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Abstract	When side-walls of a rectangular tank, which is filled up with water, are vibrating, the inside water will also make a vibratory motion. This motion of water lowers considerably the natural frequency of vibration of side-walls of the tank. This effect is conveniently expressed by "virtual mass" of water. In the previous reports, I to V, of the same title, the author has made theoretical studies about the value of "virtual mass" of water, and examined various factors affecting it. In the present report, which is the continuation of the same study, the case is examined wherein two opposite (rectangular) side-walls are vibrating in a mode which correspond to the case of "clamped four edges" Approximate formula for the fundamental frequency of free-vibration of the system (consisting of rectangular elastic plates and inside-water) is given. The result is shown as graphs, which give values of confficient M of virtual mass of water, for different values of B/L and B/H (B=breadth, L=length, H=height, of the rectangular water tank). In addition, a case is examined, wherein the upper edge-lines of the tank is sligntly vibrating, instead of being kept immovable. The result is illustrated by a numerical example. The treatment throughout is made, on the assumption that water is an incompressible, non-viscous fluid, and that the vibration amplitude is infinitesimally small.
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On Virtual Mass of Water Contained in a Rectangular Tank, whose Side-Walls are Vibrating—VI

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Abstract

When side-walls of a rectangular tank, which is filled up with water, are vibrating, the inside water will also make a vibratory motion. This motion of water lowers considerably the natural frequency of vibration of side-walls of the tank. This effect is conveniently expressed by "virtual mass" of water. In the previous reports, I to V, of the same title, the author has made theoretical studies about the value of "virtual mass" of water, and examined various factors affecting it. In the present report, which is the continuation of the same study, the case is examined wherein two opposite (rectangular) side-walls are vibrating in a mode which correspond to the case of "clamped four edges" Approxjmate formula for the fundamental frequency of free-vibration of the system (consisting of rectangular elastic plates and inside-water) is given. The result is shown as graphs, which give values of confficient M of virtual mass of water, for different values of B/L and B/H (B=breadth, L=length, H=height, of the rectangular water tank).

In addition, a case is examined, wherein the upper edge-lines of the tank is sligntly vibrating, instead of being kept immovable. The result is illustrated by a numerical example.

The treatment throughout is made, on the assumption that water is an incompressible, non-viscous fluid, and that the vibration amplitude is infinitesimally small.

I. Intoroduction

Let us consider a rectangular tank, inside of which water is filled up. When side walls of the tank vibrate, the inside water will also make a vibratory motion. Owing to this fact, there appears, on natural frequency of vibration of side-walls, an effect called the "virtual mass" of water. In the reports I to V [1)2)3)4)5)] of the

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1) F. Kito, This Proceedings, Vol. 11, No. 40 (1958)

2) F. Kito, ditto, Vol. 12, No. 46 (1959)

3) F. Kito, ditto, Vol. 12, No. 47 (1959)

4) F. Kito, ditto, Vol. 13, No. 49 (1960)

5) F. Kito, ditto, Vol. 15, No. 56 (1962)

same title as the present report, the author has made theoretical studies about this effect, and examined various factors affecting it. In the present report, which is the continuation of the same line of study, the case is examined wherein two opposite (rectangular) side-walls are vibrating in a mode which correspond to the case of "clamped four edges."

Approximate formula for the fundamental frequency of free vibration of the system (consisting of rectangular elastic plates and water) is given. The result is shown as graphs, which give value of coefficient M of virtual mass of water, for different values of B/L and B/H. It may be pointed out here, that for a case of a single rectangular elastic plate (not in contact with water), which is in free vibration under the condition of "clamped four edges," the mode of vibration is not given in rigorous expression (at least in the compact form which admit us of easy handling). So that we had to start with the approximate expression for the mode of vibration (which is given in usual text-books[6]). The question of degree of accuracy of the formula worked out below, by the author may be raised. This question will be discussed in the future report under the same title.

In addition, a case is examined, wherein the upper edge-lines of the tank is vibrating, instead of being kept immovable. The result is illustrated by numerical example.

