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A Note on Changes in the A-Matrix of Linear Programs*

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Abstract

Parametric linear programming with changes in the cost coefficients in the objective function and the restraint constant vector has been studied in some detail. This note adds to these studies simple procedures to obtain a series of solutions on varying a column or row of the constraint matrix, A, linearly with a scalar parameter.

I. Introduction

I. 1 : Consider a standard linear programming problem :

$$\begin{cases} \text{maximize} & f = \boldsymbol{cx} \\ \text{subject to} & \boldsymbol{Ax} = \boldsymbol{b} \\ \text{and} & \boldsymbol{x} \geqslant 0, \end{cases} \tag{1}$$

where $\boldsymbol{x} = (x_j)$ is an n-column variable vector; $\boldsymbol{A} = (a_{ij})$, an m by n matrix; $\boldsymbol{b} = (b_i)$, an m-column vector; $\boldsymbol{c} = (c_j)$, an n-row vector ; and $f = \boldsymbol{cx}$, an objective function to be maximized. The simplex method has been known to be an efficient computational procedure to solve linear programs [e.g. 2, 3, 5].

Consider the current tableau, with which a square submatrix, $\boldsymbol{B}$, consisting of columns of $\boldsymbol{A}$, is associated. The simplex method is an iterative procedure to replace a column of $\boldsymbol{B}$ by a column of $\boldsymbol{A}$, not belonging to $\boldsymbol{B}$, so as to maximize $\boldsymbol{c}_B \boldsymbol{x}_B$. A row vector $\boldsymbol{c}_B$ and a column vector $\boldsymbol{x}_B$ consist of c_j's and x_j's, respectively, associated with columns, $a_{\cdot j}$ of $\boldsymbol{B}$. These x_j's are called the basic variables.

I. 2 : We shall use the notations $\mathfrak{B}$ and $\bar{\mathfrak{B}}$ for the index sets defined by

$$\mathfrak{B} = \{j\,;\ x_j \text{ is a basic variable in the initial tableau}\}$$

$$\text{and} \quad \bar{\mathfrak{B}} = \{j\,;\ x_j \text{ is a basic variable in the current or final tableau}\}.$$

Notations with bars denote elements in the current tableau distinguished from

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the initial tableau elements. It is convenient to denote the i-th row and the j-th column of A by $a_{i.}$ and $a_{.j}$, respectively. By these conventions, the basis is denoted by $\boldsymbol{B}=(a_{ij})$, $(i\in\mathfrak{B}, j\in\bar{\mathfrak{B}})$, and the inverse of the basis, by $\boldsymbol{B}^{-1}=(\bar{a}_{ij})$, $(i\in\bar{\mathfrak{B}}, j\in\mathfrak{B})$.

I. 3 : The procedure to solve the problems when each of the coefficients is the function of a scalar parameter, is said to be parametric linear programming. The parametric changes in the coefficients of the objective function, $\boldsymbol{c}$, and the right-hand side of the constraints, $\boldsymbol{b}$, have been studied in some detail [3, 5, 6]. In this note, linear parametric changes in a row or column of the constraint matrix, A, are investigated as an extention of Shetty's work [8]. This investigation makes practical and easy analyses of cases such as follows.

I. 4 : The parametric changes in a column may occur, say, in a case when one of the activities in the system is controlled linearly, or when the two different devices with $a_{.j}$ and $a'_{.j}$ are available for one activity x_j and the manager wishes to know the mixture ratio of utilization of two devices.

I. 5 : In Section II, for the convenience to proceed to the problem, studies on the effects of changes in a column or row of the A-matrix by the preceeding authors [1, 8] will be abridged. In Section III the parametric A-matrix problem is investigated, with a simple numerical example at the end of the section.

II. Changes in a column or a row of the A-matrix

II. 1 : Shetty and Barnett presented the formula for the changes in elements of the inverse corresponding to changes, made after a solution is obtained, in a row or column of the original A-matrix [1, 8]. Suppose changes are made by the amount $\Delta a_{.k}$ in column k of $\boldsymbol{A}=(a_{ij})$:

$$\text{the new } a_{ij}=a^*_{ij}=\begin{cases} a_{ik}+\Delta a_{ik} \\ a_{ij}\,,\ j\neq k \end{cases} \qquad i\in\mathfrak{B}. \tag{2}$$

If $a_{.k}$ is outside the current basis $(k\notin\bar{\mathfrak{B}})$, the optimality of the basis is preserved if the simplex criterion, $\bar{w}^*_k=\boldsymbol{c}_B(a_{.k}+\Delta a_{.k})-c_k\geqslant 0$, and the current tableau must be pivoted taking x_k into the basis, if $\bar{w}^*_k<0$.

