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Author	鬼頭, 史城(Kito, Fumiki)
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On Virtual Mass of Water Contained in a Rectangular Tank, whose Side-Walls are Vibrating—VII

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Fumiki KITO*

Abstract

When side-walls of a rectangular tank, which is filled with water, are vibrating, the inside water will also make a vibratory motion. This motion of water lowers considerably the natural frequency of vibration of side-walls of the tank. This effect is conveniently expressed by "virtual mass" of water. In the previous reports, I to VI, of the same title, the author has made a theoretical study of the value of "virtual mass," and examined various factors affecting it. These calculations were made by assuming, tentatively, the mode of vibration of rectangular elastic plate which constitute the side-walls of the tank. And, the velocity potential for the vibratory motion of water caused by this vibration of side-walls was found. The question of degree of accuracy of the result obtained by this method, will naturally be raised, and some examination was made in the report II of the same title.

In the present report, this question of degree of accuracy of the author's procedure in the previous reports, is examined. Here, the equation of vibratory motion of a rectangular elastic plate subjected to lateral load (which is the water pressure caused by vibratory motion of water), is taken as the base of our discussion. And, magnitude of correction terms are deduced from it. It is concluded that, at least for rectangular plates of ordinary proportions, the approximate formula given in previous reports may be said to be fairly accurate.

I. Introduction

Let us consider a rectangular water (or, oil) tank, inside which water is filled up. When the rectangular flat plates, which constitute the side-walls of the tank, are vibrating, the inside water will make a vibratory motion. Due to this fact, there appear on the vibration of rectangular plates, the effect called the "virtual mass" of water. The author has made, in the previous reports I to VI,[1] of the same title, the study of this effect of "virtual mass," by examining in turn various

* 鬼頭史城 Dr. Eng.; Professor at Keio University.

[1] F. Kito, This Proceedings, Vol. 11, No. 40 (1958), Vol. 12, No. 46, No. 47 (1959), Vol. 13, No. 49 (1960), Vol. 15, No. 56 (1962), also Vol. 15, No. 56 (1962).

factors which affect it. The author has given approximate formula for natural frequency of vibration. In these reports, calculation was made, starting from the mode of vibration of flat-plate which was tentatively given. In the report II, some examination about the degree of approximotion of these formula was made.

Since then, it was felt desirable to make the examination of degree of accuracy of these approximate formulae by a more direct procedure, than the energy calculation. The result of the examination in this direction is given here.

As in the previous reports, the water is assumed to be an incompressible ideal fluid, and the amplitude of vibration to be of infinitely small magnitude. The rectangular plate is assumed to be a flat elastic plate. Also, the following notation, which are generally the same as in the previous reports, are used here;

ϕ=velocity potential for the vibratory motion of water, w=transverse displacement of the flat elastic plate, A=vibration amplitude of ditto, $W_0=\omega A$ (vibration amplitude of transverse velocity of ditto), ω=angular frequency of free vibration of ditto, L=length of the rectangular water tank, H=its height, B=its breadth, ρ_w=density of water, T_w=kinetic energy of water, $m=\pi/L$, $s=\pi/H$, T_m=kinetic energy of rectangular plate, P_m=strain (or, potential) energy of rectangular plate, ρ_m=density of the material composing the rectangular plate, D=flexural rigidity of the plate$=Eh^3/[12(1-\nu^2)]$, h=its thickness, E=its Young's modulus, p=water pressure, t=time, R=the ratio H/L, V=volume of the tank.

II. General considerations

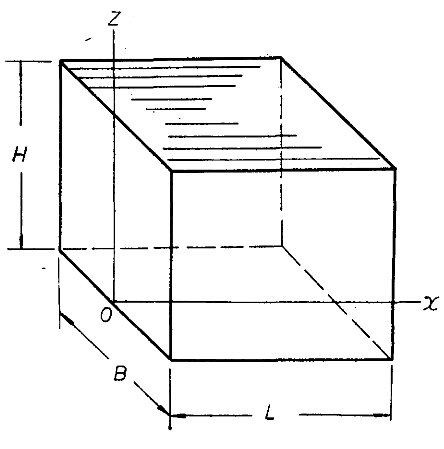

Fig. 1. Sketch of a Rectangular Water Tank.

