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On Vibration of an Elastic Bar immersed in the Water Region lying between two parallel Plane Walls—II.

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Abstract

Free vibration of an elastic bar of circular cross section, which is immersed in a water region lying between two parallel plane rigid walls, has been studied, as a problem of hydro-elasticity. The vibration is assumed to be of infinitesimally small amplitude, and the water is regarded to be an incompressible ideal fluid. The end conditions of the elastic bar are taken to be that of clamped ends. This is the continuation of previous study, by the author, of the same title, wherein the case of elastic bar with supported ends has been reported.

I. Introduction

The author has, in a previous occasion,[1] reported the result of his study, about the same subject. It was the study of an elastic (circular) bar, which is immersed in a water region lying in between two infinite (rigid) parallel plane walls. Also, the elastic bar was assumed to be fixed to the plane walls, in the state of supported ends.

In the present paper, the study is made about the same problem, except that the elastic bar is fixed in the state of fixed (clamped) ends. Namely, referring to Fig. 1, there is a water region which is bounded by two parallel (infinite) plane walls kept at a distance l apart, and extending to infinity. An elastic bar, of uniform circular section, and of length l, is attached perpendicular to two plane-walls. Both ends of the bar is in the state of fixed (clamped) ends. When the bar is vibrating, the surrounding water will also make a oscillatory motion. In the present paper, this problem of hydroelasticity will be treated, the water being assumed to be an ideal, nonviscous fluid, and the vibration being regarded as of infinitesimally small amplitude.

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[1] This PROCEEDINGS, Vol. 10, No. 39, 1957.

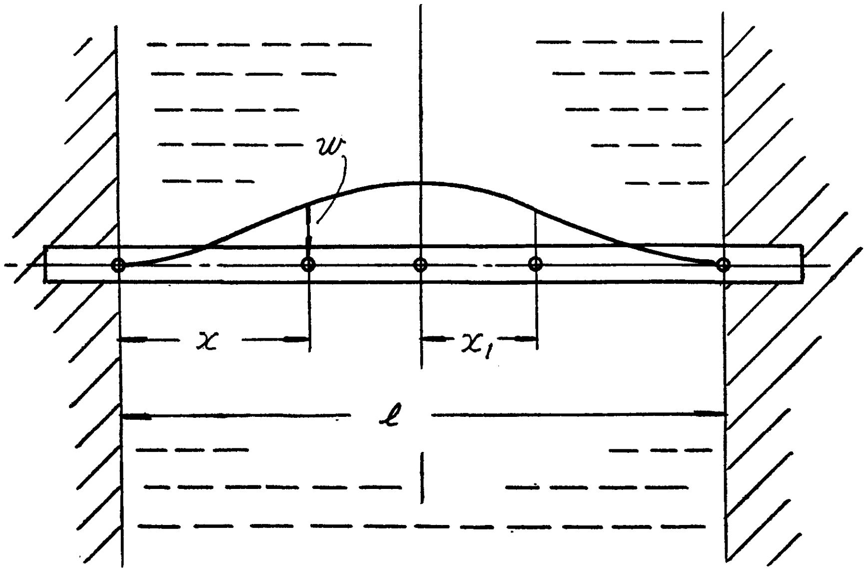

Fig. 1. Elastic bar immersed in a water region.

II. Notations

In this paper, the following notations will be used : ——

x=coordinate of a point on the center line of the circular bar, θ=angular coordinate, r=radial distance from the center line of the bar, thus (r, θ, x) constitute a system of cylindrical coordinates of a point in the water region. ω=angular frequency of vibration, a=radius of the circular bar, EI=flexural rigidity of cross-section of the circular bar, A=cross sectional area of the circular bar ($=\pi a^2$ if the bar is not hollow), w=transverse (infinitesimally small) displacement of the bar, ρ_m=density of the material composing the bar, ρ_w=density of water, l=length of the circular bar, p=hydraulic pressure, $\eta=x/l$, $m=\pi/l$, ϕ=the velocity potential which gives the vibratory motion of water, $\varphi_\mu(\eta)$=eigen-functions which correspond to transverse vibration of an elastic bar whose both ends are clamped-in, m_μ=eigen-values corresponding to $\varphi_\mu(\eta)$, μ=integers.

