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Abstract	When side-walls of a rectangular tank, which is filled with water, are vibrating, the inside water will also make a vibratory motion. This motion of water lowers considerably the natural frequency of vibration of side-walls of the tank. This effect is conveniently expressed by "virtual mass" of water. In the previous reports, I to VII, of the same title, the author has made theoretical studies on the value of "virtual mass" of water, and examined various factors affecting it. Especially, in the report VI, an approximate formula for the virtual mass was given for the case in which two opposite (rectangular) side-walls are vibrating in a mode which correspond to the case of "rectangular plate with clamped four-edges." The calculation was made, by assuming tentatively, the mode of vibration of the rectangular plate. The question of degree of accuracy of this approximate formula will naturally be raised. In the present report VIII, this question of degree of accuracy is taken up. The treatment may be said to be a case of hydro-elasticity. A set of normalized orthogonal functions (which correspond to the mode of free vibration of elastic bar with fixed ends) is used. The transverse displacement w of the rectangular plate in vibration is expressed as a double infinite series of these set of functions. Putting this expression into the equation of vibratory motion of rectangular elastic plate (wherein, the effect of vibratory water pressure is taken into account), a system of linear equations about the component amplitudes Aaβ (a,β= 1, 2, 3, ...) is obtained. And, thence, equations for Aaβ is made, by means of which the values of Aaβ can be obtained by successive (iterative) approximations. We start with A11= 1, and all the others (Aaβ)=0, which we regard as the zeroth approximation. And, we are to calculate first, second, ... , approximate values. Numerical examples for the case of (A). B : H : L=1: 1: 2, and (B). B : H : L=1: 1: 1, are shown. It is seen that the values of A12/A11, A31/A11, ... , are comparatively small, but not so small enough to say that we may neglect them at all. It is concluded that, we may use the approximate formula given in our report VI, for practical purposes, on the undererstanding that they give olny approximate values.
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On Virtual Mass of Water contained in a Rectangular Tank, whose Side-Walls are Vibrating—VIII.

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Abstract

When side-walls of a rectangular tank, which is filled with water, are vibrating, the inside water will also make a vibratory motion. This motion of water lowers considerably the natural frequency of vibration of side-walls of the tank. This effect is conveniently expressed by "virtual mass" of water. In the previous reports, I to VII,[1] of the same title, the author has made theoretical studies on the value of "virtual mass" of water, and examined various factors affecting it. Especially, in the report VI, an approximate formula for the virtual mass was given for the case in which two opposite (rectangular) side-walls are vibrating in a mode which correspond to the case of "rectangular plate with clamped four-edges." The calculation was made, by assuming tentatively, the mode of vibration of the rectangular plate.

The question of degree of accuracy of this approximate formula will naturally be raised. In the present report VIII, this question of degree of accuracy is taken up. The treatment may be said to be a case of hydro-elasticity. A set of normalized orthogonal functions (which correspond to the mode of free vibration of elastic bar with fixed ends) is used. The transverse displacement w of the rectangular plate in vibration is expressed as a double infinite series of these set of functions. Putting this expression into the equation of vibratory motion of rectangular elastic plate (wherein, the effect of vibratory water pressure is taken into account), a system of linear equations about the component amplitudes $A_{\alpha\beta}$ (α, $\beta=1, 2, 3, \cdots\cdots$) is obtained. And, thence, equations for $A_{\alpha\beta}$ is made, by means of which the values of $A_{\alpha\beta}$ can be obtained by successive (iterative) approximations. We start with $A_{11}=1$, and all the others $(A_{\alpha\beta})=0$, which we regard as the zeroth approximation. And, we are to calculate first, second, $\cdots\cdots$, approximate values.

Numerical examples for the case of (A). $B:H:L=1:1:2$, and (B). $B:H:L=1:1:1$, are shown. It is seen that the values of A_{12}/A_{11}, A_{31}/A_{11}, $\cdots\cdots$, are comparatively small, but not so small enough to say that we may neglect them at all. It is concluded that, we may use the approximate formula given in our report VI, for practical purposes, on the undererstanding that they give olny approximate values.

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I. Introduction

The author has already made seven reports under the same title as the present paper. In the reports I to VI, approximate formulae for natural frequency of vibration of rectangular elastic plates, which constitute side-walls of rectangular water tank, have been given for different cases. It is to be noted that approximate formulae in these reports were obtained by calculation of kinetic energy (of water and the plate) and the potential energy (of the plate), assuming tentatively the mode of transverse vibration of the rectangular plate.

In the report VII, a consideration was given about the degree of accuracy of the approximate formula, restricting ourselves to the case of side-walls which are in state of "rectangular elastic plate with simply supported four edges." Now, in the present paper, we shall take up the case of side walls which are in the state of "rectangular elastic plate with clamped four edges." And, shall examine the degree of accuracy of the approximate formula (already given in the report VI) for the natural frequency of vibration.

The conclusion is, in a word, that the degree of accuracy of the approximate formula given in the report VI, is not so good as in the case of "simply supported four edges," but that it may serve as a practical approximate formula. The calculation throughout is made, by assuming water (or, other fluid) to be an incompressible non-viscous fluid, and that the vibration amplitude is infinitesimally small.

Our treatment may be said to be dealing with a problem of hydro-elasticity.

