Title	Stability problem of a resonant circuit with variable parameters
Sub Title	
Author	佐藤, 力(Sato, Chikara)
Publisher	慶応義塾大学藤原記念工学部
Publication year	1962
Jtitle	Proceedings of the Fujihara Memorial Faculty of Engineering Keio University (慶應義塾大学藤原記念工学部研究報告). Vol.15, No.57 (1962.) ,p.48(12)- 56(20)
JaLC DOI	
Abstract	Parametrically-excited resonant circuits of two-loops or two-node-pairs are dealt with in this paper. Investigation is made on the stability of the circuits by an analog computer. Mathematical formulation of these circuits leads to linear variable coefficient differential equations. These diffrential equations (so-called generalized Methieu equations) are given as the following equations:[function][function] In this paper the transition curves from stability to instability are obtained under the special conditions in the above equations.
Notes	
Genre	Departmental Bulletin Paper
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=KO50001004-00150057-0012

Stability Problem of a Resonant Circuit with Variable Parameters

(Recieved March 26, 1963)

Chikara SATO*

Abstract

Parametrically-excited resonant circuits of two-loops or two-node-pairs are dealt with in this paper. Investigation is made on the stability of the circuits by an analog computer. Mathematical formulation of these circuits leads to linear variable coefficient differential equations. These differential equations (so-called generalized Methieu equations) are given as the following equations:

$$\frac{d^2y_1}{dt^2}+\Omega_1{}^2y_1=(\alpha_{11}y_1+\alpha_{12}y_2)\cos t\,,$$

$$\frac{d^2y_2}{dt^2}+\Omega_2{}^2y_2=(\alpha_{21}y_1+\alpha_{22}y_2)\cos t\,.$$

In this paper the transition curves from stability to instability are obtained under the special conditions in the above equations.

I. Introduction

Tne continuing research for the analysis of parametrically-excited circuits and their applications have been made in recent years. The mathematical formulation of these circuits leads to either linear or nonlinear variable coefficient differential equations. Van der Pol and Strutt (1) have treated the differential equation of Hill type, and MaLachlan,(2) Stoker (3) and Mandelstam (4) have treated Mathieu type equatson of the second order, and they have given the transition curves from stability to instability for the equation. Cesari,(5) Coddington,(6) Hale,(7) Zadeh (8) and Malkin (9) have investigated the linear or nonlinear differential equations of the higher order. However, none of them have given the transition curves from stability to instability for the variable coeffieicnt differential equations of the higher order. In this paper the author treated the variable coefficient differential equation of the fourth order (two simultaneous equations of the second order) by an analog computer under special conditions, and the transition curves from stability to instability are obtained.

* 佐 藤 力 Assistant Professor at Keio University.

II. Mathematical formulation

Let us consider the parametrically-excited linear resonant circuit of two-loops shown in Fig. 1. The discussion for its dual circuit shown in Fig. 2 is eliminated here since the circuit equation is the same as the former. In Fig. 1 three inductances are L_1, L_2 and L_3, and three variable capacitances are assumed to be as follows:

$$\begin{aligned} \frac{1}{C_1(t)} &= D_1 + d_1 \cos \omega t\,, \\ \frac{1}{C_2(t)} &= D_2 + d_2 \cos \omega t\,, \\ \frac{1}{C_3(t)} &= D_3 + d_3 \cos \omega t\,. \end{aligned} \tag{1}$$

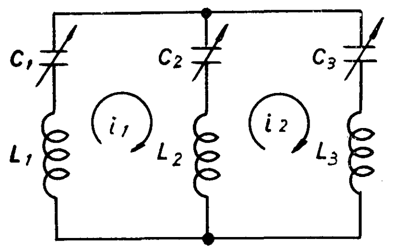

Fig. 1. Resonant circuit of two-loops containing time-variable C's.

