Title	On a nonlinear but perfectly sinusoidal oscillator
Sub Title	
Author	藤田, 広一(Fujita, Hiroichi)
Publisher	慶応義塾大学藤原記念工学部
Publication year	1963
Jtitle	Proceedings of the Fujihara Memorial Faculty of Engineering Keio University (慶應義塾大学藤原記念工学部研究報告). Vol.16, No.61 (1963.) ,p.29(1)- 34(6)
JaLC DOI	
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Notes	
Genre	Departmental Bulletin Paper
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=KO50001004-00160061-0001

On a nonlinear but perfectly sinusoidal oscillator

(Received March 23, 1964)

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Abstract

Wave form of a nonlinear oscillator is usually distorded. But in this paper some examples of nonlinear oscillators are shown whose wave form are perfectly sinusoidal. The nonlinear differential equation for such oscillator and the exact solutions are discussed.

I. Introduction

An oscillator contains generally some active elements, such as a transistor or a vacuum tube. Energy generated from such active elements compensates for the energy lost in passive elements. If the active element has a linear characteristic and the dissipation is less than the supplied energy, the amplitude increases infinitely. If the loss from resistance in the circuit and the negative loss from the linear active element were exactly balanced, then the amplitude would be constant. But in such a condition the oscillation not being structurally stable can not be realized.

An actual active element has a nonlinear characteristic. At the initial state of oscillation, the amplitude is so small that the active nonlinear element may be regarded as linear and the amplitude increases exponentially. For a considerably large amplitude, the nonlinearlity of the active element suppresses the amplitude and the oscillator is maintained constantly. So the wave form of a nonlinear oscillator is usually distorted.

However such distortion need not necessarily occur in nonlinear oscillation. In other words, there may be a nonlinear oscillator whose wave form is perfectly sinusoidal. In this paper, some examples of such sinusoidal nonlinear oscillators will be shown and discussed.

II. The nonlinear differential equation with sinusoidal solutions

Nonlinear differential equation

$$x'' + \varepsilon (x^2 + x'^2 - 1) x' + x = 0 \tag{1}$$

has sinusoidal solutions

$$x = \sin t \quad \text{or} \quad x = \cos t . \tag{2}$$

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This can be ascertained by direct subsitution. (1) above is not linear; the sum of the solutions is not its solution. But

$$x=\sin(t+\varphi) \tag{3}$$

is also a solution, where φ is an arbitrary phase angle.

This differential equation is considered as a special case of the equation

$$x''+\varepsilon(px'^2+qx^2-1)x'+x=0\,. \tag{4}$$

If $p=q=1$ in (4), it becomes (1).

If $p=0$ and $q=1$, (4) becomes van der Pol's equation.

If $p=1$ and $q=0$, it becomes Rayleigh's equation.

Amplitude A of the solution of (4) which is obtained by the harmonic balance method is

$$A=\frac{2}{\sqrt{q+3p}}\,. \tag{5}$$

$A=2$ for van der Pol's equation and $A=2/\sqrt{3}$ for Rayleigh's equation. For (1), $A=1$. The former two amplitudes are not exact, but approximate. The latter is obtained without approximation.

III. Nonlinear circuits corresponding to nonlinear differential equation (4)

What kinds of circuits may be represented by the equation (1) or (4)? The differential equation is of the second order, therefore it may be naturally supposed that the circuit contains one inductance and one capacitance. However no resonance circuit of L, C, and R, some of which have nonlinear characteristics is represented by the eq. (1) or (4). It is probably impossible to compose a circuit corresponding to such equations of simple two-terminal nonlinear elements. Then a slightly more complicated two-terminal nonlinear element must be considered. That is a mutual resistance which is defined as follows:

i) A two-terminal nonlinear element whose voltage v_n depends on the current i in some other element,

$$v_n=f(i)\,.$$

ii) A two-terminal nonlinear element whose current i_n depends on voltage v on some other element,

$$i_n=g(v)\,.$$

Such characteristics often are found in vacuum tubes or transistors.

Now consider a circuit composed of such elements and linear L and C and a

simple two-terminal nonlinear element. An example is shown in Fig. 1.
The equations of currents and voltages are,

$$Li'+v_n-v_N=0\ ,$$

$$i=Cv_n'+i_n\ .$$

The notations are shown in Fig. 2. The nonlinear characteristics are

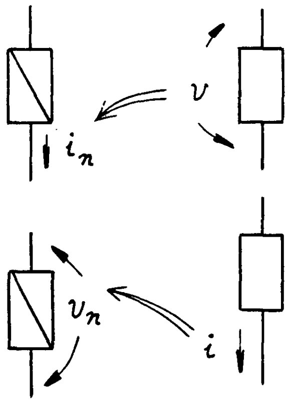

Fig. 1.
Mutual resistonec.