II. 2 : If $a_{.k}$ is in the basis $(k\in\bar{\mathfrak{B}})$, more sophisticated calculations are needed to obtain a new optimal solution.

Consider the original inverse matrix $\boldsymbol{B}^{-1}=(\bar{a}_{ij})$, $(i\in\bar{\mathfrak{B}}, j\in\mathfrak{B})$. The inverse after changes, $\boldsymbol{B}^{*-1}$, is given by the sum of $\boldsymbol{B}^{-1}$ and the changed amount, $\Delta(\boldsymbol{B}^{-1})=(\Delta\bar{a}_{ij})$:

$$\boldsymbol{B}^{*-1}=\boldsymbol{B}^{-1}+\Delta(\boldsymbol{B}^{-1})=(\bar{a}_{ij})+(\Delta\bar{a}_{ij}),\ i\in\bar{\mathfrak{B}}, j\in\mathfrak{B}. \tag{3}$$

Before and after changes are made, we have the following relations respectively :

$$\begin{cases} BB^{-1}=I \\ B^*B^{*-1}=(B+\Delta B)(B^{-1}+(\Delta B^{-1}))=I \end{cases} \tag{4}$$

or

$$\begin{cases} (a_{li})(\bar{a}_{ij})=I \\ (a_{li}+\Delta a_{li})(\bar{a}_{ij}+\Delta\bar{a}_{ij})=I \end{cases} \quad l,\ j\in\mathfrak{B},\ i\in\bar{\mathfrak{B}}, \tag{5}$$

where I is an m-identity matrix. Eliminating a_{li} from (5), $\Delta\bar{a}_{ij}$ is expressed by the values on the current tableau and Δa_{ij}, (assuming that $1+\bar{a}_{k\cdot}\Delta a_{\cdot k}>0$) :

$$\Delta\bar{a}_{ij}=-\frac{\bar{a}_{kj}\bar{a}_{i\cdot}\,\Delta a_{\cdot k}}{1+\bar{a}_{k\cdot}\,\Delta a_{\cdot k}},\quad i\in\bar{\mathfrak{B}},\ j\in\mathfrak{B}, \tag{6}$$

where $\bar{a}_{i\cdot}$ and $\bar{a}_{k\cdot}$ are m-row vectors in the inverse, and $\Delta a_{\cdot k}$, a column vector consists of corresponding m components of Δa_{ik}.

Substituting (6) for $(\Delta\bar{a}_{ij})$ in (3), the new tableau will be given by

$$\text{the new } \bar{a}_{\cdot j}=\bar{a}^*_{\cdot j}=B^{*-1}a^*_{\cdot j}=\begin{cases} B^{*-1}a^*_{\cdot j}, & j=0,\ j\notin\bar{\mathfrak{B}} \\ \bar{a}_{\cdot j}\ (=e_j), & j\in\bar{\mathfrak{B}}, \end{cases} \tag{7}$$

where e_j is the j-th column of the m-identity matrix.

If all $\bar{b}^*_i\ (=\bar{a}^*_{i0})\geqslant 0$, compute $\bar{w}_j$ with the new elements $\bar{a}^*_{\cdot j}$:

$$\bar{w}^*_j=c_B\bar{a}^*_{\cdot j}-c_j,\quad j=1,\ldots,n. \tag{8}$$

If all $\bar{w}^*_j\geqslant 0$, the tableau obtained is already optimal. Otherwise (when dual infeasible), apply the primal simplex method.

If some $\bar{b}^*_i<0$ $(i\notin\bar{\mathfrak{B}})$ (primal infeasible), apply the dual simplex method to the tableau to get an improved solution.