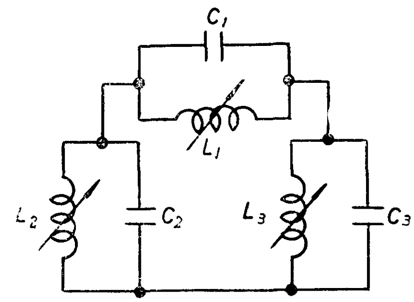

Fig. 2. Resonant circuit of two-node-pairs containing time-variable L's.

The circuit equation is given by

$$\begin{bmatrix} L_1+L_2 & -L_2 \\ -L_2 & L_2+L_3 \end{bmatrix} \cdot \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} D_1+D_2 & -D_2 \\ -D_2 & D_2+D_3 \end{bmatrix} \cdot \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \cos \omega t \begin{bmatrix} d_1+d_2 & -d_2 \\ -d_2 & d_2+d_3 \end{bmatrix} \cdot \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = 0\,, \tag{2}$$

where

$$\dot{q}_1 = \frac{dq_1}{dt} = i_1\,, \quad \dot{q}_2 = \frac{dq_2}{dt} = i_2\,.$$

Assuming that

$$\begin{vmatrix} L_1+L_2 & -L_2 \\ -L_2 & L_2+L_3 \end{vmatrix} \neq 0\,, \tag{3}$$

and if we transform q_1, q_2 to y_1, y_2 by a nonsingular linear transformation:

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \tag{4}$$

equation (2) is reduced to the form:

$$\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} + \begin{bmatrix} \Omega_1^2 & 0 \\ 0 & \Omega_2^2 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \cos \omega t \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \qquad (5)$$

where parameters Ω_1^2, Ω_2^2, α_{11}, α_{12}, α_{12}, α_{21}, and α_{22} are all obtained from the circuit parameters L_1, L_2, L_3; D_1, D_2, D_3; d_1, d_2, and d_3. In the equation (5) we can put $\omega=1$ without loss of generality, therefore the equation (5) can be rewritten as follows,

$$\begin{aligned} \ddot{y}_1+\Omega_1^2 y_1 &= (\alpha_{11} y_1+\alpha_{12} y_2) \cos t, \\ \ddot{y}_2+\Omega_2^2 y_2 &= (\alpha_{21} y_1+\alpha_{22} y_2) \cos t. \end{aligned} \qquad (6)$$

This equation (6) is a generalized Mathieu equation, because if we put $\alpha_{12}=\alpha_{21}=0$ then equation (6) is variable-separated and becomes two independent Mathieu equations of the second order.

Now we are able to reduce the problem of the parametrically-excited resonant circuit to the mathematical equation (6). Therefore we will investigate equation (6) in the following section.

III. Solutions by analog computer

The ultimate aim of this section is to obtain the transition curves from stability to instability under the special conditions. Using an analog computer we obtain the solutions y_1 and y_2 in equation (6). The solutions y_1 and y_2 are shown in Fig. 3, Fig. 4, and Fig. 5, under the common initial condition at $t=0$,

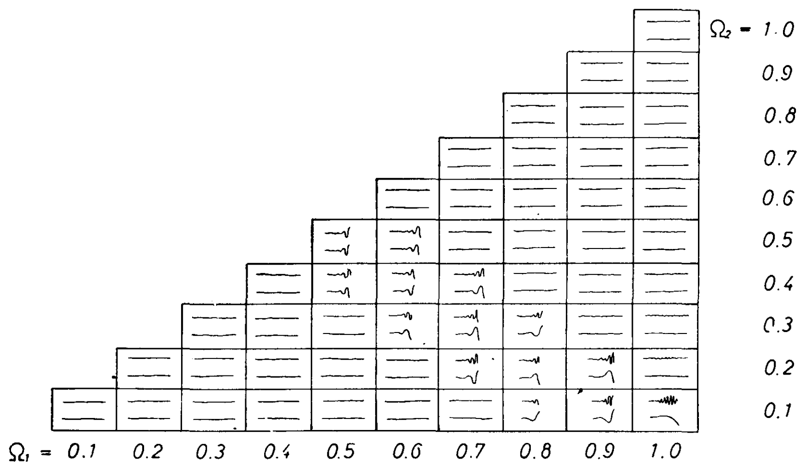

Fig. 3. Solutions y_1 (above) and y_2 (below) in equation (8) when $\alpha=0.1$.