It is to be noted that we take the origin of coordinate x at left-end of the bar. When we are considering solely the state of vibration which is symmetric about the mid-point $x=l/2$, of the bar, we take $\mu=1, 3, 5, \cdots\cdots$. The functions $\varphi_\mu(\eta)$ are then even functions of $\left(\eta-\frac{1}{2}\right)$, for $-\frac{1}{2}\leqq\left(\eta-\frac{1}{2}\right)\leqq+\frac{1}{2}$.

III. Equation of vibration and its solution

As a preliminary remark, we note that, when a function (which satisfies the condition of Dirichlet) is given for $0 \leqq x \leqq l$, (see Fig. 2), we can represent it by the Fourier series of the form

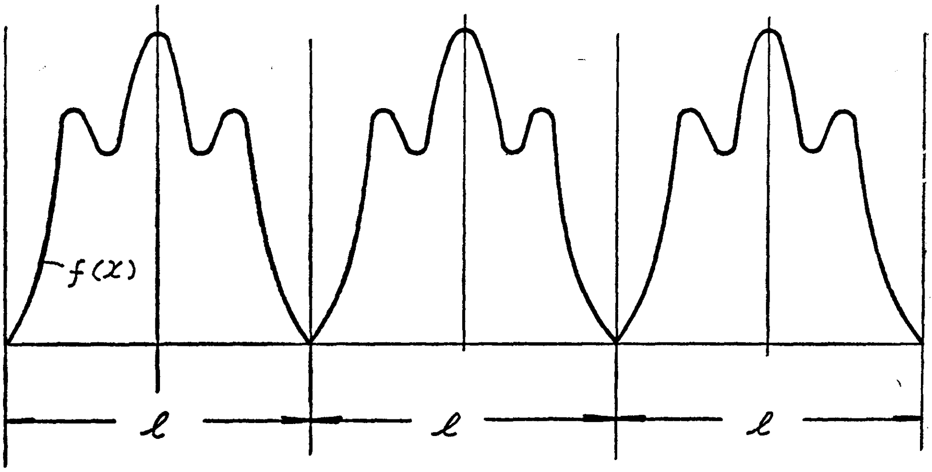

Fig. 2. Series of curves $f(x)$, arranged with a period of l.

$$b_0 + \sum_{n=1}^{\infty} a_n \sin \frac{2n\pi}{l} x + \sum_{n=1}^{\infty} b_n \cos \frac{2n\pi}{l} x .$$

Also, we note that, if the given function $f(x)$ is symmetric about the mid-point $x=l/2$, we shall have $a_n=0$.

Next, for the case in which the circular bar is vibrating in a form expressed by

$$w = \cos \frac{2n\pi}{l} x \cos \omega t = \cos mix \cos \omega t,$$

where $i=0, 2, 4, \cdots\cdots$, corresponding expression for the velocity potential ϕ may be given, in the form : ——

$$\phi_i = \Phi_i(r, x) \sin \theta \sin \omega t,$$

where we put,

$$\Phi_i(r, x) = A_i F_i(r) \cos mix \sin \theta \cos \omega t,$$

$$F_i(r) = K_1(mir), \quad A_i = -\frac{\omega}{(mi)K_1'(mia)} .$$

It is to be observed that, by the above form of velocity potential ϕ_i, we have $\nabla^2\phi=0$, and also that the conditions at two rigid plane walls, namely, at $x=(0$ and $l)$, $\partial\phi_i/\partial x=0$, are satisfied.[2]

[2] see the author's previous paper.[1]

Now, the equation of motion of the elastic bar is given by

$$EI\frac{\partial^4 w}{\partial x^4}+\rho_m A\frac{\partial^2 w}{\partial t^2}-p_a=0, \tag{1}$$

where p_a is the resultant force of the hydraulic pressure acting on the surface $r=a$ of the bar, at the position x, per unit length of the bar. Using the other variable η defined by $\eta=x/l$, this equation (1) can be rewritten in the following form :——

$$\frac{EI}{l^4}\frac{\partial^4 w}{\partial \eta^4}+\rho_m A\frac{\partial^2 w}{\partial t^2}-p_a=0. \tag{2}$$