II. 3 : By a similar discussion to the above, the charges in elements of the inverse for the changes in row h of A, i.e., for a problem with

$$a^*_{ij}=\begin{cases} a_{hj}+\Delta a_{hj}, & h\in\mathfrak{B} \\ a_{ij}, & i\neq h, \end{cases} \tag{9}$$

are given by

$$\Delta\bar{a}_{ij}=-\frac{\bar{a}_{ij}\Delta a_{h\cdot}\bar{a}_{\cdot j}}{1+\Delta a_{h\cdot}\bar{a}_{\cdot h}},\quad i\in\bar{\mathfrak{B}},\ j\in\mathfrak{B}, \tag{10}$$

where $\bar{a}_{\cdot j}$ and $\bar{a}_{\cdot h}$ are m-column vectors in the inverse of the basis, and $\Delta a_{h\cdot}$, a corresponding m-row vector.

II. 4 : The effect of changes in a column, $a_{\cdot k}$, may also be examined by adding a variable, x_{n+1}, with coefficients $a_{\cdot,n+1}=a_{\cdot k}+\Delta a_{\cdot k}$ [7]. And we modify the original objective function by setting $c_{n+1}=c_k$ and replacing c_k by a negative large number $-M$. Then, we have only to compute a column $\bar{a}_{\cdot,n+1}=B^{-1}a_{\cdot,n+1}$, $\bar{w}_k$ and $\bar{w}_{n+1}$ in order to check the optimality of the current basis for the changes. We need one more simplex iteration, of course, if the optimality is broken.

On the other hand, the procedure described above (in Sec. I. 2) always requires rewriting all elements on the tableau.

II. 5 : For variation in a single element, a_{ij}, the range of Δa_{ij} which preserves the optimality and feasibility of the current solution is given similarly in the references [5, 8].

III. Parametric A-matrix

III. 1 : Consider a problem in which column k of the $\boldsymbol{A}$-matrix is changed linearly by a scalar parameter λ :

$$\begin{cases} \max & f=\boldsymbol{cx} \\ \text{subject to} & \boldsymbol{Ax}=\boldsymbol{b} \\ \text{and} & \boldsymbol{x}\geqslant 0 \end{cases} \quad \text{with } a_{ij}=\begin{cases} a_{ik}+\lambda a'_{ik} \\ a_{ij}, \quad j\neq k, \end{cases} \tag{11}$$

where $\boldsymbol{a}'_{.k}$ is a specified proportional constant vector.

As the value of the parameter, λ, is changed, the different optimal bases are composed corresponding to ranges of values of λ : one solution remains optimal and feasible for a certain range of λ. We are now interested in obtaining a series of solutions by varying the value of λ.

Assume that for a certain value of λ, $\lambda^{(p)}$, the problem (11) is feasible and an optimal basic solution is already obtained :

$$\boldsymbol{x}^{(p)}=\boldsymbol{B}_{(p)}{}^{-1}\boldsymbol{b}=\bar{\boldsymbol{b}},$$

where $\boldsymbol{B}_{(p)}{}^{-1}$ denotes the inverse basis, with the simplex criteria, $\bar{\boldsymbol{w}}=(\bar{w}_j{}^{(p)})$, and the maximized objective function, $f^{(p)}=\boldsymbol{c}_{B(p)}\boldsymbol{x}^{(p)}$. Hereafter, $\bar{w}_j{}^{(p)}$ and $\boldsymbol{c}_{B(p)}$ will be written, in short, as $\bar{w}_j$ and $\boldsymbol{c}_B$, as they will cause less confusion.

III. 2 : Consider first a case when column k, altered by the parameter, is currently outside the basis :

$$a_{ij}=\begin{cases} a_{ik}+\lambda a'_{ik}, & k\notin\mathfrak{B} \\ a_{ij}, & j\neq k. \end{cases} \tag{12}$$