II. Notation

Generally, the same notations as in our previous reports, will be used here. It is to be noted that here we use a system $\varphi_\alpha(\xi_1)$ of normalized orthogonal functions, as defined below, and some coefficients $a_{\alpha\beta}$, $F_{\mu i}/L$, $G_{\nu j}/H$, $Q^{\alpha\beta}$, etc., which are related to them are also used. We refer to a rectangular water tank as shown in Fig. 1.

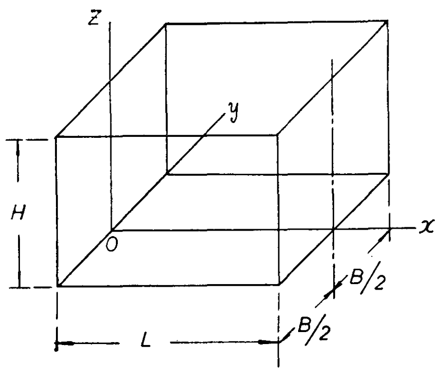

Fig. 1. *A* **sketch of rectangular water tank.**

L = length of the rectangular water tank, H = its height, B = its breadth, V = its inside volume, ϕ = velocity potential which represent the vibratory motion of water. (x, y, z) = rectangular coordinates, w = transverse displacement of the rectangular elastic plate, which constitute side-wall of the tank, A = amplitude of vibration of the rectangular plate, $w_0 = \omega A$ = amplitude of diplacement velocity of ditto, ω = angular frequency of natural vibration, t = time, ρ_w = density of water,

ρ_m=density of the material composing the plate, h=thickness of the plate, D=its flexural rigidity$=(1/12)Eh^3/(1-\nu^2)$, E=Young's modulus, ν=Poisson's ratio, p=water pressure due to vibratory motion of water, ∇^2=Laplacian operator$=(\partial^2/\partial x^2)+(\partial^2/\partial z^2)$, $m_1, m_2, \ldots\ldots$=eigenvalues defined below, $\xi_1=(x/L)-(1/2)$, $\zeta_1=(z/H)-(1/2)$. $m=\pi/L$, $s=\pi/H$, $n_{ij}=[(im)^2+(js)^2]^{1/2}$.

III. Fundamental equations of our problem

Let us consider a rectangular water tank, as shown in Fig. 1, inside which water is filled up. In this report VIII, we mainly take up the case of two opposite side-walls $(L\times H)$ vibrating in phase, or in opposite phase, each other. The top surface at $z=H$ will be considered as a free surface. We restrict ourselves to the case of infinitesimally small vibrations of an incompressible, non-viscous fluid. Furthermore, we assume that two panels $(L\times H)$ of side-walls are in the state of "rectangular elastic plate with clamped four edges."

The fundamental equation for our problem of free vibration is taken in the form: —

$$D\nabla^2\nabla^2 w+\rho_m h\frac{\partial^2 w}{\partial t^2}-p=0, \tag{1}$$

$$p=-\rho_w\frac{\partial\phi}{\partial t} \tag{2}$$

$$=\omega\sum_i\sum_j B_{ij}f_{ij}\left(\frac{B}{2}\right)\cos mix\ \cos siz\ \sin\omega t, \tag{3}$$

$$w=\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}A_{\mu\nu}\varphi_\mu(\xi_1)\varphi_\nu(\zeta_1)\sin\omega t, \tag{4}$$

$$\phi=\sum_i\sum_j B_{ij}f_{ij}(y)\ \cos mix\ \cos sjz\ \cos\omega t. \tag{5}$$

Equation (1) is the equation of transverse vibration of the a rectangular elastic plate, wherein the effect of (vibratory) water pressure p acting on it is taken into account. Equation (2) gives the value of the water pressure p in terms of the velocity potential ϕ, for the case of small vibrations. (5) is the solution of Laplace's equation $\nabla^2\phi=0$, together with boundary condition of our problem. (3) gives the value of p as derived from (5). We are to take $i=0, 2, 4, \ldots\ldots$; $j=1/2, 3/2, 5/2, \ldots\ldots$: We take

$$f_{ij}(y)=\cosh(n_{ij}y),$$

for the case of opposite-phase vibration of side-walls, while we take sinh $(n_{ij}y)$, instead of cosh $(n_{ij}y)$. Also, we have

$$n_{ij}B=\left[(iB/L)^2+(jB/H)^2\right]^{1/2}\pi.$$

We use a doubly-infinite series of the form (4) to express the transverse displacement w of the rectangular plate. The set of functions $\varphi_\mu(\xi_1)$, $\varphi_\nu(\zeta_1)$ are defined below:

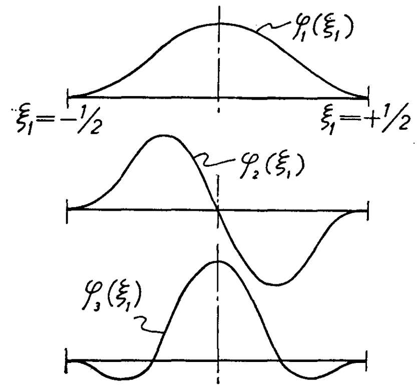

Fig. 2. A Set of fuuctions $\varphi_\alpha(\xi_1)$.

The set of functions $\varphi_\mu(\xi_1)$ are to be solutions of linear differential equations

$$\frac{d^4\varphi_\mu(\xi_1)}{d\xi_1^4}=m_\mu^4\varphi_\mu(\xi_1),$$

together with the boundary conditions

$$\varphi_\mu\left(\pm\frac{1}{2}\right)=0, \quad \varphi_\mu'\left(\pm\frac{1}{2}\right)=0,$$

where we take $\mu=1, 2, 3, \ldots\ldots$. m_μ are the value of μ-th eigenvalue. For odd μ, φ_1, φ_3, ... are even functions, while for even μ, φ_2, φ_4, ... are odd functions. (see Fig. 2).