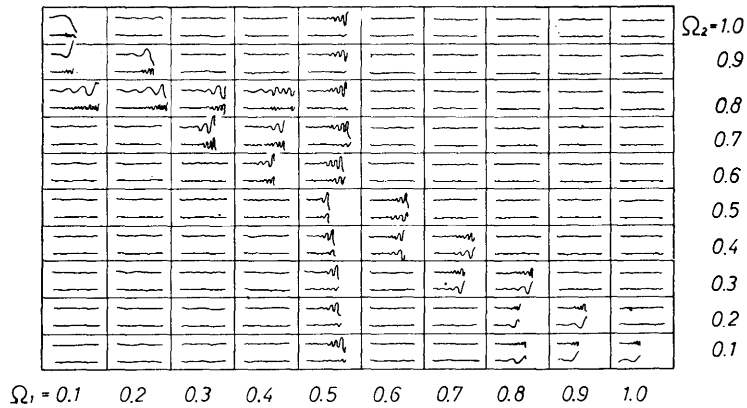

Fig. 4. Solutions y_1 (above) and y_2 (below) in equation (9) when $\alpha=0.1$.

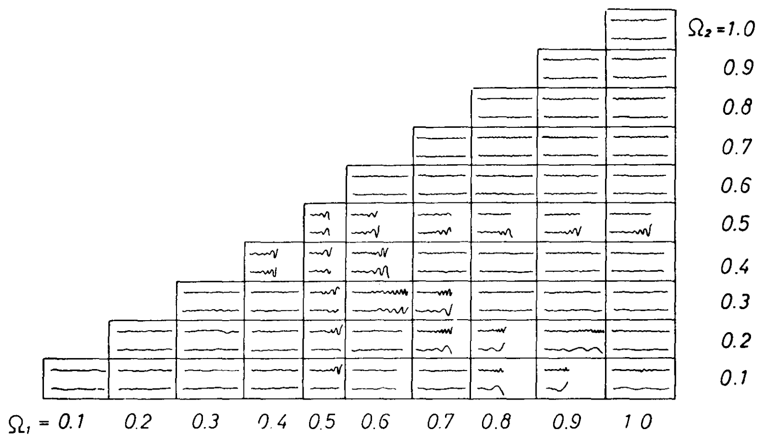

Fig. 5. Solutions y_1 (above) and y_2 (below) in equation (10) when $\alpha=0.1$.

$$y_1=y_2=1\,,\quad \dot{y}_1=\dot{y}_2=0\,. \tag{7}$$

Putting $\alpha_{11}=\alpha_{22}=0$ and $\alpha_{12}=\alpha_{21}=\alpha$ in equation (6), we have the following equation,

$$\begin{aligned}\ddot{y}_1+\Omega_1{}^2 y_1&=\alpha y_2 \cos t\,,\\ \ddot{y}_2+\Omega_2{}^2 y_2&=\alpha y_1 \cos t.\end{aligned} \tag{8}$$

The solution curves of equation (8) with the initial condition (7) and with $\alpha=0.1$ are shown in Fig. 3. Putting $\alpha_{11}=\alpha_{12}=\alpha_{21}=\alpha$ and $\alpha_{22}=0$, we have the following equation,

$$\begin{aligned}\ddot{y}_1+\Omega_1{}^2 y_1&=\alpha (y_1+y_2) \cos t\,,\\ \ddot{y}_2+\Omega_2{}^2 y_2&=\alpha y_1 \cos t\,,\end{aligned} \tag{9}$$

and their solution curves with $\alpha=0.1$ are shown in Fig. 4. Putting $\alpha_{11}=\alpha_{12}=\alpha_{21}=\alpha_{22}=\alpha$ we have the equation