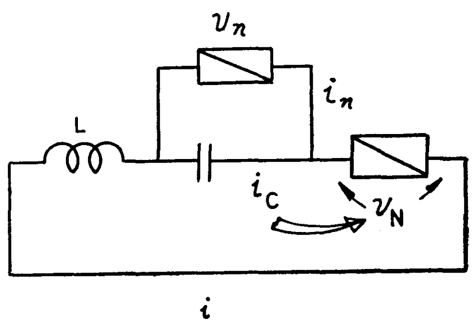

Fig. 2.

$$i_n=-g_1v_n+g_3v_n^3\ ,$$

$$v_N=f(i_c)=\beta v_n'^3\ .$$

After simple calculation from these equations, a second order nonlinear differentinal equation for v_n is obtained,

$$v_n''+\frac{3g_3}{C}v_n^2v_n'+\frac{\beta}{LC}v_n'^3-\frac{g_1}{LC}v_n'+\frac{1}{LC}v_n=0\ . \tag{5}$$

Normalize the time t by $\omega t=\tau$, where $\omega^2=1/LC$ and voltage by $x=v_n/v_0$ where v_0 is unit voltage.
Then the equation (5) is reduced to

$$x''+\varepsilon(px^2+qx'^2-1)x'^2+x=0\ , \tag{6}$$

where $p=(3g_3/g_1)L$,

$q=(\beta/g_1)/\sqrt{LC}$ and $\varepsilon=g_1/\sqrt{LC}$.

The dual circuit to this is shown in Fig. 3. It is also represented by the same type differential equation.
The plate tuned vacuum tube oscillator circuit is represented by such an equation. In Fig. 4

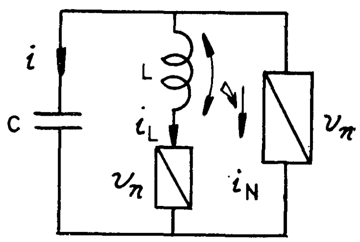

Fig. 3.

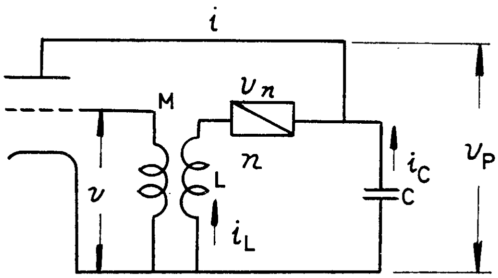

Fig. 4. Sinusoidal plate tuned oscillator.

a nonlinear element has a saturation characteristic as

$$v_n = \alpha i_L^3 .$$

The characteristic of the vacuum tube is represented by

$$i_p = -gv + g_3 v^3 .$$

The fundamental equations are

$$i_c + i_L = i_p ,$$

$$v_p = v_n + L i_L' ,$$

$$i_c = C v_p' .$$

From these equations, the second order differential equation for i_L is obtained;

$$i_L'' + \frac{3\alpha}{L} i_L^2 i_L' + \frac{g_3 M}{LC} i_L'^2 - \frac{gM}{LC} i_L' + \frac{1}{LC} i_L = 0. \tag{7}$$

Normalize the time $t = \tau/\omega$ and transform the constants in this equation as follows

$$i_L = i_0 x , \qquad \alpha_0 = \sqrt{\frac{g}{3\alpha} \frac{M}{C}} ,$$

$$p = \frac{3\alpha}{g} \frac{C}{M} i_0^2 , \qquad q = \frac{g_3}{g} \frac{M^2}{LC} i_0^2 ,$$

$$\varepsilon = gM\omega .$$

Then (7) is reduced to (6). If $M^3 g_3 = 3\alpha LC$, then $p = q = 1$ and the solution of (7) is perfectly sinusoidal.

IV. Forced oscillation to the equation (4)

In forced oscillation of (4), the locking phenomenon of frequency is observed as in forced oscillation to the van der Pol's or Rayleigh's oscillator.

The response curves are shown in Fig. 5. These curves are obtained by the harmonic balanced method. Substitute