$\varphi_\mu(\xi_1)$ ($\mu=1, 2, 3, 4, \ldots\ldots$) are to conststute a set of normalized orthogonal functions. Actual expressions for $\varphi_\mu(\xi_1)$ are as shown below:

(i). Case of even functions ($\mu=1, 3, 5, \ldots\ldots$),

$$\varphi_\mu(\xi_1)=A_\mu\cos m_\mu\xi_1+B_\mu\cosh m_\mu\xi_1, \tag{6}$$

$$A_\mu=(1/C_\mu)\sinh\left(\frac{1}{2}m_\mu\right),\quad B_\mu=(1/C_\mu)\sin\left(\frac{1}{2}m_\mu\right),$$

$$[2C_\mu]^2=\cosh m_\mu-\cos m_\mu+3(\cosh m_\mu-1)\frac{\sin m_\mu}{m_\mu}+3(1-\cos m_\mu)\frac{\sinh m_\mu}{m_\mu}.$$

(ii). case of odd functions ($\nu=2, 4, 6, \ldots\ldots$),

$$\varphi_\nu(\xi_1)=A_\nu\sin m_\nu\xi_1+B_\nu\sinh m_\nu\xi_1, \tag{7}$$

$$A_\nu=(1/C_\nu)\sinh\left(\frac{1}{2}m_\nu\right),\quad B_\nu=-(/C_\nu)\sin\left(\frac{1}{2}m_\nu\right),$$

$$[2C_\nu]^2=\cosh m_\nu-\cos m_\nu-3(1-\cos m_\nu)\frac{\sinh m_\nu}{m_\nu}-(\cosh m_\nu-1)\frac{\sin m_\nu}{m_\nu}.$$

For both cases of (6) or (7), by putting the boundary conditions to them, we see that m_μ must be roots of the equation

$$1=\cosh m\cdot\cos m. \tag{8}$$

IV. Coefficients appearing in the expansion about $\varphi_\mu(\xi_1)$, $\varphi_\nu(\zeta_1)$

The values of m_μ have already been given, in connection with the problem of free-vibration of an elastic bar with fixed ends. So, we have

$$m_1=4.7300=\frac{3\pi}{2}+0.0176,$$

$$m_2=7.853=\frac{5\pi}{2}-0.0009, \qquad m_3=10.996=\frac{7\pi}{2}+0.00001,$$

$$m_4=14.137 \doteqdot \frac{9\pi}{2}, \qquad m_5=17.279 \doteqdot \frac{11}{2}\pi.$$

That the set of functions $\varphi_\mu(\xi_1)$, $\varphi_\nu(\xi_1)$ are orthogonal to each other, and that we can expand a given function $F(\xi_1)$ in an infinte seres of Fourier's type (of course, on some restrictions about the nature of function $F(\xi_1)$), have already been pointed out. (see, for example, A. Kneser, Integralgleichungen). In the following calculations, we require some coefficients which depend on $\varphi_\mu(\xi_1)$. We shall here mention them, giving their expressions.

$$a_{\mu\alpha}=\int_{-1/2}^{+1/2}\varphi''_\mu(\xi_1)\varphi_\alpha(\xi_1)d\xi_1,$$

$$a_{\nu\beta}=\int_{-1/2}^{+1/2}\varphi''_\nu(\zeta_1)\varphi_\beta(\zeta_1)d\zeta_1.$$

It is to be noted that $a_{\mu\alpha}=a_{\alpha\mu}$. Also, that (i) when α is odd but μ is even, or (ii) when α is even but μ is odd, we shall have $a_{\mu\alpha}=0$. (iii) when μ, α are odd (that is, φ_μ and φ_α are even functions), we have

$$a_{\mu\alpha}=\frac{(m_\mu)^2}{C_\mu C_\alpha}\cdot A,$$

where $A=A_1+A_2+A_3+A_4$, with

$$A_1+A_4=-\left[\sinh\frac{m_\mu}{2}\sinh\frac{m_\alpha}{2}\right]\cdot\left[\frac{\sin\frac{1}{2}(m_\mu+m_\alpha)}{m_\mu+m_\alpha}+\frac{\sin\frac{1}{2}(m_\mu-m_\alpha)}{m_\mu-m_\alpha}\right]$$
$$+\left[\sin\frac{m_\mu}{2}\sin\frac{m_\alpha}{2}\right]\cdot\left[\frac{\sinh\frac{1}{2}(m_\mu+m_\alpha)}{m_\mu+m_\alpha}+\frac{\sin\frac{1}{2}(m_\mu-m_\alpha)}{m_\mu-m_\alpha}\right],$$

$$A_2+A_3=\frac{-2}{(m_\mu)^2+(m_\alpha)^2}\left[\sinh\frac{m_\alpha}{2}\sinh\frac{m_\mu}{2}\left\{m_\alpha\sin\frac{m_\alpha}{2}\cos\frac{m_\mu}{2}-m_\mu\sin\frac{m_\mu}{2}\cos\frac{m_\alpha}{2}\right\}\right.$$
$$\left.+\sin\frac{m_\alpha}{2}\sin\frac{m_\mu}{2}\left\{m_\mu\sinh\frac{m_\mu}{2}\cosh\frac{m_\alpha}{2}-m_\alpha\sinh\frac{m_\alpha}{2}\cosh\frac{m_\mu}{2}\right\}\right]\cdot$$