As λ is changed, only the simplex criterion for column k, $\bar{w}^*_k$, is affected :

$$\begin{aligned} \bar{w}^*_k &= \bar{w}_k+\Delta\lambda \boldsymbol{c}_B\boldsymbol{B}_{(p)}{}^{-1}a'_{.k}, \quad \Delta\lambda=\lambda-\lambda^{(p)} \\ &= \bar{w}_k+\Delta\lambda\bar{\boldsymbol{u}}\, a'_{.k}, \end{aligned} \tag{13}$$

where $\bar{\boldsymbol{u}}=\boldsymbol{c}_B\boldsymbol{B}_{(p)}{}^{-1}$ is an m-row vector whose components are the simplex criteria, $\bar{w}_j$'s, for all $j\in\mathfrak{B}$, and $a'_{.k}$, an m-column vector composed of a'_{ik} $(i\in\mathfrak{B})$. The optimality of the current basis is maintained provided that $\bar{w}^*_k\geqslant 0$, as is well-known. Since $\bar{w}_k\geqslant 0$, the range of $\Delta\lambda$ without affecting the basis is given by

$$\underline{\Delta}\lambda_1\leqslant\Delta\lambda\leqslant\bar{\Delta}\lambda_1, \tag{14}$$

where

$$\underline{\Delta}\lambda_1 = \max \begin{cases} -\dfrac{\bar{w}_k}{\bar{u}a'_{\cdot k}}, & \bar{u}a'_{\cdot k} > 0 \\ -\infty, & \bar{u}a'_{\cdot k} \leqslant 0 \end{cases} \tag{15}$$

$$\bar{\Delta}\lambda_1 = \min \begin{cases} +\infty & \bar{u}a'_{\cdot k} \geqslant 0 \\ -\dfrac{\bar{w}_k}{\bar{u}a'_{\cdot k}}, & \bar{u}a'_{\cdot k} < 0\,. \end{cases} \tag{16}$$

For $\Delta\lambda$ within the limits of (14), the current basis remains optimal without affecting the values of the variables and the objective function.

III. 3 : Next we shall suppose column k is in the basis :

$$a_{ij} = \begin{cases} a_{ik} + \lambda a'_{ik}, & k \in \mathfrak{B} \\ a_{ij} & j \neq k. \end{cases} \tag{17}$$

In this case, the basic matrix is affected as λ is changed. For $\Delta\lambda$ $(=\lambda - \lambda^{(p)})$, the changes in elements of the inverse, $\Delta\bar{a}_{ij}$, are given by (6), putting $\Delta a_{\cdot k} = \Delta\lambda a'_{\cdot k}$,

$$\bar{a}^*_{ij} = \bar{a}_{ij} + \Delta\bar{a}_{ij} = \bar{a}_{ij} - \frac{\Delta\lambda \bar{a}_{i\cdot} a'_{\cdot k} \bar{a}_{kj}}{1 + \Delta\lambda \bar{a}_{k\cdot} a'_{\cdot k}}, \quad i \in \mathfrak{B},\ j \in \mathfrak{B} \tag{18}$$

where $\bar{a}_{i\cdot}$ and $\bar{a}_{k\cdot}$ are m-row vectors in the inverse of the basis as before, and $a'_{\cdot k}$ is a constant column vector in (17). The use of the formula (6) in analysing effects of a simple change or replacement of a column of A is not always efficient, but (6) plays an important role here.

Substituting $\bar{a}^*_{ij}$ of (18) for $B^{*-1} = (\bar{a}^*_{ij})$ in $\bar{b}^*_i = (\bar{a}^*_{i\cdot})b$ and $\bar{w}^*_j = c_B B^{*-1} a_{\cdot j} - c_j$, we test the signs of $\bar{b}^*_i$'s and $\bar{w}^*_j$'s. The feasibility and optimality are maintained provided that $\bar{b}^*_i \geqslant 0$ for all $i \in \mathfrak{B}$ and $\bar{w}^*_j \geqslant 0$ for all $j \notin \mathfrak{B}$. Thus, the range of $\Delta\lambda$ without violating the optimality and feasibility of the basis is given by

$$\underline{\Delta}\lambda_2 \leqslant \Delta\lambda \leqslant \bar{\Delta}\lambda_2\,, \tag{19}$$

where

$$\underline{\Delta}\lambda_2 = \max \begin{cases} \dfrac{\bar{b}_i}{(\bar{b}_k \bar{a}_{i\cdot} - \bar{b}_i \bar{a}_{k\cdot})\, a'_{\cdot k}} & \text{denominator} < 0,\ i \in \mathfrak{B} \\ \dfrac{\bar{w}_j}{(\bar{a}_{kj}\bar{u} - \bar{w}_j \bar{a}_{k\cdot})\, a'_{\cdot k}} & \text{denom.} < 0,\ j \notin \mathfrak{B} \\ -\infty & \text{denom.} \geqslant 0 \end{cases} \tag{20}$$