In order to obtain the solution of this equation (2), which corresponds to the case of free vibration of the bar, both ends of which are rigidly clamped, let us put

$$w=\sum_{\mu=1}^{\infty} G_\mu \varphi_\mu(\eta)\cos\omega t, \tag{3}$$

where $\varphi_\mu(\eta)$ are solutions of the equation

$$\frac{d^4\varphi_\mu(\eta)}{d\eta^4}=m_\mu{}^4\varphi_\mu(\eta), \quad (\mu=1,\ 2,\ 3,\ \cdots\cdots)$$

which satisfy the end-conditions that at $\eta=0$ or $\eta=1$, we have $\varphi_\mu(\eta)=0$, $\varphi_\mu'(\eta)=0$.

We observe that, $\varphi_\mu(\eta)$, being eigenfunctions, will also satisfy the conditions of normalized orthogonality ; ——

$$\int_0^1 \varphi_\mu(\eta)\varphi_\nu(\eta)\,d\eta=0, \quad (\mu\neq\nu)$$

$$\int_0^1 [\varphi_\mu(\eta)]^2\,d\eta=1.$$

Also, we note that the function $w(\eta)$ given by (3) will also satisfy the condition that at $\eta=0$ and $\eta=1$, we have $w(\eta)=0$, $w'(\eta)=0$.

As we are discussing the case of vibration-mode which is symmetrical about the mid-point $\eta=1/2$, of the bar, we shall take only functions $\varphi_\mu(\eta)$ of odd numbers of μ $(\mu=1,\ 3,\ 5,\ \cdots\cdots)$.

Substituting the expression (3) into the equation (2), we obtain,

$$\sum_\mu\left[\frac{EI}{l^4}m_\mu{}^4-\rho_m A\omega^2\right]G_\mu\varphi_\mu(\eta)\cos\omega t-p_a=0. \tag{4}$$

Multiplying, by $\varphi_\mu(\eta)$, the both sides of this equation (4), and integrating, from $\eta=0$ to $\eta=1$, we have, due to the above-mentioned normalized orthogonality of functions $\varphi_\mu(\eta)$: ——

$$\left[\frac{EI}{l^4}m_\mu{}^4-\rho_m A\omega^2\right]G_\mu\cos\omega t=\int_0^1 p_a\varphi_\mu(\eta)\,d\eta, \qquad (\mu=1,\ 3,\ 5,\cdots\cdots). \tag{5}$$

The hydraulic pressure p is given by

$$p=-\rho_w\frac{\partial\phi}{\partial t}.$$

So that we have, by taking its resultant force over the face $r=a$ of the bar

$$p_a=-\left[\rho_w\frac{\partial\phi}{\partial t}\right]_{r=a}\left[-\int_0^{2\pi} a\sin^2\theta\,d\theta\right]=\pi a\rho_w\left[\frac{\partial\phi}{\partial t}\right]_{r=a}. \tag{6}$$

Now, in the above, the transverse displacement w was expressed by (3). But, regarding $w(x)$ to be a periodic function of period l, we may also express $w(x)$, or $w(\eta)$, as follows;

$$w(\eta)=\sum_{i=0}^{\infty} w_i\cos(\pi i\eta)\cos\omega t, \tag{7}$$

where we take $i=0, 2, 4, \cdots\cdots$. w_i are constants representing the amplitudes of component waves. Corresponding to it, the velocity potential ϕ may be constructed, as follows : ——

$$\phi=\sum_{i=0}^{\infty} w_i\,\Phi_i(r,x)\sin\theta\sin\omega t, \tag{8}$$

($i=0, 2, 4, \cdots\cdots$). And, the value of the hydraulic force p_a will be given by,

$$p_a=(\pi\rho_w\omega a)\sum_{i=0}^{\infty} w_i\,\Phi_i(a,x)\cos\omega t. \tag{9}$$

From the expression (7), we deduce that,

$$w_i=2e_i\int_0^1 w(\eta)\cos(i\pi\eta)\,d\eta\cos\omega t, \tag{10}$$

where we have $e_0=1/2$, and $e_i=1$ if i=non-zero even integer.