Also, following coefficients $F_{\mu i}/L$, for $i=0, 2, 4, \ldots\ldots$, and $G_{\nu j}/H$, for $j=1/2, 3/2, 5/2, \ldots\ldots$, are required;

$$\frac{F_{\mu i}}{L}=\int_0^L\varphi_\mu(\xi_1)\cos(mix)dx\cdot\frac{1}{L}, \qquad \frac{G_{\nu j}}{H}=\int_0^H\varphi_\nu(\zeta_1)\cos sjz dz\cdot\frac{1}{H}\cdot$$

Their actual values are:
For an odd value of μ,

$$\frac{F_{\mu i}}{H}=(-1)^{i/2}A_\mu\left[\frac{\sin\frac{m_\mu+\pi i}{2}}{m_\mu+\pi i}+\frac{\sin\frac{m_\mu-\pi i}{2}}{m_\mu-\pi i}\right]$$
$$+(-1)^{i/2}B_\mu\cdot\frac{2}{(\pi i)^2+(m_\mu)^2}m_\mu\sinh\frac{m_\mu}{2}\cos\frac{\pi i}{2},$$

For an odd value of ν,

$$\frac{G_{\nu j}}{H}=A_\nu\cos\frac{\pi j}{2}\left[\frac{\sin\left(\frac{m_\nu}{2}+\frac{\pi j}{2}\right)}{m_\nu+\pi j}+\frac{\sin\left(\frac{m_\nu}{2}-\frac{\pi j}{2}\right)}{m_\nu-\pi j}\right]$$

$$+B_\nu\cos\frac{\pi j}{2}\cdot\frac{2}{(m_\nu)^2+(\pi j)^2}\left[m_\nu\sinh\frac{m_\nu}{2}\cos\frac{\pi j}{2}+\pi j\cosh\frac{m_\nu}{2}\sin\frac{\pi j}{2}\right].$$

(iv). When ν and α are even (that is, φ_ν and φ_α are odd functions), we have

$$a_{\nu\alpha}=a_{\alpha\nu}=\frac{(m_\nu)^2}{C_\nu C_\alpha}\cdot A,$$

where $A=A_\nu+A_2+A_3+A_4$, with

$$A_1+A_4=-\left[\sinh\frac{m_\nu}{2}\sinh\frac{m_\alpha}{2}\right]\cdot\left[\frac{\sin\frac{1}{2}(m_\nu-m_\alpha)}{m_\nu-m_\alpha}-\frac{\sin\frac{1}{2}(m_\nu+m_\alpha)}{m_\nu+m_\alpha}\right]$$

$$+\left[\sin\frac{m_\nu}{2}\sin\frac{m_\alpha}{2}\right]\left[\frac{\sinh\frac{1}{2}(m_\nu+m_\alpha)}{m_\nu+m_\alpha}-\frac{\sinh\frac{1}{2}(m_\nu-m_\alpha)}{m_\nu-m_\alpha}\right],$$

$$A_2+A_3=\frac{2}{(m_\nu)^2+(m_\alpha)^2}\left[\sinh\frac{m_\nu}{2}\sinh\frac{m_\alpha}{2}\cdot\left\{-m_\nu\sin\frac{m_\alpha}{2}\cos\frac{m_\nu}{2}+m_\alpha\sin\frac{m_\nu}{2}\cos\frac{m_\alpha}{2}\right\}\right.$$

$$\left.+\sin\frac{m_\alpha}{2}\sin\frac{m_\nu}{2}\left\{m_\alpha\sinh\frac{m_\nu}{2}\cosh\frac{m_\alpha}{2}-m_\nu\sinh\frac{m_\alpha}{2}\cosh\frac{m_\nu}{2}\right\}\right].$$

Also, we have, for even values of ν, ($\varphi_\nu(\xi_1)$being odd functione),

$F_{\mu i}=0$ $(i=0,\ 2,\ 4,\ \ldots\ldots)$,

$$\frac{G_{\nu j}}{H}=-A_\nu\sin\frac{\pi j}{2}\left[\frac{\sin\frac{1}{2}(m_\nu-\pi j)}{m_\nu-\pi j}-\frac{\sin\frac{1}{2}(m_\nu+\pi j)}{m_\nu+\pi j}\right]$$

$$-B_\nu\sin\frac{\pi j}{2}\left[m_\nu\sin\frac{\pi j}{2}\cosh\frac{m_\nu}{2}-\pi j\cos\frac{\pi j}{2}\sinh\frac{m_\nu}{2}\right]\cdot\frac{2}{(m_\nu)^2+(\pi j)^2}\cdot$$

$(j=\frac{1}{2},\ 3/2,\ 5/2,\ \ldots\ldots)$

Lastly, we have,
for odd values of α;

$$a_{\alpha\alpha}=\frac{1}{2}(m_\alpha B_\alpha)^2\left[1+\frac{\sinh m_\alpha}{m_\alpha}\right].\ -\frac{1}{2}(m_\alpha A_\alpha)^2\left[1+\frac{\sin m_\alpha}{m_\alpha}\right],$$

while for even values of ν;

$$a_{\nu\nu}=\frac{1}{2}(m_\nu B_\nu)^2\left[\frac{\sinh m_\nu}{m_\nu}-1\right]$$

$$-\frac{1}{2}(m_\nu A_\nu)^2\left[1-\frac{\sin m_\nu}{m_\nu}\right].$$

V. The solution of fundamental equation of vibration in form of a doubly infinite series

At first, we transform the value of water pressure p which act on the face of side-wall, as given by (3), into a doubly infinite senies of the form of expsession