Let us consider a rectangular water tank, as shown in Fig. 1, inside which water is filled up. When side-walls of the tank vibrate, the inside water will also make a vibratory motion. In order to examine this vibratory motion of water, we must first give boundary condition. Among many possible cases of this boundary conditions, let us take up the case in which water is almost full, but there is left on top a free surface. Also let us take the case in which two opposite side-walls ($L\times H$) are vibrating in opposite phases each other, while the remaining two end-walls and bottom-plate do not vibrate. In what follows, the discussion will be made about this case, but it may be noticed that the same line of reasoning can be applied to other cases of boundary condition.

For a rectangular elastic plate, which is in vibration, the transverse displacement will be denoted by w, which is a function of x, z, and t. This displacement w must satisfy the fundamental equation of vibratory motion, viz.;

$$D\nabla^2\nabla^2 w+\rho_m h\frac{\partial^2 w}{\partial t^2}-p=0, \tag{1}$$

where ∇^2 is the Laplace operator $\partial^2/\partial x^2+\partial^2/\partial z^2$, and p is the value of water pressure which acts, as a transverse load, on the rectangular plate. Originally, p is caused by the vibratory motion of side-walls. Namely, assuming that the vibratory motion of water is a kind of potential flow, its velocity potential ϕ is determined from the values of $\partial w/\partial t$ on the face of side-walls. And, from which, the value of water pressure p is given by

$$p=-\rho_w\frac{\partial\phi}{\partial t}\ . \tag{2}$$

Thus, we may say that the value of p at the face of side-wall is a linear functional of $\partial^2 w/\partial t^2$.

As to the velocity potential ϕ, it must satisfy Laplace equation inside the water region, together with the boundary condition as mentioned above. So that, according to potential theory, ϕ may be written (at least formally) in the following form;

$$\phi(x,\ z)=\int_0^L\int_0^H K_1(x,\ z;\ \xi,\ \zeta)\frac{\partial w(\xi,\ \zeta)}{\partial t}d\xi d\zeta. \tag{3}$$

Here, $(x,\ z)$ and $(\xi,\ \zeta)$ are rectangular coordinates of two points on the face of the rectangular plate. $K_1(x,\ z;\ \xi,\ \zeta)$ is a kernel function. It being so, the equation of vibratory motion (1) will become,

$$D\nabla^2\nabla^2 w(x,\ z)+\rho_m h\frac{\partial^2 w(x,\ z)}{\partial t^2} + \rho_w\int_0^L\int_0^H K_1(x,\ z;\ \xi,\ \zeta)\frac{\partial^2 w(\xi,\ \zeta)}{\partial t^2}d\xi d\zeta. \tag{4}$$

If we consider only the case of periodic (free) vibration, putting

$$w=f(x,\ z)\sin\omega t, \tag{5}$$

we shall have following equation for the function $f(x,\ z)$;

$$D\nabla^2\nabla^2 f(x,\ z)-\rho_m h\omega^2 f(x,\ z) -\rho_w\omega^2\int_0^L\int_0^H K_1(x,\ z;\ \xi,\ \zeta)f(\xi,\ \zeta)d\xi d\zeta. \tag{6}$$

Thus the problem of free-vibration of the rectangular plate, which is in contact with water, will be solved if the above integro-differential equation (6) could be solved (under the suitable boundary condition for $f(x,\ z)$). This procedure, which may not be announced to be impossible, yet no doubt it will require very much complicated work of calculation in order to obtain numerical values suited for practical uses.

Thus, as a second recourse, we are naturally led to use some method of successive approximations. One such a method is to first assume tentatively the value of transverse displacement w, and obtain the corresponding value of the water pressure p. Thence, putting this value of water pressure p into the equation (1), we evaluate the trasverse displacement w, which gives the more accurate value than the tentative one. Thus the first correction to w will be obtained. If the amount of this correction is found to be very small, then it could be said that the assumed value of w and the result deduced from it are fairly accurate. In what follows, the result of calculations made by the author, along this line of thoughts, will be given. (Of course, if the amount of correction to w was found to be not very small, it will mean that further task of successive approximations must be carried on.)