Observing that, the values of w as given by (3) and (7) must be equal each other, and putting the expression (3) into (10), we have,

$$w_i=2e_i\sum_{\nu=1}^{\infty}G_\nu\left[\int_0^1\varphi_\nu(\eta)\cos(i\pi\eta)\,d\eta\right]\cos\omega t$$

$$=\sum_{\nu=1}^{\infty}G_\nu a_{i\nu}\cos\omega t,\qquad [\nu=1, 3, 5, \cdots\cdots]$$

where we put

$$a_{i\nu}=2e_i\int_0^1\varphi_\nu(\eta)\cos(i\pi\eta)\,d\eta;$$

($i=0, 2, 4, \cdots\cdots$; $\nu=1, 3, 5, \cdots\cdots$). Inserting this value of w_i into the equation (9), we obtain,

$$\int_0^1 p_a \varphi_\mu(\eta)\, d\eta = (\rho_w \omega \pi a) \cos \omega t \sum_i \sum_\nu G_\nu a_{i\nu} c_{i\mu},$$

where we put

$$c_{i\mu} = \int_0^1 \varphi_\mu(\eta) \Phi_i(a, x)\, d\eta = \omega a D_i \frac{a_{i\mu}}{2e_i},$$

$$D_i = \frac{A_i F_i(a)}{\omega a}.$$

Summing up the above mentioned calculations, we obtain the following system of equations;

$$[(Bm_\mu{}^4)\lambda^2 - 1]\, G_\mu = \varepsilon \sum_i \sum_\nu G_\nu a_{i\nu} a_{i\mu} \left(\frac{D_i}{2e_i}\right), \tag{5a}$$

where we put

$$B = \left[\frac{EI}{l^4}\right] \Big/ (\rho_m A), \qquad \varepsilon = [\rho_w \pi a^2]/[\rho_m A], \qquad \lambda = \frac{1}{\omega}.$$

Also, we are to take $i = 0, 2, 4, \cdots\cdots$; and $\mu,\ \nu = 1, 3, 5, \cdots\cdots$.

This equation (5a) is a system of simultaneous linear equations, with respect to unknown constants G_μ. If the determinant composed of the coefficients of G_μ in (5a), is put equal to zero, we shall obtain an equation which determines λ (and, in turn, ω), giving us the angular frequency of free vibration of the system composed of the elastic bar and the surrounding water.

IV. Approximate formula for the natural frequency of vibration of the given system

The above-mentioned determinantal equation may be written in the form,

$$\begin{vmatrix} k-1-\varepsilon B_{11}, & -\varepsilon B_{31}, & \cdots\cdots\cdots \\ -\varepsilon\left(\frac{m_1}{m_3}\right)^4 B_{13}, & k-\left(\frac{m_1}{m_3}\right)^4 - \varepsilon\left(\frac{m_1}{m_3}\right)^4 B_{33}, & \cdots\cdots \\ -\varepsilon\left(\frac{m_1}{m_5}\right)^4 B_{15}, & -\varepsilon\left(\frac{m_1}{m_5}\right)^4 B_{35}, & \cdots\cdots\cdots \end{vmatrix} = 0, \tag{10}$$

where we put, as the new unknown quantity

$$k = (Bm_1{}^4)\lambda^2 = \frac{Bm_1{}^4}{\omega^2}.$$

Moreover, we put

$$B_{\nu\mu} = \sum_i a_{i\nu}\, a_{i\mu} \left(\frac{D_i}{2e_i}\right). \tag{11}$$