$$\ddot{y}_1+\Omega_1^2 y_1=\alpha(y_1+y_2)\cos t,$$
$$\ddot{y}_2+\Omega_2^2 y_2=\alpha(y_1+y_2)\cos t, \qquad (10)$$

and their solution curves with $\alpha=0.1$ are shown in Fig. 5. From these figures we can easily recognize whether solutions y_1 and y_2 are stable or unstable (divergent). The stable solutions oscillate within the values of order 1 which is equal the initial value, on the other hand unstable (divergent) solutions grow up oscillatory beyond the value of order 1. The typical solutions are shown in detail in Fig. 9. According to these solution curves the transition curves from stability to instability are obtained in Fig. 6, Fig. 7, and Fig. 8, using several other solution curves near the transition curves, although these curves are not shown in Fig. 3, Fig. 4 and Fig. 5. For symmetricity some solution curves are abreviated in Fig. 3 and Fig, 5.

From the transition curves in Fig. 6, we know that instability occurs near $\Omega_1+\Omega_2=1$. From the transition curves in Fig. 7, we know that instability occurs near $\Omega_1+\Omega_2=1$ and near $\Omega_1=0.5$. The instability near $\Omega_1+\Omega_2=1$ is of the same kind as in Fig. 6, on the other hand the instability near $\Omega_1=0.5$ belongs to the instability of 1/2 subharmonic in Mathieu equation of the second order, because of the coefficient $\alpha_{11}\neq 0$ in equation (6). Similarly, it is easily known that the following equation

$$\ddot{y}_1+\Omega_1^2 y_1=\alpha y_2\cos t,$$
$$\ddot{y}_2+\Omega_2^2 y_2=\alpha(y_1+y_2)\cos t, \qquad (11)$$

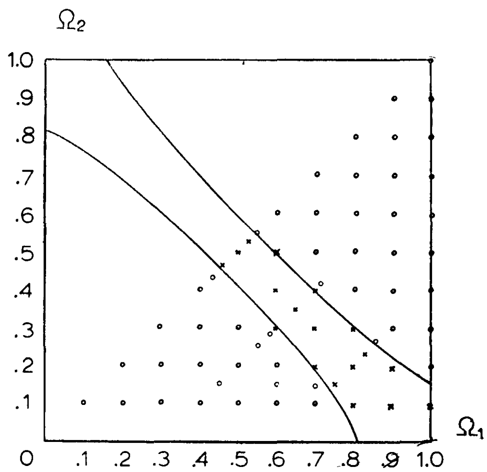

Fig. 6. Transition curves from stability to instability in equation (8) when $\alpha=0.1$.
o stable point, × unstable point.

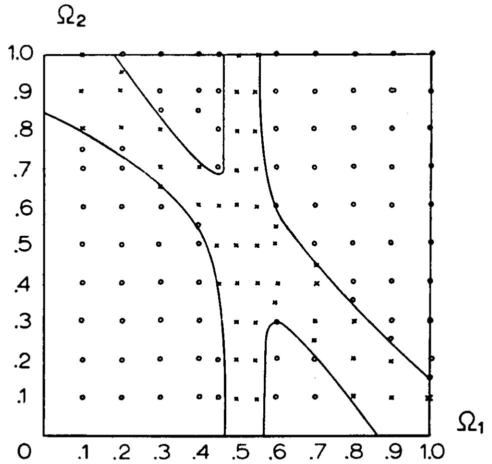

Fig. 7. Transition curves from stability to instability in equation (9) when $\alpha=0.1$.
o stable point, × unstable point.

