuit theory is discussed with respect to the optimal problems. Here we will treat only a simple L–C resonant circuit with a variable parameter. Mathematical treatment used here depends on both maximum principle of Pontryagin and transversality condition.

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* 佐藤　力 Assistant Professor at Keio University

II. Mathematical Analysis

Parametric oscillation can be induced in an L–C resonant circuit by varying the capacitance. Conversely an induced oscillation in the circuit can be suppressed by varying the capacitance. In this paper investigation will be made to determine the optimal variation of the capacitance, in order to induce or suppress the oscillation. From both engineering application and mathematical interest it is supposed that the value of the capacitance $C(t)$ is positive and bounded between two finite values. Under these assumptions the ultimate objective is to find the optimal trajectories so that an initial phase point moves toward a terminal in a minimum time. The terminal is a circle with its center at origin. The circle is employed as a simple region including the origin inside. The circle also coincides with one of the constant energy level in a special case. Those are the reasons why we take a circle as the terminal. Using maximum principle and transversality condition optimal trajectories will be obtained on the phase plane.

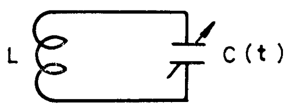

Fig. 1
L–C Resonant circuit

From the circuit shown in Fig. 1 the corresponding differential equation is given by

$$\ddot{x}+[\omega^2+u(t)]x=0, \tag{1}$$

where x is a charge in the capacitance $C(t)$, $\ddot{x}=d^2x/dt^2$, and parameters ω^2 and $u(t)$ are given by reactances L and $C(t)$. We can put $\omega^2=1$ without loss of generality. Function $u(t)$ is supposed to be controllable under the condition

$$u_{\min}\leq u(t)\leq u_{\max}. \tag{2}$$

The function $u(t)$ which is possibly piecewise continuous function is said an admissible control function, after Pontryagin. If we put that

$$x=x^1,\qquad \dot{x}=x^2,$$

the original differential equation (1) and its auxiliary equation are followings,

$$\dot{x}^1=x^2,\qquad \dot{x}^2=-[1+u(t)]\,x^1, \tag{3}$$

$$\dot{\psi}_1=[1+u(t)]\,\psi_2,\qquad \dot{\psi}_2=-\psi_1. \tag{4}$$

Hamiltonian function H is given by

$$H=\psi_1x^2-[1+u(t)]\,\psi_2x^1. \tag{5}$$

In order to maximize the function H, $u(t)$ must be maximum value if $-\psi_2x^1$ is positive and $u(t)$ must be minimum if $-\psi_2x^1$ is negative, that is :

$$\begin{aligned} u(t)&=u_{\max}\quad \text{for}\quad -\psi_2x^1>0,\\ u(t)&=u_{\min}\quad \text{for}\quad -\psi_2x^1<0, \end{aligned} \tag{6}$$

and $u(t)$ is not determined by these treatments if $-\psi_2 x^1=0$. So the function $u(t)$ is expected to be of bang-bang type.

Now, right-hand terminal condition is on a circle with radius R, that is

$$(x^1)^2+(x^2)^2=(R)^2. \tag{7}$$

We assume that at $t=0$, the vector (x^1, x^2) reaches the terminal circle (7), and that at $t=0$, $(x^1, x^2)=(R\cos\alpha, R\sin\alpha)$. According to the transversality condition, at $t=0$ the vector (ψ_1, ψ_2) must be normal to the circle (7) at the point $(x^1, x^2)=(R\cos\alpha, R\sin\alpha)$. Therefore we can take at $t=0$ that

$$(\psi_1, \psi_2)=\pm(R\cos\alpha, R\sin\alpha), \tag{8}$$

because the equation (4) is homogeneous for ψ_1 and ψ_2. Thus we have two transversality conditions depending on the signs in expression (8). Using these two transversality conditions synthesis will be made below.

III. Synthesis

The aim of this section is to have optimal trajectories from an arbitrary point to the terminal circle in a minimum time. The method of obtaining optimal trajectories and their switching curves will be explained using an example when $u_{max}=0.5$ and $u_{min}=-0.5$.