First, let us examine the special case, for which we have $a/l \to 0$. In this case, we have $D_i=1$. So that we have, according to theorem of Parceval,

$$B_{\nu\mu}=\sum_i\left[\frac{1}{2e_i}a_{i\mu}a_{i\nu}\right]=1(\text{if } \nu=\mu)\ ;\ =0\ (\text{if}\, \nu\neq\mu),$$

$$[k-1-\varepsilon]\cdot\left[k-\left(\frac{m_1}{m_3}\right)^4(1+\varepsilon)\right]\cdot\left[k-\left(\frac{m_1}{m_5}\right)^4(1+\varepsilon)\right]\cdots\cdots=0,$$

and its roots will be given by

$$k_1=1+\varepsilon,\qquad k_3=(1+\varepsilon)\left(\frac{m_1}{m_3}\right)^4,\qquad \text{etc., etc.} \tag{12}$$

Keeping this roots for the special case in mind, we shall discuss the approximate solution for the root of the equation (10), for the general case in which the value of the ratio a/l is not zero, but has a small value in comparison with unity. For this purpose, some values of numerical coefficients D_i and $a_{i\mu}$ are shown.

Table 1. Values of D_i.

$a/l=$	1/10	1/5	1/2.5
D_0	1.00	1.00	1.00
D_2	0.725	0.517	0.321
D_4	0.517	0.321	0.179
D_6	0.398	0.230	0.124
D_8	0.321	0.179	0.0945

Table 2. Values of $a_{i\mu}$.

	$\mu=1$	$\mu=3$	$\mu=5$	$\mu=7$
$i=0$	0.828	-0.364	0.223	-0.1650
2	-0.784	-0.818	0.4556	-0.3322
4	-0.0342	1.040	0.632	-0.362
6	-0.0062	0.00962	-1.120	-0.570
8	-0.0020	0.0278	-0.1394	1.164
10			-0.0468	-0.1612
12			-0.0244	0.0630
14				0.0342

Using these values of Tables 1 and 2, we can obtain the values of $B_{\nu\mu}$, as defined by (11), for a given value of a/l. For instance, some values of $B_{\mu\nu}$ for the case of $l/a=5$ $[l/(2a)=2.5]$ are shown in Table 2.

Table 3. Values of $B_{\mu\nu}=B_{\nu\mu}$, for the case of $l/a=5$.

	$\mu=1$	$\mu=3$	$\mu=5$
$\nu=1$	0.844	−0.141	0.085
3	−0.141	0.379	−0.032
5	0.089	−0.032	0.314

Moreover, since

$$m_1=4.730, \qquad m_3=10.996, \qquad m_5=17.279,;$$

we have,

$$\left(\frac{m_1}{m_3}\right)^4 \doteqdot \frac{1}{29}, \qquad \left(\frac{m_5}{m_1}\right)^4 \doteqdot \frac{1}{180}, \qquad \text{etc., etc}$$

The value of ε, or

$$\varepsilon = \frac{\rho_w}{\rho_m}\frac{\pi a^2}{A},$$

is usually less than 1. Even in the case of a hollow tube, ε will have values which is comparable with unity. Under these circumstances, the determinantal expression on the left hand side of equation (10) will have values as follows,

$$\begin{vmatrix} \circledcirc & \triangle & \triangle & \triangle & \cdots\cdots \\ \times & \circledcirc & \times & \times & \cdots\cdots \\ \times & \times & \circledcirc & \times & \cdots\cdots \\ \times & \times & \times & \circledcirc & \cdots\cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots\cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots\cdots \end{vmatrix},$$

where the mark $\triangle$ represents quantities of order of 0.10, while the mark $\times$ represents quantities of order of 1/300, or much less than that. Under these circumstances, we may put, approximately,

$$k=1+\varepsilon B_{11}=\lambda^2(Bm_1^4),$$

or

$$\omega^2=\frac{1}{\lambda^2}=\frac{Bm_1^4}{(1+\varepsilon B_{11})}, \qquad (\omega=2\pi f)$$

for angular frequency ω of fundamental mode of free vibration (for the case of vibration of infinitesimally small amplitude).

V. Another solution

The author has derived another mothod of solution, for the same problem. The final result showed us that this second solution is less convenient for the numerical evaluation. Nevertheless, this solution is given below, thinking that it may afford us some theoretical interests.