The treatment throughout is made on the assumption that water is an incompressible, non-viscous fluid, and that the vibration amplitude is of infinitesimally small magnitude. The following notations (which is mainly the same as for the previous reports) is used here:—

ϕ=velocity potential of the vibratory motion of the water, w=transverse displacement of the plate, A=vibration amplitude of ditto. ω=angular frequency of free vibration of ditto, L=length of rectangular water tank, H=its height, B=its breadth, ρ_m=density of material composing the flat plate, ρ_w=density of water, $W_0=\omega A$=amplitude of transverse velocity of vibration of the plate, T_m=kinetic energy of vibratory motion of the plate, T_w=kinetic energy of vibration of water, P_m=strain energy of the rectangular (elastic) plate, D=flexural rigidity of the plate, h=thickness of the plate, $V=LBH$=volume of the rectangular tank. Also we put

$$m=\pi/L,\quad s=\pi/H,\quad K=\omega^2H/g,\quad n_{ij}=\left[(mi)^2+(s_j)^2\right]^{1/2}.$$

II. Evaluaton of the kinetic energy of water, for the case of plate with clamped edges

Referring to Fig. 1, we take the case in which two side-walls with the dimension $H\times L$ are vibrating (together with the inside water). Each one of the plates

[6] S. Timoshenko, Theory of Elastic stability, 1936, p. 364 (Mc Graw-Hill Co.)

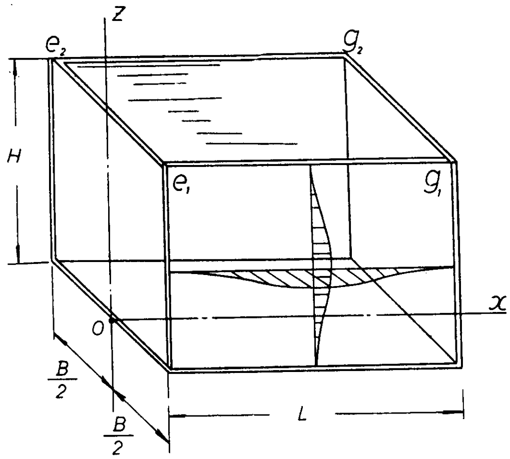

Fig. 1. Sketch of the Rectangular Water Tank.

being in state of "clamped four edges" the mode of vibration will be as sketched in Fig. 2. For the case of a single plate (when there is no water), this state of vibration is represented approximately by the following expression[6] :—

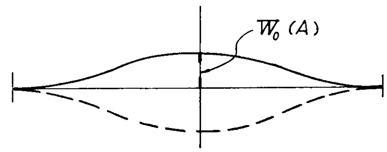

Fig. 2. Mode of Vibration of clamped-ends Type.

$$\frac{dw}{dt} = W_0 \cos \omega t \cdot \frac{1}{4}\left(1-\cos\frac{2\pi x}{L}\right)\left(1-\cos\frac{2\pi z}{H}\right). \qquad (1)$$

It is to be noted that this is not the exact expression, but is an approximate one. But since we are given no exact expression, which enable us of easy handling, it was thought that we must be content to start whih this approximate expression (1).

One panel at $y = +B/2$ (see Fig. 1) of side walls being in state of vibration given by (1), let us firstly take the case in which the other panel at $y = -B/2$ is vibrating with the same mode but in "opposite phase," that is, minus sign attached to the expression (1). The remaining two end-walls and the bottom wall being assumed to be immovable, the motion of water inside the tank, which is caused by the vibratory motion of two side-walls may be expressed by a velocity potential of the following form (as in the previous reports):—

$$\phi = \sum_i \sum_j B_{ij} \cosh(n_{ij}y) \cos(mix) \cos(s_j z), \qquad (2)$$

($i = 0, 1, 2, \ldots\ldots; j = 1, 2, 3, \ldots\ldots$), which saisfies the following boundary conditions:—

$$\text{at } x=0, \text{ or } =L, \qquad \partial\phi/\partial x=0,$$

$$\text{at } z=0, \qquad \partial\phi/\partial z=0,$$

$$\text{at } z=H, \qquad -\partial^2\phi/\partial t^2=g(\partial\phi/\partial z).$$

(condition of free-surface)

According to the last condition, we have $s_j=\xi_j/H$, where ξ_j are roots of the equation $\xi_j=-K\cot\xi_j$ $(K=\omega^2H/g)$. Expressing the fact that transverse velocity of rectangular plate coinside with that of water, at two faces $y=+B/2$ and $y=-B/2$, we have the equation,

$$\sum_i\sum_j B_{ij}n_{ij}\sinh(n_{ij}B/2)\cos(mix)\cos(s_jz)$$
$$=W_0\cos\omega t\cdot\frac{1}{4}\left(1-\cos\frac{2\pi x}{L}\right)\left(1-\cos\frac{2\pi z}{H}\right). \tag{3}$$