(4). For that purpose, we write down the condition that on side-wall we have $\partial\phi/\partial y=\partial w/\partial t$, from the equations (4) and (5), thus;

$$\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}A_{\mu\nu}\varphi_{\mu}(\xi_1)\varphi_{\nu}(\zeta_1)\cdot\omega\cos\omega t$$
$$=\sum_{i}\sum_{j}B_{ij}f'_{ij}\left(\pm\frac{B}{2}\right)\cos(mix)\cos(sjz)\cos\omega t$$

Next, multiplying by cos (mix) cos (sjz), the both sides of this equation, and integrating for $x=0$ to $x=L$, and $z=0$ to $z=H$, we have,

$$\frac{LH}{\varepsilon}B_{ij}f'_{ij}\left(\frac{B}{2}\right)=\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}A_{\mu\nu}F_{\mu i}G_{\nu j}\omega,$$

wherein we are to take $\varepsilon=2$ if $i=0$, but $\varepsilon=4$ if $i\neq0$. Putting the values of B_{ij} thus obtained into the equation (5), we have,

$$\phi=\sum_{i}\sum_{j}\frac{\varepsilon}{LH}\frac{f_{ij}(B/2)}{f'_{ij}(B/2)}\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}A_{\mu\nu}F_{\mu i}G_{\nu j}$$
$$\cos(mix)\cos(sjz)\cdot\omega\cos\omega t \tag{9}$$
$$=\sum_{\alpha}\sum_{\beta}C_{\alpha\beta}\varphi_{\alpha}(\xi_1)\varphi_{\beta}(\zeta_1)\,\omega\cos\omega t \tag{9a}$$

The coefficients $C_{\alpha\beta}$ can be obtained by multiplying both sides of equation (9) by $\varphi_{\alpha}(\xi_1)$ $\varphi_{\beta}(\zeta_1)$ and integrating, thus;

$$C_{\alpha\beta}=\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}A_{\mu\nu}M_{\mu\nu}^{\alpha\beta}, \tag{10}$$

where we put,

$$M_{\mu\nu}^{\alpha\beta}=\sum_{i}\sum_{j}\left[\frac{\varepsilon f_{ij}(B/2)}{f'_{ij}(B/2)}F_{\mu i}G_{\nu j}F_{\alpha i}G_{\beta j}\right]\frac{1}{L^2H^2}\cdot$$

As before, we are to take $i=0, 2, 4, \ldots\ldots$; $j=1/2, 3/2, 5/2, \ldots\ldots$. α and β are integers. We put the value of (9a) into the fundamental equation (1), taking into accaunt the equation (2). Furthermore, we have, by multiplying by $\varphi_{\alpha}(\xi_1)\cdot\varphi_{\beta}(\zeta_1)$, and by integration;

$$\left[D\left(\frac{m_4^4}{L_4}+\frac{m_\beta^4}{H^2}\right)-k\omega^2\right](LH)A_{\alpha\beta}$$
$$+\frac{2D}{LH}\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}a_{\mu\alpha}a_{\nu\beta}A_{\mu\nu}-\rho_w\omega^2(LH)C_{\alpha\beta}=0 \tag{11}$$

where we put $k=\rho_m h$.

This equation (11) is a homogeneous linear equation with respect to cofficients $A_{\alpha\beta}$, which represent different component amplitudes of vibration of flat plate (constituting side-wall of rectangular water tank). (11) has a system of non-all-zero values of $A_{\alpha\beta}$ as the solution only for specified values of ω. This eigen-value of ω is a root of determinantal equation $\Delta=0$, the determinant Δ being constructed by the array of coefficients of $A_{\alpha\beta}$ in equation (11). This determmant Δ is of infnite orders. For a practical estimation, we shall have to take only finite number of coefficients, for $\alpha=1, 2, \ldots\ldots n$, and $\beta=1, 2, \ldots\ldots n$, thus making the determinantal equation $\Delta=0$ an algebraic equation of order n with respect to ω^2.

In the numerical illustration given in the following section, we do not adopt this procedure, in order to avoid heavy numerical calculations. We can show that the approximate evaluation given in our report VI, is just the same as assuming $A_{11}=1$ and all the other $A_{\alpha\beta}$'s$=0$, in the equation (11), and thence obtaining an approximate estimate for ω^2. So that, if we could show, by some means, that actually the values of the ratios of component amplitudes $A_{\alpha\beta}/A_{11}$ (for $\alpha>1$ and $\beta>1$) are very small, the approximate formula given in report VI may be said to be accurate. Thus the values of $A_{\alpha\beta}/A_{11}$ (for $\alpha>1$, $\beta>1$) would give us a measure about the degree of accuracy of our approximate formula given ın report VI. The estimation of numerical values along this line of thoughts will be given here.