In Fig. 2 (a) and (b), a phase plane for (x^1, x^2) and a phase plane for (ψ_1, ψ_2) are shown respectively. In Fig. 2 (a) there is a circuit with radius R, and its center is at origin. This circle is the terminal which corresponds to equation (7). By taking the negative direction of time t, optimal trajectories can be traced from the terminal state P_0 on the circle toward a initial point. This P_0 is an arbitrary point. Corresponding to P_0, a point P'_0 can be set on a circle on the (ψ_1, ψ_2)plane by transversality condition. If the angle $\angle P_0 O\ x^1$ is taken to be α then the angle $\angle P'_0 O-\psi_1$ makes an angle α. At first the trajectory proceeds from P_0 to P_1 with control $u=0.5$. This trajectory is a part of an ellipse. When the trajectory crosses x^2-axis at P_1, control u changes its sign from positive to negative value, that is, u becomes from -0.5 to 0.5. From P_1 to P_2 the trajectory proceeds along another ellipse. At point P_2 the sign of x^1 does not change, but at corresponding point P'_2 the sign of ψ_2 changes from negative to positive. Likewise switching is made at points P_1, P_2, P_3, P_4,…… Among them switching points P_1, P_3, P_5,…… are on x^2-axis. On the other hand switching points P_2, P_4, P_6,…… are not in general on x^1-axis. It is important to notice that three points P_2, O and P_4 lie on a straight line. This can be easily proved, because the figure $O\ P_2\ P_3\ P_4$ is similar to the figure $O\ P'_2\ P'_3\ P'_4$. The other optimal trajectories and their switching curves can be obtained by geometric expansion, for outside the circle as shown in Fig. 3. Likewise optimal trajectories can be obtained for inside of the circle, by using another transversality condition. Now the synthesis is made for both outside and inside of the circle as shown in Fig. 4. Since all the trajectories consist of two kinds of ellipse, synthesis can be accomplished by geometric procedures. To test the validity of this geometric synthesis, analog computation-technique is used. The curves shown in Fig. 5 are optimal solutions of x^1, x^2, ψ_1 and ψ_2. These curves agree well with the above results.

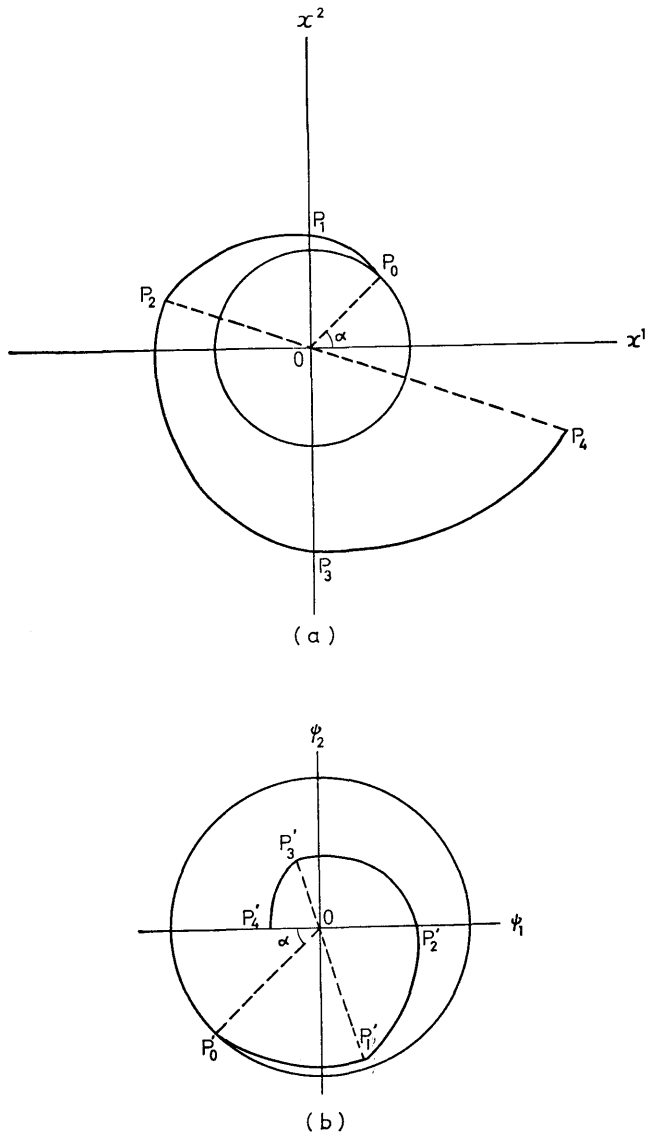

Fig. 2 Geometric method of synthesis

It must be noticed that Pontryagin's maximum principle provides only necessary condition, and does not provide sufficient condition. Furthermore, uniqueness is not always satisfied. In the present case sufficient condition and uniqueness are not guaranteed analytically, but only uniqueness is known from the geometric figure, because there are no overlapping trajectories in Fig. 4.