III. Calculation of correction for w

As mentioned above, we shall be mainly concerned with the case in which two opposite side-walls are vibrating in opposite phases each other, while the top surface is in the state of free surface. When two opposite side-walls $(H\times L)$ are regarded to be rectangular elastic plates with supported four edges, the displacement w may, at first, be given tentatively by;

$$w=A\sin\omega t\sin mx\sin sz, \tag{7}$$

the transverse velocity $\partial w/\partial t$ being given by

$$\frac{\partial w}{\partial t}=W_0\cos\omega t\sin mx\sin sz. \tag{8}$$

The corresponding value of velocity potential ϕ, for the vibratory motion of inside water, was already obtained in the previous report **I**, thus;—

$$\phi=A\omega B\cos\omega t\left(\frac{2}{\pi^3}\right)\sum_i\sum_j E_{ij}\cos(mix)\cos(sjz), \tag{9}$$

where we put,

$$E_{ij}=\frac{1}{Z_{ij}}\cdot\frac{\varepsilon}{(i^2-1)(j^2-1)}, \tag{10}$$

(with $\varepsilon=2$ for $\varepsilon=0$, but $\varepsilon=4$ for $i\neq 0$.)

$$Z_{ij}=\frac{1}{\pi}(n_{ij}B)\tanh(n_{ij}B/2), \tag{11}$$

$$n_{ij}=\left[(mi)^2+(sj)^2\right]^{1/2}, \tag{12}$$

For the sign $\sum\sum$ of summation in the equation (9), we are to take the sum for

$i=0, 2, 4, \ldots\ldots$; $j=1/2, 3/2, 5/2, \ldots\ldots.$ Also, it is assumed that the value of $\omega^2 H/g$ is very large in comparison with unity.

The water pressure p, which act upon side-walls, being obtained by (9) and (2), is given by,

$$p = A\omega^2 \rho_w B \sin \omega t \left(\frac{2}{\pi^3}\right) \sum_i \sum_j E_{ij} \cos(mix) \cos(sjz). \tag{9a}$$

In order to put this value of p into the equation (1), it is convenient to expand it into the double Fourier series, as follows;—

$$\begin{aligned} &p=\omega^2 AK_a \sin\omega t \cdot \Phi(x, z), \\ &K_a = B\rho_w(2/\pi^3), \\ &\Phi(x, z) = \sum_\lambda \sum_\mu \Phi_{\lambda\mu} \sin(\lambda mx) \sin(\mu sz), \end{aligned} \tag{13}$$

where we take, $\lambda=1, 3, 5, \ldots\ldots$; $\mu=1, 2, 3, \ldots\ldots.$ Only odd integers for λ are taken, because w and p are symmetrically distributed about the center line $x=L/2$, of the rectangular plate.

The transverse displacement w, will no longer be given by (7), but will be of the form,

$$\begin{aligned} &w=A\sin\omega t \cdot f(x, z), \\ &f(x, z) = \sum_\lambda \sum_\mu F_{\lambda\mu} \sin(\lambda mx) \sin(\mu sz), \end{aligned} \tag{14}$$

it being understood that, here we are considering the case of simply supported four edges. Putting these expressions (13) and (14) into the equation (1), and making comparison of coefficients of each terms $\sin(\lambda mx)$ $\sin(\mu sz)$, we obtain the following equation :—

$$F_{\lambda\mu} = \omega^2 K_a \Phi_{\lambda\mu} \div \Big[D\{(m\lambda)^2 + (s\mu)^2\}^2 - \omega^2 \rho_m h \Big] \cdot \tag{15}$$

Now, let us express the angular frequency of free vibration ω in the following form

$$\omega^2 = \frac{D\{m^2+s^2\}^2}{\rho_m hS}, \tag{16}$$

where S is a numerical factor. We have $S=1$, if there is no water, the formula (16) giving us the fundamental (angular) frequency of free vibration of rectangular elastic plate whose four edges are simply supported. When water is filled up in the tank, putting

$$S = \frac{M_m + M_w}{M_m}, \tag{17}$$

the formula (16) gives us the fundamental frequency of free vibration for that case.