For this purpose, it is convenient to rewrite the equation (11) in the following form: —

$$\left[\left(\frac{H}{L}\right)^4 m_\alpha{}^4+m_\beta{}^4-\frac{kH^4}{D}\omega^2\right]A_{\alpha\beta}+2\left(\frac{H}{L}\right)^2\sum_\mu\sum_\nu a_{\mu\alpha}a_{\nu\beta}A_{\mu\nu}$$

$$-\rho_w\omega^2\frac{H^4}{(LH)D}\left[\frac{C_{\alpha\beta}}{B}\right](BLH)=0 \tag{12}$$

VI. Tables of numerical coefficients

For making the numerical evaluation mentioned in the previous section, we require numerical values of various coefficients involved in equation (12). First, putting

$$H_{ij}=\frac{f'_{ij}(B/2)}{f_{ij}(B/2)}=\frac{\coth(n_{ij}B/2)}{(n_{ij}B)}\cdot B,$$

where we have

$$n_{ij}B=\pi\left[\left(\frac{iB}{L}\right)^2+\left(\frac{jB}{H}\right)\right]^{1/2},$$

we need the numerical values of the coefficients defined by

$$H_{ij}=\frac{B}{\pi Z_{ij}}\,;\ Z_{ij}=\frac{n_{ij}B}{\pi}\tanh\left(\frac{1}{2}n_{ij}B\right).$$

We take up two cases (A); $B:H:L=1:1:2$, and (B); $B:H:L=1:1:1$. Numerical values of Z_{ij} for the case A is given in Table 1,. while Table 2 gives us those values of Z_{ij} for the Case B. Next, the numerical values of $Q^{\alpha\beta}$, defined by

$$Q^{\alpha\beta}=\left(\frac{H}{L}\right)^4 m_\alpha{}^4+m_\beta{}^4$$

are given, in Table 3 for the Case A, and in Table B for the Case B, respectively. Values of $a_{\mu\nu}$, $F_{\mu i}/L$ and $G_{\nu j}/H$ are shown in Tables 5, 6 and 7 respectively. It is to be noted that numerical values of Tables 5, 6 and 7 are independent of the proportion B: H: L of the rectangular tank. Lastly, some numerical values of the coefficients $\overset{\alpha\beta}{M}_{\lambda\mu}$ have been calculated for the Case A, and are given in Table 8.

Table 1. Value of Z_{ij}.

Case A; B: H: L=1: 1: 2.

j \ i	$i=0$	2	4	6	8
1/2	0.3279	1.053	2.062	2.06	4.03
3/2	1.473	1.791	2.50	3.37	4.27
5/2	2.50	2.692	3.202	3.90	4.72
7/2	3.50	3.64	4.03	4.60	5.32
9/2	4.50	4.61	4.93	5.41	6.03

Table 2. Value of Z_{ij}.

Case B; B: H: L=1: 1: 1.

j \ i	$i=0$	2	4	6	8
1/2	0.3274	2.062	4.031	6.02	8.01
3/2	1.473	2.50	4.272	6.18	8.15
5/2	2·50	3.202	4.717	6.50	8.38
7/2	3.50	4.03	5.25	6.95	8.73
9/2	4.50	4.93	6.03	7.50	8.98

Table 3. Value of $Q^{\alpha\beta}$.

Case A; $B/L=0.50$, $B/H=1.0$

	$\alpha=1$	$\alpha=2$	$\alpha=3$	$\alpha=4$
$\beta=1$	533	752	1255	3130
2	3833	4050	4555	6430
3	12100	12350	12860	14730
4	39500	39750	40260	42100

Table 4. Value of $Q^{\alpha\beta}$.

Case B; $B/L=1.0$, $B/H=1.0$

	$\alpha=1$	$\alpha=2$	$\alpha=3$	$\alpha=4$
$\beta=1$	1000	4300	12600	40100
2	3800	7600	15900	43300
3	12100	15900	24200	51600
4	39500	43300	51600	79000

Table 5. Value of $\boldsymbol{a}_{\mu\nu}$.

	$\mu=1$	$\mu=2$	$\mu=3$	$\mu=4$
$\alpha=1$	−12.3	0	−5.46	0
2	0	−108.8	0	−0.038
3	−5.46	0	−99.9	0
4	0	−0.038	0	−257.0

Table 6. Value of $F_{\mu i}/L$.

	$\mu=1$	$\mu=2$	$\mu=3$	$\mu=4$
$i=0$	0.828	0	−0.364	0
2	−0.392	0	−0.409	0
4	−0.0171	0	+0.520	0
6	−0.0031	0	+0.0481	0
8	−0.0010	0	+0.0139	0

Table 7. Value of $G_{\nu j}/H$.

	$\nu=1$	$\nu=2$	$\nu=3$	$\nu=4$
$j=1/2$	0.563	−0.0335	−0.209	+0.0019
3/2	−0.391	−0.666	−0.1076	+0.025
5/2	−0.170	+0.521	−0.422	+0.0393
7/2	−0.0198	+0.375	+0.456	+0.280
9/2	−0.0286	−0.1123	+0.238	−0.557

Table 8. Value of $M_{\lambda\mu}^{\alpha\beta}$ (Case A).