In actual cases which we meet in practice, the value of K is very large in comparison with unity. Confining ourselves to such cases, we have approximately,

$$s_j=\frac{\pi}{H}\left(\frac{2\sigma+1}{2}\right). \qquad (\sigma=0,\ 1,\ 2,\ \ldots\ldots)$$

Treating the equation (3) quite similar to that case of double Fourier series, we obtain,

$$B_{ij}=\frac{W_0\cos\omega t(LHB)}{(n_{ij}B)\sinh(n_{ij}B/2)}\cdot\left(\frac{\varepsilon}{4\pi^2}\cdot\frac{2\pi}{LH}\right)\cdot\left(\pm\frac{1}{2}J_\sigma\right), \tag{4}$$

where $i=0$ and 2; $j=(2\sigma+1)/2$, $(\sigma=0,\ 1,\ 2,\ 3,\ \ldots\ldots)$ The $+$sign in the r. h. s. of eq. (4) is to be taken for $i=0$, while we have to take the $-$sign for $i=2$.

Kinetic energy T_w of water (alloted to each one panel of side-wall) will be given by

$$T_w=\frac{1}{2}\rho_w\int_0^L dx\int_0^H\phi\frac{\partial\phi}{\partial y}dz,$$

from which we have, after evaluation of the integral,

$$T_w=\frac{1}{2}\rho_w(W_0\cos\omega t)^2(LHB)\cdot\sum_i\sum_\sigma\frac{\varepsilon}{16\pi^2}[J_\sigma]^2\frac{\coth(n_{ij}B/2)}{n_{ij}B} \tag{5}$$

($i=0$ and 2; $\sigma=0,\ 1,\ 2,\ \ldots\ldots$; $\varepsilon=2$ for $i=0$, $\varepsilon=4$ for $i=2$). Thus the kinetic energy of water can be expressed in the form;

$$T_w=\frac{1}{2}\rho_w(W_0\cos\omega t)^2MV, \tag{6}$$

where we put,

$$M=\sum_i\sum_\sigma\frac{\varepsilon}{16\pi^2}[J_\sigma]\frac{\coth(n_{ij}B/2)}{n_{ij}B}. \tag{7}$$

This coefficient M is the coefficient of virtual mass. It is to be remarked that the expression (6), (7) gives only an approximate value, in two senses. Namely, the mode of vibration (1), which was our starting point, gives only an approximate value. Next, the modification of mode of vibration due to the effect of water is not taken into account. (It may be expected to be of small amount, as we saw in report II.) Nevertheless, we ventured this calculation to be carried out, because there was demand for practical formula for virtual mass coefficient, even though it may be a rough estimate. The author is now preparing a report, wherein the degree of approximation of these rough estimates is discussed, and which, the author hopes, will appear in the near future.

Lastly, for the case of vibration in which two side-walls are in uibration "in-phase" each other, the formula similar to (6), (7) holds, the only difference being that we should take tanh instead of coth in the formula (7).

III. Remarks about the natural frequency of the system

For a rectangular elastic plate, which is in vibration prescribed by the eqation (1), the potential (or strain) energy P_m is given by,

$$P_m=\pi^4[A\sin\omega t]^2\frac{DL}{8H^3}\cdot\left[3\left(\frac{H}{L}\right)^4+2\left(\frac{H}{L}\right)^2+3\right], \qquad (8)$$

while its kinetic energy is given by,

$$T_m=\frac{1}{2}(W_0\cos\omega t)^2\frac{9}{16}LHh. \qquad (9)$$

The natural angular frequency ω of the system consisting of elastic side-walls and inside water, will be given by the equation

$$\overline{P}_m=\overline{T}_m+\overline{T}_w, \qquad (10)$$

where the bar — above the letters mean their (timely) mean values. If the mode of vibration (1) was exactly given, the equation (10) will furnish us exact value of the natural (angular) frequency ω of the system. But, due to above-mentioned reason, it will give only