$$\bar{\Delta}\lambda_2 = \min \begin{cases} \dfrac{\bar{b}_i}{(\bar{b}_k \bar{a}_{i\cdot} - \bar{b}_i \bar{a}_{k\cdot})\, a'_{\cdot k}} & \text{denom.} > 0,\ i \in \mathfrak{B} \\ \dfrac{\bar{w}_j}{(\bar{a}_{kj}\bar{u} - \bar{w}_j \bar{a}_{k\cdot})\, a'_{\cdot k}} & \text{denom.} > 0,\ j \notin \mathfrak{B} \\ +\infty & \text{denom.} \leqslant 0. \end{cases} \tag{21}$$

Hence,

$$x^*_i = \bar{b}^*_i = \bar{b}_i + \Delta\bar{b}_i,$$

$$\Delta\bar{b}_i = \begin{cases} -\dfrac{\Delta\lambda\bar{a}_{i.}a'_{.k}\bar{b}_k}{1+\Delta\lambda\bar{a}_{k.}a'_{.k}}, & i\in\mathfrak{B} \\ 0, & i\notin\mathfrak{B} \end{cases} \tag{22}$$

yields the maximum for all $\Delta\lambda$ of (19) :

$$f^* = f^{(p)} + \boldsymbol{c}_B\Delta\bar{\boldsymbol{b}} \tag{23}$$

III. 4 : We wish to change λ and get a series of solutions for each value of λ. Consider first to increase λ starting with $\lambda^{(p)}$, hence $\Delta\lambda = \lambda - \lambda^{(p)} \geqslant 0$.

In what follows, take $\Delta\lambda_1$ for $\Delta\lambda$ if the column concerned is outside the current basis, and take $\Delta\lambda_2$ for $\Delta\lambda$, if in the basis.

If $\overline{\Delta}\lambda = +\infty$, the problem has an unbounded solution for $\lambda \geqslant \lambda^{(p)}$ and the problem is over. If $\overline{\Delta}\lambda < +\infty$, the current basis is optimal for a range of $\lambda^{(p)} \leqslant \lambda < \lambda^{(p+1)}$ $(=\lambda^{(p)}+\overline{\Delta}\lambda)$. But for $\lambda > \lambda^{(p+1)}$, it becomes infeasible or non-optimal. It is necessary to change the basis by the primal simplex algorithm taking a column $a_{.s}$ into the basis if the minimum in (21) is attained for $s\in\mathfrak{B}$, or by the dual algorithm dropping a column $a_{.r}$ out of the basis if the minimum is for $r\notin\mathfrak{B}$. If the simplex algorithm shows our problem has no finite solution, we are finished†.

The new basis thus obtained is certainly feasible and optimal since we have followed the simplex procedure and $\bar{b}_i \geqslant 0$ for all $i\in\mathfrak{B}$, and $\bar{w}_j \geqslant 0$ for all $j\in\mathfrak{B}$ hold for the new basis.

And the value of the objective function makes no change by this pivoting :

$$f^{(p+1)} = f^{(p)} - \bar{b}_s^{(p+1)}\bar{w}_s^{(p)} = f^{(p)}. \tag{24}$$

Indeed, if we assume that either only one of $\bar{b}_i$ or $\bar{w}_j$ becomes zero at $\lambda = \lambda^{(p+1)}$, i.e., the minimum in (21) is attained uniquely for one index, either of the following is true at the characteristic value :

i) for $\lambda^{(p)} < \lambda = \lambda^{(p+1)}$,
$\bar{b}_r^{(p)} = 0$; $\bar{b}_i^{(p)} \geqslant 0,\ i \neq r$; $\bar{w}_j^{(p)} \geqslant 0,\ j\notin\mathfrak{B}$, and
for $\lambda^{(p+1)} = \lambda < \lambda^{(p+2)}$
$\bar{b}_s^{(p+1)} = \bar{b}_r^{(p)}/\bar{a}_{rs} = 0$; $\bar{b}_i^{(p+1)} \geqslant 0,\ i \neq s$; $\bar{w}_j^{(p+1)} \geqslant 0$.