We saw that the numerical value of S may be as large as 10 in practical cases (such as rectangular panel of a super-tanker.) Also, the equation (15) may be rewritten as;—

$$F_{\lambda\mu}=\frac{\omega^2 K_a H^4}{\pi^4 D}\cdot\frac{\Phi_{\lambda\mu}}{G_{\lambda\mu}}, \tag{15a}$$

where we put,

$$G_{\lambda\mu}=\left\{(\lambda R)^2+\mu^2\right\}^2-\frac{(R^2+1)^2}{S}, \tag{18}$$

and $R=H/L$.

Lastfy, the coefficients $\Phi_{\lambda\mu}$ are found to have the following values,

$$\Phi_{\lambda\mu}=\sum_i\sum_j\left(\frac{8}{\pi^2}\right)E_{ij}\left[\frac{\lambda\mu}{(\lambda^2-i^2)(\mu^2-j^2)}\right], \tag{19}$$

with $i=0,\ 2,\ 4,\ \ldots\ldots;\ j=1/2,\ 3/2,\ 5/2,\ \ldots\ldots;\ \lambda=1,\ 3,\ 4,\ \ldots\ldots;\ \mu=1,\ 2,\ 3,\ \ldots\ldots.$

It may be conjectured that the value (14) of the transverse displacement w will give more accurate one, than that given tentatively by (7). If necessary, we could obtain the corrected value of ϕ, by using (14), and so on. Now let us denote by ω_1, the value of angular frequency when the tentative value of (7) was used. If this was accurate enough, we should have, in (15), $F_{11}=1$, $F_{31}=0$, ..., when we put $\omega=\omega_1$ into (15). So that we shall have from (15), (by putting $\lambda=1$, $\mu=1$),

$$\omega_1^2=\frac{D\{m^2+s^2\}^2}{\rho_m h+K_a\Phi_{11}},$$

which we shall write, for convenience,

$$\omega_1^2=\frac{(m^2+s^2)^2 D}{\rho_m h S}.$$

Then we have

$$S=\frac{\rho_m h+K_a\Phi_{11}}{\rho_m h}=\frac{\frac14\rho_m hLH+\frac14 K_a LH\Phi_{11}}{\frac14\rho_m hLH}.$$

Now, (1/4) $\rho_m hHB$ is the mass (vibrational) of the flat plate, while we have

$$\tfrac14 K_a LH\Phi_{11}=BLH\rho_w\frac{4}{\pi^4}\cdot$$

$$\sum\sum\frac{\coth(n_{ij}B/2)}{(n_{ij}B)}\varepsilon\cdot\left[\frac{1}{(i^2-1)(j^2-1)}\right]^2=V\rho_w M \tag{20}$$

This coefficient M has just the value of coefficient of virtual mass which we have deduced (in the report I) by energy calculation. We must next estimate the amount of correction to be imposed on it. The result will be that ω is slightly different from ω_1, and that F_{11} will slightly differ from unity. This will be examined numerically, in the next section.

IV. Numerical check about the results of preceding section

In order to make numerical check about the result arrived at in the preceding section, let us take up the case in which B: H: $L=1$: 1: 2. In this case, we have $R=H/L=0.50$. Moreover, we shall assume that $S=10$. This is done only as a check calculation. But, we can see that the main conclusion will not be altered whether we have $S=5$, or $S=15$.

Values of coefficients Z_{ij} and E_{ij}, as indicated by equations (11) and (10), were evaluated numerically, and the result is shown in Table 1 and Table 2. As we see from the Table 2, values for $i=6$, or $j=7/2$, are so small that we can neglect them, at least for a practical use.

Table 1. Value of Z_{ij}

	$j=\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$
$i=0$	0.3279	1.473	2.50	3.50
2	1.053	1.791	2.692	3.62
4	2.062	2.50	3.202	4.00
6	3.03	3.34	3.90	4.60

Table 2. Value of E_{ij}

	$j=\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$
$i=0$	+8.12	−1.088	−0.153	−0.0508
2	−1.68	+0.598	+0.0945	+0.0305
4	−2.33	+0.0857	+0.0155	+0.00593
6	−0.050	+0.0365	+0.0054	+0.0022