The same notation as before, will be used, the only difference being that, here the independent variable x_1 is used instead of x. This new variable x_1 is taken as the distance from mid-point of the bar, so that we have $x_1 = x - \frac{1}{2}l$. Also, we put $\xi = x_1/l$. Naturally we have, $\xi = \eta - \frac{1}{2}$.

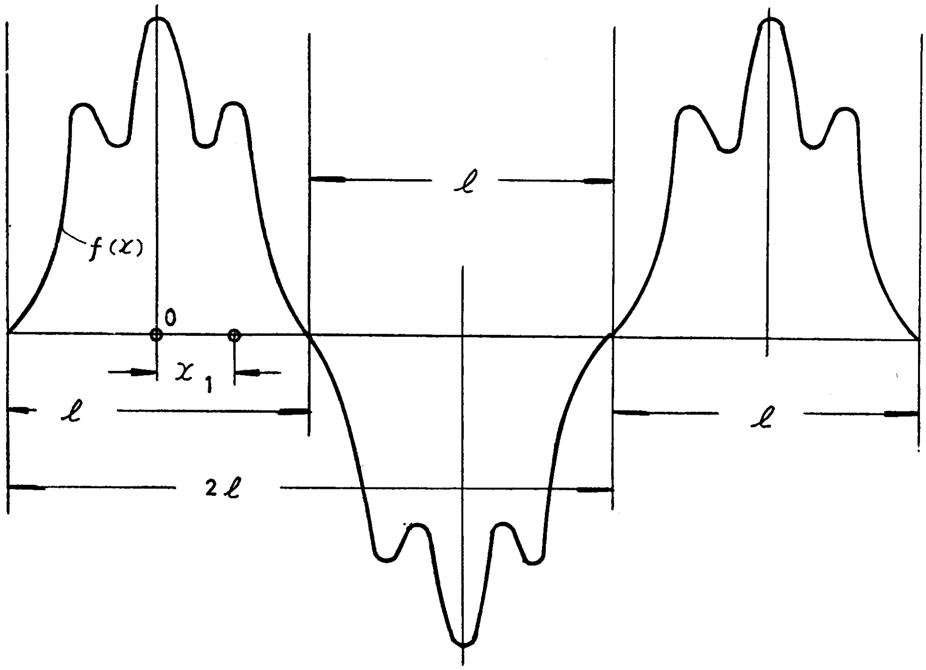

Fig. 3. Series of curves $f(x)$, arranged with a period of $2l$.

Here, we note that the given function $f(x)$ (for $0 \leqq x \leqq l$) may be regarded as arranged in a series of curves, as shown in Fig. 3, obtaining thus, a series of periodic functions with a period of $2l$ (instead of l, as formerly done). Thus we may expand it in the Fourier Series of the form;

$$\sum_{n=1}^{\infty} a_n \cos\left(n\pi \frac{x_1}{l}\right), \qquad (n=1, 2, 3, \cdots\cdots).$$

Also, we remind that, when the displacement w of the bar is given by,

$$w = \cos knx_1 \cos \omega t \qquad \left(k=\frac{\pi}{l},\ n=1, 2, 3, \cdots\cdots\right),$$

the corresponding form of the velocity potential ϕ_n, giving the vibratory motion of the surrounding water, has already been obtained in the following form[3];

$$\phi_n = \Phi_n(r, x_1) \sin\theta \sin\omega t,$$

where we have put

$$\Phi_n(r, x_1) = A_n F_n(r) \cos knx_1 + \int_0^\infty s f_n(s) S_1(s, r) \cosh sx\, ds,$$

$$A_n = \frac{-\omega}{kn\, K_1'(kna)},$$

$$F_n(r) = K_1(knr),$$

$$S_1(s, r) = J_1(sr) Y_1'(sa) - Y_1(sr) J_1'(sa),$$

$$f_n(s) = \frac{\omega nk}{s^2 M(s, s) \sinh(sl/2)} \cdot \frac{1}{s^2 + (nk)^2},$$

$$M(s, s) = [Y_1'(sa)]^2 + [J_1'(sa)]^2.$$

The equation of vibration of the elastic bar, will be taken, as before,

$$EI \frac{\partial^4 w}{\partial x^4} + \rho_m A \frac{\partial^2 w}{\partial t^2} - p_a = 0, \tag{1'}$$