ii) for $\lambda^{(p)} < \lambda = \lambda^{(p+1)}$,
$\bar{b}_i^{(p)} \geqslant 0$; $\bar{w}_s^{(p)} = 0$; $\bar{w}_j^{(p)} \geqslant 0,\ j \neq s$, and
for $\lambda^{(p+1)} = \lambda < \lambda^{(p+2)}$
$\bar{b}_i^{(p+1)} \geqslant 0$; $\bar{w}_r^{(p+1)} = \bar{w}_s^{(p)} - \bar{w}_r^{(p)}/\bar{a}_{rs} = 0$; $\bar{w}_j^{(p+1)} \geqslant 0,\ j \neq r$.

Thus, $\bar{b}_s^{(p+1)} = 0$ or $\bar{w}_s^{(p)} = 0$ in (24).

† If all elements in column s of the current tableau, $\bar{a}_{is} < 0$, in the primal algorithm, or all elements in row r, $\bar{a}_{rj} > 0$, in the dual algorithm, the problem has no finite solution.

III. 5 : For $\lambda < \lambda^{(p)}$ the same discussion can be made. When $\underline{\Delta}\lambda > -\infty$, the basis for $\lambda = \lambda^{(p)}$ is optimal for the range $\lambda^{(p-1)} \leqslant \lambda \leqslant \lambda^{(p)}$. And for $\lambda < \lambda^{(p-1)}$, it is necessary to change the basis.

In this manner, starting with $\lambda^{(0)} = 0$, say, we can increase the value of λ until $\bar{\Delta}\lambda = +\infty$, or until it is shown by the simplex method that no finite solution exists for the problem.

III. 6 : We next consider a problem where row h of the A-matrix is changed by a scalar parameter λ :

$$\begin{cases} \max & f = \boldsymbol{cx} \\ \text{subject to} & \boldsymbol{Ax} = \boldsymbol{b} \qquad \text{with } a_{ij} = \begin{cases} a_{hj} + \lambda a'_{hj} \\ a_{ij}, \ i \neq h, \end{cases} \\ \text{and} & \boldsymbol{x} \geqslant 0 \end{cases} \tag{25}$$

where $a'_{h\cdot}$ is a proportional constant vector.

Suppose the above problem is feasible for a certain value of λ, $\lambda^{(p)}$, and suppose an optimal basic solution is already obtained. For $\lambda \neq \lambda^{(p)}$, new elements of the inverse basis corresponding to the changes are given by putting $\Delta a_{h\cdot} = \Delta\lambda a'_{h\cdot}$, $\Delta\lambda = \lambda - \lambda^{(p)}$ in (10) :

$$\bar{a}^*_{ij} = \bar{a}_{ij} + \Delta\bar{a}_{ij} = \bar{a}_{ij} - \frac{\Delta\lambda \bar{a}_{ih} a'_{h\cdot}\bar{a}_{\cdot j}}{1 + \Delta\lambda a'_{h\cdot}\bar{a}_{\cdot h}}, \ i \in \mathfrak{B}, \ j \in \mathfrak{B} \tag{26}$$

where $\bar{a}_{\cdot j}$, and $\bar{a}_{\cdot h}$ are m-column vectors in the inverse, and $a'_{h\cdot}$ is a constant row vector in (25).

$\bar{b}^*_i$ and $\bar{w}^*_j$ corresponding to the changes are calculated and the feasibility and optimality of the basis are tested. The range of $\Delta\lambda$ for which the current basis is maintained is given by

$$\underline{\Delta}\lambda \leqslant \Delta\lambda \leqslant \bar{\Delta}\lambda \tag{27}$$

$$\underline{\Delta}\lambda = \max \begin{cases} \dfrac{\bar{b}_i}{a'_{h\cdot}(\bar{a}_{ih}\bar{\boldsymbol{b}} - \bar{b}_i\bar{a}_{\cdot h})}, & \text{denominator} < 0, \ i \in \mathfrak{B} \\ \dfrac{\bar{w}_j}{a'_{h\cdot}(\bar{z}_h\bar{a}_{\cdot j} - \bar{w}_j\bar{a}_{\cdot h})}, & \text{denom.} <, \ j \notin \mathfrak{B} \\ -\infty, & \text{denom.} \geqslant 0 \end{cases} \tag{28}$$