Putting these values of E_{ij} into the equation (9), the pressure distribution on the face of rectangular plate (side-wall) can be obtained. This is shown in Fig. 2 and Fig. 3. Moreover, evaluating $G_{\lambda\mu}$ and $\Phi_{\lambda\mu}$ by means of equations (18) and (19), we obtain tne values shown in Table 3 and Table 4. Lastly, the values of ratio $\Phi_{\lambda\mu}/G_{\lambda\mu}$ are shown in Table 5. According to equations (14) and (17), the values of coefficients $F_{\lambda\mu}$ will be proportional to $\Phi_{\lambda\mu}/G_{\lambda\mu}$. *If* F_{31}, F_{12}, ... were negligibly small in comparison with F_{11}, it could be said that the tentative assumption of (7) was fairly accurate. Looking at the Table 5, we observe that F_{11}: F_{31}: F_{12} is in proportion of 1 : 0.025 : 0.021. Thus, we see that F_{31}, F_{12}, ... are very small in comparison with F_{11}, even if it may not be said to be negligibly small.

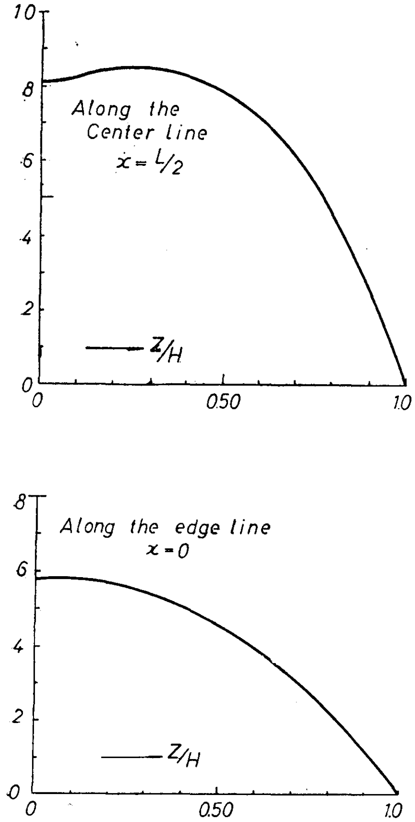

Fig. 2. Value of Φ (x, z) — (1).

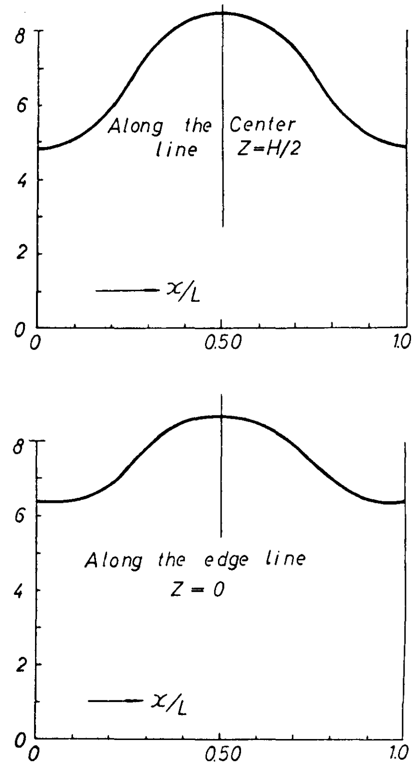

Fig. 3. Value of Φ (x, z) — (2).

Table 3. Value of $G_{\lambda\mu}$

	$\lambda=1$	3	5
$\mu=1$	1.40	10.74	52.3
2	17.84	39.0	105.2
3	85.3	126.5	232.0

Table 4. Value of $\Phi_{\lambda\mu}$

	$\lambda=1$	3	5
$\mu=1$	10.15	1.97	1.205
2	2.60	0.738	0.446
3	1.79	0.482	0.306

Table 5. Value of $\Phi_{\lambda\mu}/G_{\lambda\mu}$

	$\lambda=1$	3	5
$\mu=1$	7.25	0.183	0.0230
2	0.149	0.0188	0.00423
3	0.0210	0.00381	0.00132

The above inference is made about the case in which $B : H : L=1 : 1 : 2$, which is rather long in horizontal direction. The author has made similar checks for the dase of $B : H : L=1 : 2 : 1$, which is rather long in vertical direction, and observed that the accuracy of the tentative assumption (7) is slightly worse than in the case of $B : H : L=1 : 1 : 2$, but, of course, not so bad to affect the conclusion mentioned in the above.