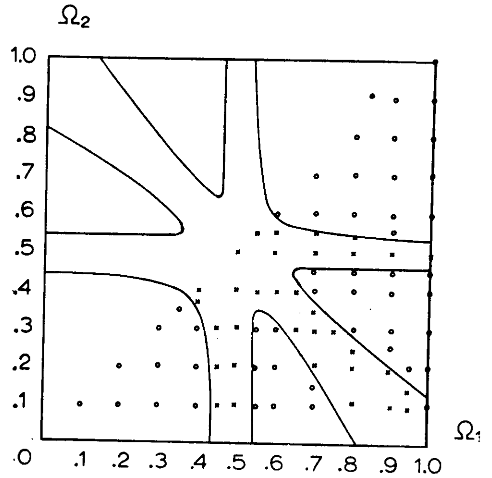

Fig. 8. Transition curves from stability to instability in equation (10) when $\alpha=0.1$.
o staɒle point, × unstable point.

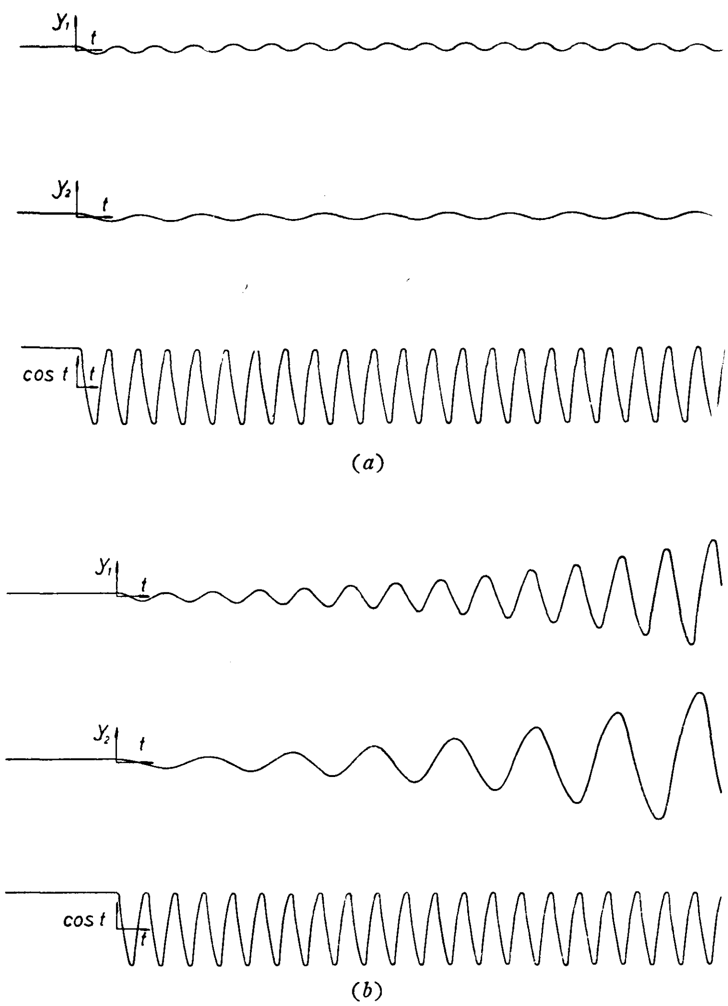

Fig. 9. Solutions y_1 and y_2 in equation (8), when $\alpha=0.1$, $\Omega_1=0.8$, $\Omega_2=0.5$ for (a), when $\alpha=0.1$, $\Omega_1=0.65$, $\Omega_2=0.35$ for (b).

has instability near $\Omega_1+\Omega_2=1$ and near $\Omega_2=0.5$. Fig. 8 shows three kinds of instability near $\Omega_1+\Omega_2=1$, near $\Omega_1=0.5$ and near $\Omega_2=0.5$. Therefore equation (10) has mixed character of instability of both equations (9) and (11).