$$\bar{\Delta}\lambda = \min \begin{cases} \dfrac{\bar{b}_i}{a'_{h\cdot}(\bar{a}_{ih}\bar{\boldsymbol{b}} - \bar{b}_i\bar{a}_{\cdot h})}, & \text{denom.} > 0, \ i \in \mathfrak{B} \\ \dfrac{\bar{w}_i}{a'_{h\cdot}(\bar{z}_h\bar{a}_{\cdot j} - \bar{w}_j\bar{a}_{\cdot h})}, & \text{denom.} > 0, \ j \notin \mathfrak{B} \\ +\infty, & \text{denom.} \leqslant 0, \end{cases} \tag{29}$$

where $\bar{z}_h$ is a scalar $\bar{z}_h = \boldsymbol{c}_B\bar{a}_{\cdot h}$. For $\Delta\lambda$ of (27), the solution is

$$x^*_i = \bar{b}^*_i = \bar{b}_i + \Delta\bar{b}_i, \qquad \Delta\bar{b}_i = \begin{cases} -\dfrac{\Delta\lambda \bar{a}_{ih} a'_{h\cdot}\bar{b}}{1 + \Delta\lambda a'_{h\cdot}\bar{a}_{\cdot h}}, & i \in \mathfrak{B} \\ 0, & i \notin \mathfrak{B} \end{cases} \tag{30}$$

and

$$f^{*}=f^{(p)}+\boldsymbol{c}_{B}\Delta\bar{b}. \tag{31}$$

III. 7 : Consider a case of increasing λ from $\lambda^{(p)}$. If all denominators in (29) are nonpositive, $\bar{\Delta}\lambda=+\infty$ and the problem is over. If there are some positive denominators, a finite $\bar{\Delta}\lambda$ exists and the solution (30) is optimal for a range of $\lambda^{(p)}\leqslant\lambda\leqslant\lambda^{(p)}+\bar{\Delta}\lambda$. But it becomes infeasible or non-optimal for $\lambda\geqslant\lambda^{(p)}+\bar{\Delta}\lambda$ and we must change the basis by the primal algorithm, if the minimun in (28) is attained for $j\notin\mathfrak{B}$, or by the dual algorithm, if the minimum is for $i\in\mathfrak{B}$. If a feasible solution exists for $\lambda\geqslant\lambda^{(p)}+\bar{\Delta}\lambda$, repeat the above procedure. If the simplex method shows no finite solution exists, the problem is over.

For a case of decreasing λ, the similar discussions may be made.

Thus, problems with a parametric changes in a single row of the A-matrix can be solved for each value of λ, starting with $\lambda^{(0)}=0$, by the simplex method with a modification.

III. 8 : Changes in more than two columns or rows require too cumbersome computations to update the inverse, and are not dealt with here.

III. 9 : Let's solve the following problem for $0\leqslant\lambda\leqslant\infty$ as an example:

$$\begin{cases} \text{max} & 9x_1+7x_2+5.5x_3 \\ \text{subject to} & (6+0.5\lambda)x_1+5x_2+4x_3\leqslant 18 \\ & (5+\lambda)x_1+4x_2+3x_3\leqslant 14 \\ & x_1,\ x_2,\ x_3\geqslant 0 \end{cases}$$

In this problem, $A=\begin{pmatrix}6+0.5\lambda & 5 & 4\\ 5+\lambda & 4 & 3\end{pmatrix}$ and $a'_{\cdot 1}=\begin{pmatrix}0.5\\ 1\end{pmatrix}$.