		$\beta=1$	$\beta=2$	$\beta=3$
$\alpha=1$, $\lambda=1$	$\mu=1$	1.740	0.169	−0.471
	2	0.169	0.780	0.222
	3	−0.471	0.222	0.363
$\alpha=1$, $\lambda=3$ }; $\alpha=3$, $\lambda=1$ }	$\mu=1$	−0.396	0.012	0.144
	2	0.012	−0.035	−0.010
	3	0.144	−0.010	−0.058
$\alpha=3$, $\lambda=3$	$\mu=1$	0.882	0.158	−0.140
	2	0.158	0.624	−0.067
	3	−0.140	−0.067	0.210

VII. Numerical example of the coefficients $A_{\lambda\mu}$ of the component amplitudes of vibration

Writing for shortnes

$$K_2=\frac{kH^4}{D}=\frac{\rho_m hH^4}{D},\quad K_3=\left(\frac{H}{L}\right)^2,\quad K_4=\frac{\rho_w H^4 B}{D},$$

the equation (12) can be given in the form; —

$$[Q^{\alpha\beta}-K_2\omega^2]A_{\alpha\beta}=K_4\omega^2[C_{\alpha\beta}/B]-2K_3\sum_\mu\sum_\nu a_{\mu\alpha}a_{\nu\beta}A_{\mu\nu}, \tag{13}$$

where we have,

$$C_{\alpha\beta}/B=\sum_\mu\sum_\nu\left(M^{\alpha\beta}_{\mu\nu}/B\right)A_{\mu\nu} \tag{14}$$

By virtue of symmetry about $x=L/2$, we have $A_{\alpha\beta}=0$, for even values of α. Assuming, in equation (13), that α is an even integer, and noting that $A_{\mu\nu}=0$ for $\mu=$ an even integer, the equation (13) becomes $0=0$, justifying the above mentioned inference. From equation (13), we have,

$$A_{\alpha\beta}=\left[(K_4\omega^2)\sum_\mu\sum_\nu(M^{\alpha\beta}_{\mu\nu}/B)A_{\mu\nu}-2K_3\sum_\mu\sum_\nu a_{\mu\alpha}a_{\nu\beta}A_{\mu\nu}\right]\div\left[Q^{\alpha\beta}-K_2\omega^2\right] \tag{15}$$

(0). The zeroth approximation,

If we assume that $A_{11}=1$, while all the other coefficients $A_{\mu\nu}$ are equal to zero, we have, by putting $\alpha=1$, $\beta=1$ into the equation (15),

$$\left[Q^{11}-K_2\omega^2\right]+2K_3(a_1^2)=K_4\omega\,(M^{11}_{11}/B) \tag{16}$$

From which, we can derive an approximate formula for $K_4\omega^2$. We can see by retracing the calculation given above, that this value of $K_4\omega^2$ will be just the value obtained by calculation of energies, the transverse displacement w being given tentatively by,

$$w=\varphi_1(\xi_1)\,\varphi_1(\xi_1)\sin\omega t.$$

Actual value of $K_4\omega^2$ will be slightly different from this approximete value.

(I). The first approximation.

Putting the values $A_{11}=1$, other $A_{\alpha\beta}$'s $=0$, and also the approximate value of $K_4\omega^2$ found above, into the "right hand side" of the equation (15), we obtain,

$$A_{\alpha\beta}=\left[K_4\omega^2(M^{\alpha\beta}_{11}/B)-2K_3(a_{1\alpha}a_{1\beta})\right]\div\left[Q^{\alpha\beta}-K^2\omega^4\right]. \tag{17}$$

This is a set of values of $A_{\alpha\beta}$, which we may call the first approximate values.

(II). The second approximation.

Next, putting the first approximate values of $A_{\alpha\beta}$ into the "right hand side" of the equation (15), we obtuin values of $A_{\alpha\beta}$ which may be called the second approximation.

We proceed further this process of the iterative evaluation, if necessary.

It is to be pointed out that, for every step of calculation, we must also readjust the value of $K_4\omega^2$, in order to be exact. By carrying out some numerical estimations, we find that, so long as we are seeking after the order of magnitudes of ratios $A_{\alpha\beta}/A_{11}$, and not necessarily their exact values, we may stop at the second approximation, wherein the value for $K_4\omega^2$ is used as the value given by the equation (16). We carried out this process of calculation for the Case A and Case B, as mentioned above. The values for the first approximation are shown in Table 9 and Table 10. The second approximate values of $A_{\alpha\beta}$ are shown in Table 11, which correspond to Case A. (The same table for the Case B is omitted, because it offers us no different inference.) Looking at Tables 9 and 11, we note that actual values are considerably different, but that the order of magnitudes remain

Table 9. Value of $A_{\lambda\mu}$.

Case A; First approximation.

	$\mu=1$	$\mu=2$	$\mu=3$
$\lambda=1$	1	0.0145	−0.0154
$\lambda=3$	−0.133	0.0158	0.0025

Table 10. Value of $A_{\lambda\mu}$.

Case B; First approximation.

	$\mu=1$	$\mu=2$	$\mu=3$
$\lambda=1$	1	0.0325	−0.0177
$\lambda=3$	−0.0436	0.0097	0.00305

Table 11. Value of $A_{\lambda\mu}$

Case A; Second approximation.

	$\mu=1$	$\mu=2$	$\mu=3$
$\lambda=1$	1	0.0104	−0.0109
$\lambda=3$	−0.098	−0.020	0.00393

the same. Also, we see, faom Table 11, that there appear a value of A_{31} which is about equal to -0.100, and which may be not said to be quite small.

VIII. Concluding remark which follow from the numerical example

As was mentioned obove, the values of $A_{\alpha\beta}(\alpha>1,\ \beta>1)$ are small compared with A_{11}(which is taken$=1$), but, as there appear the value A_{31}, which amount to -0.10, we may not claim that we can quite neglect all of $A_{\alpha\beta}(\alpha>1,\ \beta>1)$ in comparison with A_{11}. This fact shows us that the value of $K_4\omega^2$ given by (16) [which correspond to the case of $A_{11}=1$, all the other $A_{\alpha\beta}$'s being$=0$], could not be said to be very exact. The state of matters is rather different from the case of rectangular plates (side-walls) whose four edges are simply supperted (and discussed in the report VII.) But, as we infer from the calculation of strain energy (given in the next section), the equation (16) may serve as an approximate formula for practical uses, without claiming that it is very accurate.