The next step required for our investigation is the calculation of the corrected value of velocity potential ϕ. The expression (9) for ϕ was that value corresponding to the wave form $\sin mx \sin sz$ only, of the transverse displacement w. So that we must make similar calculation as in the report II, correponding to the added correction terms $\sin mx \sin 2sz$, $\sin 3mx \sin sz$, etc., for w. But, it can easily be inferred that this correction to ϕ will be very slight owing to two-fold reasons. The first reason in this reasoning lies on the fact that F_{12}, F_{31}, ... are small in comparison with unity, while the second reason is deduced from the fact that value of ϕ will become smaller if the number of nodal lines (wrinkles) of w is increased. Thus, we shall here contend ourselves by not giving actual values to this correction of velocity potential ϕ.

V. Kinetic energy and strain energy of the rectangular plates

The strain energy P_m of the rectangular elastic plate can be obtained from the formula

$$P_m=\frac{1}{2}D\int_0^L dx\int_0^H\Big[\Big(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial z^2}\Big)^2 -2(1-\nu)\Big\{\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial z^2}-\Big(\frac{\partial^2 w}{\partial x\partial z}\Big)^2\Big\}\Big]dz.$$

When the displacement w is given by (14), we have, by putting it into the above formula,

$$P_m=(A\sin\omega t)^2\frac{DLH}{8}\Big[(m^2+s^2)^2\big(F_{11}\big)^2 +(m^2+4s^2)^2(F_{12})^2+(9m^2+s^2)^2(F_{13})^2+\ldots\ldots\Big]. \tag{21}$$

While the kinetic energy of the rectangular plate is found, from the formula,

$$T_m=\frac{1}{2}\rho_m h\int_0^L dx\int_0^H\Big(\frac{\partial w}{\partial t}\Big)^2 dz,$$

to be given by,

$$T_m=\frac{1}{8}\rho_m hBL\Big[(F_{11})^2+(F_{12})^2+(F_{31})^2+\ \ldots\ldots\Big]\cdot(W_0\cos\omega t)^2 \tag{22}$$

For the case taken up by us in the previous section, viz.; the case in which $B : H : L=1 : 1 : 2$, the values of P_m and T_m has been calculated. The result is shown in Table 6, wherein proportional amounts about wave components $\sin mx \sin sz$, $\sin mx \sin 2sz$, and $\sin 3mx \sin sz$ are shown.

As to the kinetic energy of water, the part corresponding to wave form of

Table 6. Table of Proporttonal Distribution of Energy.

	$\lambda=1,\ \mu=1.$	$\lambda=1,\ \mu=2.$	$\lambda=3,\ \mu=1.$
Proportion of Strain Energy	1	0.0072	0.0030
Proportion of Kinetic Energy	1	0.000625	0.000441
Proportion of Amplitude of Vibration	1	0.025	0.021

$(\lambda=1,\ \mu=2)$, $(\lambda=3,\ \mu=1)$, ... will be of amount less than 1 % of the part corresponding to $(\lambda=1,\ \mu=1)$, from the inference mentioned in the end of previous section. Summing up these results, we infer that the difference in value of energies (kinetic— and strain—), between the case in which we take solely the term F_{11}, and the case in which we take F_{11}, F_{12}, and F_{31} into consideration, will be less than 1 %.

Consequently, we may safely conclude that the estimated values of natural frequency of vibration, will have a difference which is less than 1 %.

VI. Concluding remarks

From the above discussions, it may be allowed to conclude that the method of calculation of virtual mass of water, given by the author, is a fairly accurate one. It must be remarked that we are concerned only with the case of rectangular flat plate with simply supported four edges. Also, it must be noted that, if the rectangular plate was extremely long in horizontal or vertical direction, the accuracy will be considerably impaired.

Naturally, it will be questioned what will be the case, if the restangular plate was of clamped four edges (instead of being simply supported). The main line of thoughts will not be altered. But, since the normal modes of free-vibration of a rectangular plate is not given (at least for the purpose of numerical estimation), in the form convenient to handling, actual estimation will not be an easy task. The author is now carrying on, this tedious task of calculation, and wishes that the opportunity will be given, of reporting the result in this journal, as the continuation of the same study as the present report.