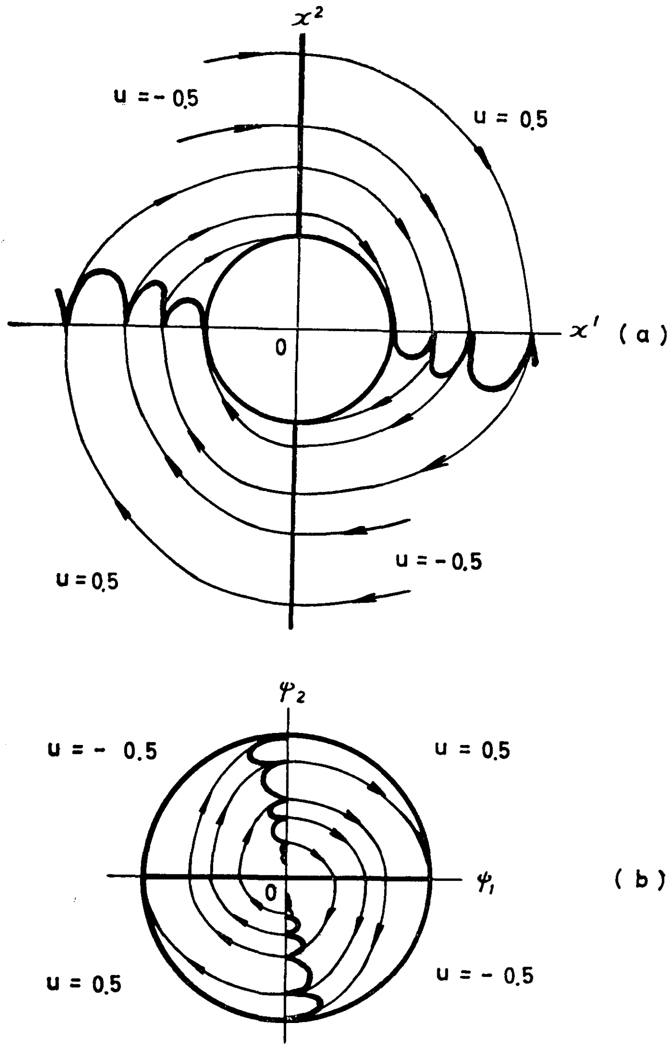

Fig. 3 Synthesis of optimal parametric excitation for outside of the circle, when $\ddot{x}+[1+u(t)]x=0,\ |u|\leq 0.5$

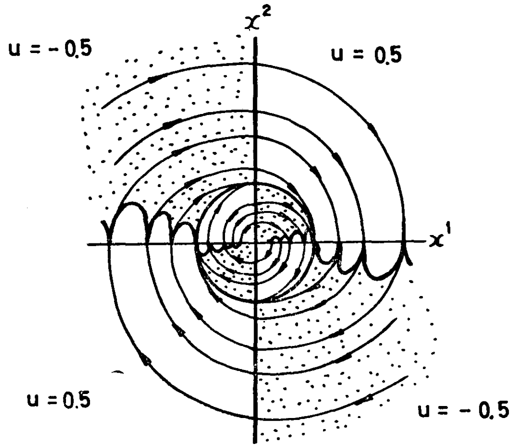

Fig. 4 Synthesis of optimal parametric excitation for outside and inside of the circle, when
$\ddot{x}+[1+u(t)]x=0,\ |u|\leq 0.5$
$u=-0.5$ for shaded regions, and
$u=0.5$ for other regions.

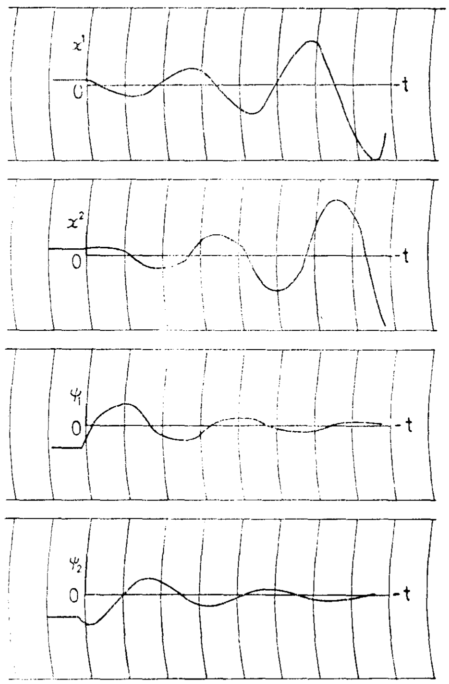

Fig. 5 Optimal solutions by analog computer

Conclusion

An L-C resonant circuit with variable capacitance is dealt with when the circuit corresponds to the differential equation of the second order. The objective is to determine the optimal variation of the capacitance in order to induce or suppress the oscillation. From both engineering application and mathematical interest it is supposed that the value of the capacitance $C(t)$ is positive and bounded between two finite values. Under these assumptions mathematical formulation of this problem coincides with the optimal problem of Pontryagin. Thus, Pontryagin's maximum principle and transversality condition are used in order to obtain the optimal trajectories and their switching curves so that an initial phase point may move toward a terminal circle in a minimum time. Synthesis is made on the phase plane for both outside and inside of the terminal circle.

The value of the capacitance takes either minimum value or maximum value, that is, the capacitance varies in bang-bang type. From the synthesis it is obtained that the capacitance changes its value four times when the argument on the phase plane increases by 2π. In other words it is 1/2 subharmonic type of oscillation. Thus it is most effective to make parametric oscillation of 1/2 subharmonic type in order to induce or suppress the oscillation in a minimum time.

Mathematical treatment depends mainly on Pontryagin's maximum principle that provides only necessary condition. Therefore, the synthesis obtained here consists of the necessary condition for this time optimal problem. The sufficient condition and uniqueness are not proved analytically.

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(2) R. Bellman : On the Bang-bang Control Problem, Quart. Appl. Math. 14, pp. 11–18, 1956.