IX. Strain energy of deformation

When the displacement of the rectangular plate is given by (4), its strain energy will be obtained by making estimation of the formtla,

$$P_m=\frac{1}{2}D\int\int\left[\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial z^2}\right)^2-2(1-\nu)\left\{\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial z^2}-\left(\frac{\partial^2 w}{\partial x\partial z}\right)^2\right\}\right]dxdz. \tag{18}$$

We have, for instance,

$$L^4\left(\frac{\partial^2 w}{\partial x^2}\right)^2=(\sin^2\omega t)\sum_\lambda\sum_\mu\sum_\alpha\sum_\beta A_{\alpha\beta}A_{\lambda\mu}\varphi_\alpha''(\xi_1)\varphi_\lambda''(\xi_1)\varphi_\beta(\zeta_1)\varphi_\mu(\zeta_1),$$

$$L^2H^2\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial z^2}\right)=(\sin^2\omega t)\sum_\lambda\sum_\mu\sum_\alpha\sum_\beta A_{\lambda\mu}A_{\alpha\beta}\varphi_\alpha''(\xi_1)\varphi_\lambda(\xi_1)\varphi_\beta(\zeta_1)\varphi_\mu''(\zeta_1)$$

In carrying out the integration indicated by the formula (18), we note that we have,

$$\int_{-1/2}^{1/2}\varphi_\alpha''(\xi_1)\varphi_\lambda''(\xi_1)d\xi_1=m_\lambda^4\int_{-1/2}^{1/2}\varphi_\alpha(\xi_1)\varphi_\lambda(\xi_1)d\xi_1$$
$$=\begin{cases}m_\lambda^4, & \text{if } \alpha=\lambda,\\ 0, & \text{if } \alpha\neq\lambda,\end{cases}$$

by virtue of the equation

$$\varphi_\lambda^{(IV)}(\xi_1)=m_\lambda^4\varphi_\lambda(\xi_1),$$

and the boundary condition that $\varphi_\lambda(\pm\frac{1}{2})=0$ and $\varphi_\lambda'(\pm\frac{1}{2})=0$. Also, we have,

$$\int_{-1/2}^{1/2}\varphi_\alpha'(\xi_1)\varphi_\lambda'(\xi_1)d\xi_1=-\int_{-1/2}^{1/2}\varphi_\alpha(\xi_1)\varphi_\lambda''(\xi_1)d\xi_1=-a_{\alpha\lambda}.$$

Thus, it is seen that,

$$\int_{-1/2}^{1/2} d\xi_1 \int_{-1/2}^{1/2} d\zeta_1 \left[\left(\frac{\partial^2 w}{\partial x^2} \right) \left(\frac{\partial^2 w}{\partial z^2} \right) - \left(\frac{\partial^2 w}{\partial x \partial z} \right)^2 \right] = 0.$$

In this way, we obtain the final formula;

$$P_m = \frac{1}{2} \frac{DL}{H^3} (\sin^2 \omega t) \Big[\sum_\alpha \sum_\beta \left\{ \left(\frac{H}{L} \right)^4 m_\alpha{}^4 + m_\beta{}^4 \right\} A_{\alpha\beta}^2 + 2 \left(\frac{H}{L} \right)^2 \sum_\alpha \sum_\beta \sum_\lambda \sum_\mu a_{\alpha\lambda} a_{\beta\mu} A_{\alpha\beta} A_{\lambda\mu} \Big] \tag{19}$$

Also we have, for the kinetic energy T_m of the vibration of the rectangular plate;

$$T_m = \frac{1}{2} \rho_m h \int_{-1/2}^{1/2} d\xi_1 \int_{-1/2}^{1/2} d\zeta_1 \left(\frac{\partial w}{\partial t} \right)^2 = \frac{1}{2} \rho_m h (\omega \cos \omega t)^2 \Big[\sum_\alpha \sum_\beta (A_{\alpha\beta})^2 \Big] \tag{20}$$

Suppose, for example, that $A_{11} = 1$, $A_{\alpha\beta} = \delta$, and all the other $A_{\alpha\beta}$'s are all zero. Then we have

$$P_m = \frac{1}{2} \frac{DL}{H^3} (\sin^2 \omega t) \Big[\left\{ \left(\frac{H}{L} \right)^4 m_1{}^4 + m_1{}^4 \right\} + \left\{ \left(\frac{H}{L} \right)^4 m_\alpha{}^4 + m_\beta{}^4 \right\} \delta^2 + \frac{1}{2} \left(\frac{H}{L} \right)^2 \left\{ (a_{11})^2 + 2 (a_{1\alpha} a_{1\beta}) \delta + (a_{\alpha\beta})^2 \delta^2 \right\} \Big]$$

and

$$T_m = \tfrac{1}{2} \rho_m h (\omega \cos \omega t)^2 [1 + \delta^2]$$

These formula may serve to judge what was said in the end of the last section.

X. Acknowledgements

Thus far, the author has carried out a series of studies about virtual mass of water contained in a rectangular tank, whose side-walls are vibrating, and could report successively the result of his study in this Proceedings.

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