IV. Mechanical models

Let us consider a mechanical model and an electro-mechanical model of the above electrical circuits. A mechanical resonant system of two degrees of freedom with varible parameters has also three kinds of instability if the conditions are satisfied. A coupled swing shown in Fig. 10 is a mechanical model, if at least one of the boys swings his body periodically. If the frequency of vertical periodic motion by the boy is equal to twice a resonant frequency, then the oscillation of the coupled swing grows up. If the frequency of the vertical motion is equal to the sum of two resonant frequencies, also oscillation of coupled mode grows up. When the amplitudes of these oscillations become large then nonlinearity in the system must be considered.(10)

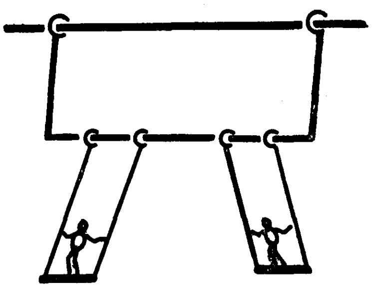

Fig. 10. Mechanical model.

An electro-mechanical model is shown in Fig. 11. A coupled resonator has a variable parameter by the *AC*-electromagnet. In these cases mathematical formulation is the same as an electrical circuits shown in Fig. 1. Many other dynamic models may be thought which corresponds to the equation (6).

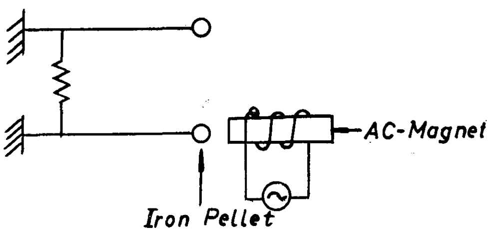

Fig. 11. Electro-mechanical model.

V. Conclusion

We have treated the solutions of equation (6) under the three special conditions. Under these special conditions it is possible for the equation (6) to have three kinds of instability: (i) near $\Omega_1=0.5$, (ii) near $\Omega_2=0.5$ and (iii) near $\Omega_1+\Omega_2=1$. The results obtained by an analog computer would be applicable only to the restricted conditions and would not be wide unless analytical approach were made. Fortunately, these results obtained here agree with the analytical results in the previous paper (10) which assumes all the variable parameters α_{11}, α_{12}, α_{21} and α_{22} are small.

Aknowledgement

The author wishes his thanks to Professors M. Mashima, S. Sasaki, H. Tomizawa, S. Mizushima, J. Yamauti, O. Nishino and J. Nagumo.

References

(1) B. van der Pol and M. J. O. Strutt, "On the stability of the solutions of Methieu's equation," Phil. Mag., vol. 5, pp. 18–38: 1928.

(2) N. W. McLachlan, "Theory and Application of Mathieu Functions," Clarendon Press, Oxford, England: 1947.

(3) I. J. Stoker, "Nonlinear Vibrations," Interscience Publishers Inc., New York, N. Y.: 1950.

(4) L. Mandelstam, N. Papalexi, A. Andronov, S. Chaikin, and A. Witt, "Exposé des recherches récentes sur les oscillations non linéarires," Tech. Phys. U. S. S. R., vol. 2, pp. 81–134: 1935.

(5) L. Cesari, "Asymptotic Behavior and Stability Problems in Ordinary Difierential Equations," Springer-Verlag, Munich, Germany, p. 75: 1959.

(6) E. A. Coddington and L. Levinson, "Theory of Ordinary Differential Equations, "McGraw-Hill Book Co., New York, N. Y., pp. 218, 356: 1955.

(7) J. K. Hale, "On the Behavior of the Solutions of Linear Periodic Differential Systems near Resonance Points," Research Institute for Advanced Study, Baltimore, Md., Rias Tech. Rept.: 1958.

(8) L. A. Zadeh. "Circuit Analysis of Linear Varying-parameter Networks," J. Applied Phys., vol. 21, pp. 1171–1177: 1950.

(9) I. G. Malkin, " Certain Problems of Nonlinear Oscillations," Moscow, pp. 225–310: 1956.

(10) C. Sato, "Stability Conditions for Resonant Circuits with Time-Variable Parameters," IRE Transactions, vol. CT–9, pp. 340–349: 1962.