and its solution in the form;

$$w = \sum_{\mu=1}^{\infty} F_\mu \varphi_\mu(\xi) \cos\omega t, \tag{3'}$$

and also,

$$w = \sum_{n=1}^{\infty} w_n \cos(n\pi\xi) \cos\omega t, \tag{7'}$$

where we put, $\xi = x_1/l$, $\mu = 1, 3, 5, \cdots\cdots$, $n = 1, 2, 3, \cdots\cdots$.
The resultant force p_a, of the hydraulic pressure p, becomes,

$$p_a = \rho_w \pi a \left[\frac{\partial\phi}{\partial t}\right]_{r=a} \tag{6'}$$

$$= (\pi\rho_w \omega a) \sum_{n=1}^{\infty} w_n \Phi_n(a, x) \cos\omega t. \tag{9'}$$

Substituting the expression (3′) into the equation of vibration (1′), we obtain,

$$\sum_\mu \left[\frac{EI}{l^4} m_\mu^4 - \rho_m A\omega^2\right] F_\mu \varphi_\mu(\xi) \cos\omega t - p_a = 0. \tag{4'}$$

whence we have, as before,

$$\left[\frac{EI}{l^4} m_\mu^4 - \rho_m A\omega^2\right] F_\mu \cos\omega t = \int_{-1/2}^{+1/2} p_a \varphi_\mu(\xi)\, d\xi, \tag{5'}$$

[3] see the previous paper, by the author.[1]

($\mu=1, 3, 5, \cdots\cdots$).

Next, by comparison of expressions (3′) and (7′), we have

$$w_n=\sum_{\mu=1}^{\infty} F_\mu a_{n\mu},$$

where we put,

$$a_{n\mu}=2\int_{-1/2}^{+1/2} \varphi_\mu(\xi)\cos(n\pi\xi)\,d\xi,$$

($n=1, 2, 3, \cdots\cdots$; $\mu=1, 3, 5, \cdots\cdots$). Actually we take only values for n=odd integers. Also, we have,

$$\int_{-1/2}^{+1/2} p_a\varphi_\mu(\xi)\,d\xi=(\rho_w\pi a\omega)\cos\omega t\cdot\sum_n\sum_\nu F_\nu a_{n\nu}[\omega a b_{n\mu}],$$

where we have put,

$$b_{n\mu}=\frac{1}{\omega a}\int_{-1/2}^{+1/2} \varphi_\mu(\xi)\,\Phi_n(a, x_1)\,d\xi.$$

Summing up these calculations, we find that the equation (5′) may be written in the following form, in which we put,

$$B=\left(\frac{EI}{l^4}\right)\Big/(\rho_m A), \quad \varepsilon=\frac{\rho_w\pi a^2}{\rho_m A}, \quad \frac{1}{\omega}=\lambda\,;$$

$$[(Bm_\mu^4)\lambda^2-1]F_\mu=\varepsilon\sum_{n=1}^{\infty}\sum_{\nu=1}^{\infty}F_\nu a_{n\nu}b_{n\mu}, \tag{5'a}$$

($n=1, 3, 5, \cdots\cdots$; μ, $\nu=1, 3, 5, \cdots\cdots$).

The equation (5′a) is a system of linear simultaneous equations (which is homogeneous in F_μ). And it has all-nonzero solutions F_μ only if the determinant Δ_1, whose elements are coefficients of F_μ in the equation (5′a), is equal to zero. In this way we can also obtain values of angular frequency ω corresponding to free vibration of the system, composed of elastic circular bar and the surrounding fluid. For making the numerical evaluation, it is convenient to write,

$$\Phi_n(a, x_1)=A_nF_n(a)\cos(knx_1)$$
$$+\int_0^\infty \frac{(kna)(\omega a)}{[\xi^2M(\xi)]\sinh\{l\xi/(2a)\}}\cdot\frac{\cosh(\xi x_1/a)}{(nka)^2+\xi^2}\,d\xi\,,$$

where we put $\xi=sa$.