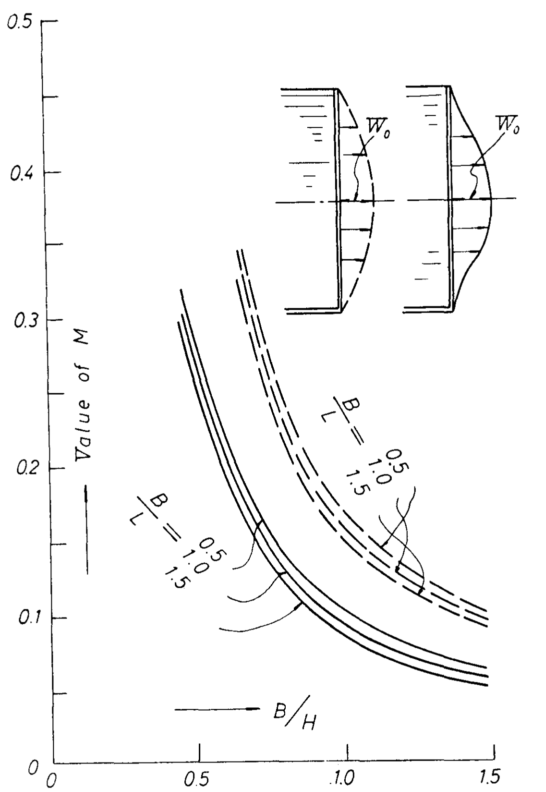

Fig. 3. Value of coefficient M of Virtual Mass, when side-walls are in state of clamped Four Edges. (Case of Opposite phase Vibration)

an approximate value. In Figures 3, and 4, the numerical values of the coefficient M of the virtual mass are shown as graphs.

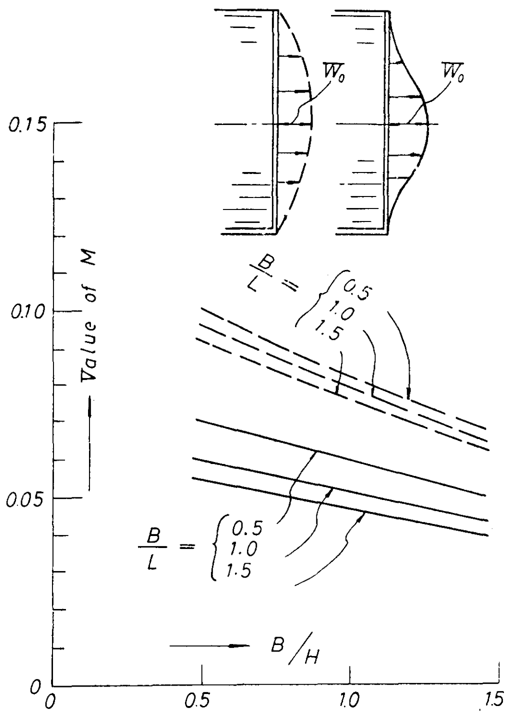

Fig. 4. Value of coefficent M of Virtual Mass, when Side-Walls are in state of clamped Faur Edges. (Case of In-phase Vibration)

IV. Evaluation of the kinesic energy of water, for the case in which upper edge-lines are slightly vibrating

In this section, the case will be considered, wherein (a) two opposite side-walls are vibrating, either "in phase" or "opposite phase" each other, the edges being in state of supported edges, (b) but the upper edge lines are slightly vibrating, instead of being kept immovable. The water will be supposed to be filled up, the upper (top) surface being in a state of free surface. For this case. we assume that the rectangular plate situated at $y=+B/2$ is vibrating in a mode which is expressed by,

$$\frac{dw}{dt} = W_0 \sin mx \sin Sz \cos \omega t, \tag{11}$$

where $S=\pi/(H+\varDelta H)$. The velocity potential ϕ of the vibratory motion of water caused by the vibration of side-walls, will be expressed by (2), as before.

The condition of coincidence of transverse velocities of plate and water, at the surface $y=+B/2$, is given by,

$$\frac{dw}{dt}=\sum_i\sum_j B_{ij}n_{ij}\cos(n_{ij}B/2)\cos(mix)\cos(s_j z), \tag{12}$$

$(i=0,\ 1,\ 2,\ \ldots\ldots;\ j=1,\ 2,\ 3,\ \ldots\ldots)$ From this equation, the value of coefficient B_{ij} is found to be,

$$B_{ij}=\frac{\varepsilon}{\pi}\left[\frac{V_j}{(i^2-1)}\right]\frac{W_0\cos\omega t}{k_j n_{ij}\cosh(n_{ij}B/2)}, \tag{13}$$

where we put,

$$U_j=\frac{1+\cos(\xi_j+\beta)}{\xi_j+\beta-\pi}-\frac{1+\cos(\xi_j-\beta)}{\xi_j-\beta+\pi},\quad k_j=1-\frac{1}{K}\sin^2\xi_j,$$

ξ_j being roots of the equation $\xi_j=-K\cos\xi_j$, and $K=\omega^2H/g$, $S=(\pi-\beta)/H$.