For $\lambda^{(0)}=0$, the solution is, as shown in Table 1a, $x_1=1.0$, $x_2=0$, $x_3=3.0$, $x_4=x_5=0$

Table 1

	Basis	b	x_1	x_2	x_3	x_4	x_5	
a	x_3	3.0	0	0.5	1	2.5	−3.0	$\lambda^{(0)}=0$
	x_1	1.0	1	0.5	0	−1.5	2.0	
	w	25.5	0	0.25	0	0.25	1.5	$\Delta\lambda=0.5$
b	x_3	3.5380	0	0.7691	1	1.6923	−1.9231	$\lambda^{(1)}=0.5$
	x_1	0.6154	1	(0.3077)	0	−0.9231	1.2308	
	w	24.9976	0	−0.0006	0	0.9998	0.5001	
c	x_3	1.9998	−2.4745	0	1	3.9996	−4.9995	
	x_2	2.0	3.2174	1	0	−3.0	4.0	
	w	24.9988	0.019	0	0	0.9980	0.5025	

with $f=25.5$. We shall consider increasing λ. As the column concerned, column 1, is currently in the basis matrix, denominators in (21) are given as follows :

$$\text{for } i\in\mathfrak{B},\ i=3\quad (1.0\quad (2.5\quad -3.0)-3.0\quad (-1.5\quad 2.0))\begin{pmatrix}0.5\\1\end{pmatrix}=-5.5<0$$

$$i=1\quad (1.0\quad (-1.5\quad 2.0)-1.0\quad (-1.5\quad 2.0))\begin{pmatrix}0.5\\1\end{pmatrix}=0$$

$$\text{for } j\notin\mathfrak{B},\ j=2\quad (1.0\quad (0.25\quad 1.5)-0.25(-1.5\quad 2.0))\begin{pmatrix}0.5\\1\end{pmatrix}=0.5>0$$

$$j=4\quad (-1.5(0.25\quad 1.5)-0.25(-1.5\quad 2.0))\begin{pmatrix}0.5\\1\end{pmatrix}=-2.75<0$$

$$j=5\quad (2.0\quad (0.25\quad 1.5)-1.5\quad (-1.5\quad 2.0))\begin{pmatrix}0.5\\1\end{pmatrix}=1.375>0$$

Then, $\bar{\varDelta}\lambda=\min\left\{\frac{0.25}{0.5}, \frac{1.5}{1.375}\right\}=0.5$, for $j=2$. The changes in the inverse for $\lambda^{(1)}=0.5$ are given by (18) :

$$\varDelta\bar{a}_{34}=-\frac{(0.5)(-1.5)(2.5\quad -3.0)\begin{pmatrix}0.5\\1\end{pmatrix}}{1+0.5(-1.5\quad 2.0)\begin{pmatrix}0.5\\1\end{pmatrix}}=-0.8077,$$

$$\varDelta\bar{a}_{35}=1.0769,\quad \varDelta\bar{a}_{14}=0.5769,\quad \varDelta\bar{a}_{15}=-0.7692.$$

With these changes, the new inverse becomes as shown in Table 1*b*.

Since the minimun in (21) is attained for $j=2$, pivoting with a pivot $\bar{a}_{12}$ (an encircled element) leads to Table 1*c*, with the solution $x_1=0$, $x_2=2.0$, $x_3=2.0$, $x_4=x_5=0$.

Since x_1 has become non-basic and $\bar{u}a'_{\cdot 1}=(0.99\quad 0.50)\begin{pmatrix}0.5\\1\end{pmatrix}\geqslant 0$, the problem is over.

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References

1. Barnett, S. : "Stability of the Solution to a Linear Programming Problem," *Oper. Res. Quart.*, **13**, 3 (1962), 212–228.
2. Charnes, A., W. W. Cooper and A. Henderson : *An Introduction to Linear Programming*, Wiley, New York, 1953.
3. Dantzig, G. B. : *Linear Programming and Extensions*, Princeton Univ. Press, Princeton, N. J., 1963.
4. Gass, S. and T. Saaty : "The Computational Algorithm for the Parametric Objective Function," *Nav. Res. Log. Quart.*, **2**, 1/2 (1955), 39–45.
5. Gass, S. : *Linear Programming —— methods and applications*, 2nd ed., McGraw, New York, 1964.
6. Graves, R. L. : "Parametric Linear Programming," R. L. Graves and P. Wolfe eds, *Recent Advances in Mathmatical Programming*, McGraw, New York, 1963, 201–210.
7. Osawa, Y. : "*Senkei-keikaku-hō no Kihon Mondai* (Fundamentals of linear Programming)," (in Japanese), Sophia University, Tokyo, 1963, Section 4.2.
8. Shetty, C. M. : "On Analyses of the Solution to a Linear Programming Problem," *Oper. Res. Quart.*, **12**, 2 (1961), 89–104.