$$x = A \sin \omega t + B \cos \omega t$$

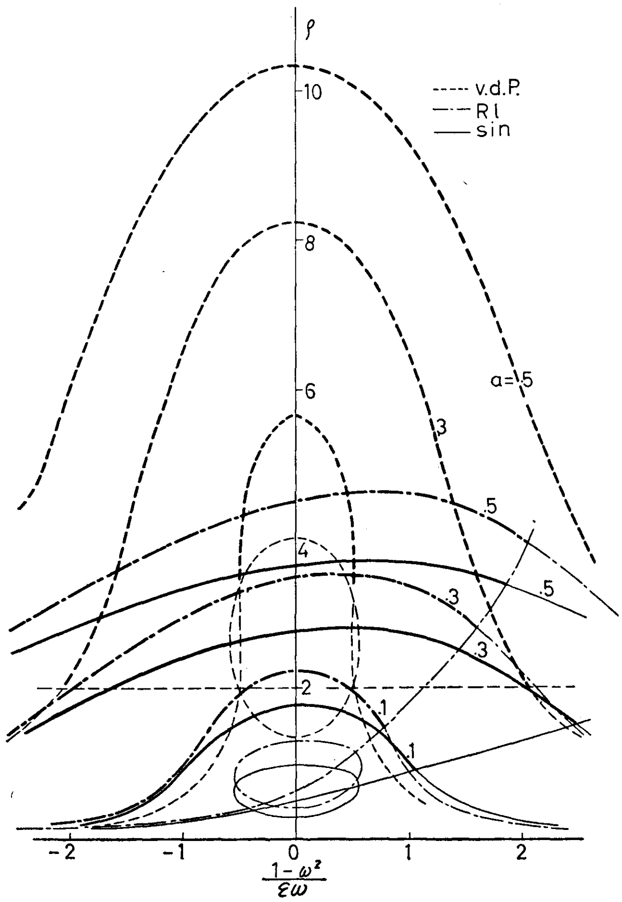

Fig. 5.

to the differential equation

$$x''+\varepsilon(x^2+x'^2-1)x'+x=a\sin\omega t\,,$$

where A and B are slowly varying time functions. Let $A^2+B^2=\rho$ and the response ρ for frequency ω is expressed

$$a^2=(1-\omega^2)\rho+(\varepsilon/16)\omega^2\rho\{1+3\omega^2)\rho-4\}^2$$

for (4). For van der Pol's equation

$$a^2=(1-\omega^2)\rho+(\varepsilon/16)\omega^2\rho\{\rho-4\}^2\,,$$

and for Rayleigh's equation

$$a^2=\omega^2\rho+(\varepsilon/16)\omega^2\rho\{3\omega^2\rho-4\}^2\,.$$

The condition of stability for ρ are

$$\sqrt{1+3\omega^2}\,\rho>2\,,$$

and

$$(1/16)\{3(1+3\omega^2)^2\rho^2-16(1+3\omega^2)\rho+16\}+\frac{(\omega^2-1)^2}{\varepsilon^2\omega^2}>0.$$

The condition of stability for van der Pol's equation are

$$\rho>2, \qquad (1/16)\{3\rho^2-16\rho+16\}+\frac{(\omega^2-1)^2}{\varepsilon^2\omega^2}>0,$$

and for Rayleigh's equation,

$$\sqrt{3\omega^2}\,\rho>2, \qquad (1/16)\{3(3\omega^2)\rho^2-16(3\omega^2)\rho+16\}+\frac{(\omega^2-1)^2}{\varepsilon^2\omega^2}>0.$$

V. The trajectories in the phase plane

The trajectories on the phase plane for self-excited oscillation are shown in Fig. 6. a) is for the sinusoidal oscillator, b) is for Rayleigh's equation and, c) is for van der Pol's equation. These are obtained by analogue computer.

The distortion of van der Pol's oscillator is larger than that of Rayleigh's oscillator. The amplitude of van der Pol's oascillator is supressed by the nonlinear characteristic for x while in Rayleigh's oscillator x' suppresses its amplitude and x is its integration so it has a smaller distortion than x'. In a perfectly sinusoidal oscillator, $x^2+x'^2$ control its amplitude; there is no distortion.

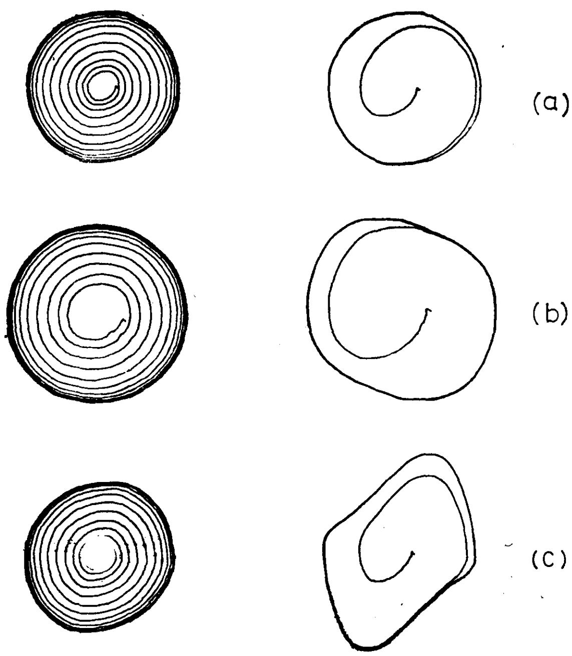

Fig. 6.

VI. Acknowledgement

The author wishes to acknowledge with thanks, Mr. T. Kitazumi, Mr. S. Mori and Mr. M. Yokoyama for the caluculation by computers.