Corresponding value of kinetic energy of water contained in the tank can be expressed in the following form, as before;

$$T_w=\frac{1}{2}\rho_w\Big[W_0\cos\omega t\Big]^2(LBH)\cdot M, \tag{14}$$

where the coefficient M has the value;

$$M=\frac{1}{\pi^2}\sum_i\sum_j\left[\frac{U_j}{(i^2-1)}\right]^2\frac{\varepsilon}{k_j(n_{ij}B)}\coth(n_{ij}B/2) \tag{15}$$

for the case in which two side-walls are vibrating in opposite phase each other. The summation $\sum\sum$ in the above expression (14) is to be taken for $i=0, 2, 4, \ldots\ldots$; $j=1, 2, 3, \ldots\ldots$, and $\varepsilon=2$ for $i=0$, but $\varepsilon=4$ for $i\neq 0$. Kinetic energy T_w of eq. (14) is the value alloted to each one panel of side-wall.

For the case in which two side-walls are vibrating "in phase" each other, the same formula applies, the only difference being that we have to write tanh instead of coth in the expression (15). Moreover, it is to be remarked that, the value (15) was obtained by evaluation of the integral

$$T_w=\frac{1}{2}\rho_w\int_0^L dx\int_0^H \phi\frac{\partial\phi}{\partial y}dz.$$

There is left an integral over the top (free-surface). This value can be estimated as in the previous report,[4] and we have, for the additional term,

$$\Delta T_w=\frac{1}{2}\rho_w\Big[K\Big](LBH)\left(\frac{B}{H}\right)^2[W_0\cos\omega t]^2$$
$$\cdot\left[\sum_i\sum_j\sum_k\frac{2}{\varepsilon}\left(\frac{B_{ij}}{B}\right)\left(\frac{B_{ik}}{B}\right)K_{ijk}\cos\xi_j\cos\xi_k\right], \tag{16}$$

where we have $i=0, 2, 4, \ldots\ldots$; $j, k=1, 2, 3, \ldots\ldots$ and $\varepsilon=2$ if $i=0$ but $\varepsilon=4$ if $i\neq 0$. The coefficient K_{ijk} has the same value as for the case of report IV. This expression (16) correspond to the case of opposite phase vibration of side-walls. For the case of in-phase vibration, we have to put $\boldsymbol{M}_{ijk}$ instead of $\boldsymbol{K}_{ijk}$ in the above expression (16).

We have, for the case of opposite-phase vibration,

$$\frac{B_{ij}}{B}=\frac{\varepsilon}{\pi k_j}\left[\frac{U_j}{(i^2-1)}\right]\frac{1}{(n_{ij}B)\sinh(n_{ij}B/2)}$$

while we have, for the case of in-phase vibration,

$$\frac{B_{ij}}{B}=\frac{\varepsilon}{\pi k_j}\left[\frac{V_j}{(i^2-1)}\right]\frac{1}{(n_{ij}B)\cosh(n_{ij}B/2)} .$$

V. Remarks about the natural frequency of the system, for the case of previous secction

Since the kinetic energy of the water was evaluated, the approximate value of (fundamental) natural frequency of vibration of the systen can be obtained, as was pointed out in section III. In connection with this, the value of strain energy of flat plate is given by,

$$P_m=\pi^4[A\sin\omega t]^2\frac{DL}{8H^3}\cdot\left[\left\{\left(\frac{H}{L}\right)^2+1\right\}^2+(1-\nu)\left(\frac{H}{L}\right)^2(1-\mu)\right] \tag{17}$$

where we have put,

$$\mu=1+\frac{\sin 2\beta}{2(\pi-\beta)} \tag{18}$$

Also we have, for the kinetic energy T_m of the plate,

$$T_m=\frac{1}{2}\rho_m[W_0\cos\omega t]^2\frac{LHh}{4}\mu \tag{19}$$

Now, we saw in previous reports, that when the value of K is large in comparison with unity, we may replace the boundary condition at the top surface, namely

$$\text{at}\quad z=H,\ -\partial^2\phi/\partial t^2=-g\,(\partial\phi/\partial z),$$

by a more simple condition that at $z=H$, $\phi=0$. In this case, the matter is much simplified, since we have $\xi_j=(2\sigma+1)/2$, $(\sigma=0,1,2,\ldots\ldots)$, and the additional terms ΔT_w disappear. For this simplified case, the author has made some numerical calculations and has shown the result in graphs of Fig. 5.

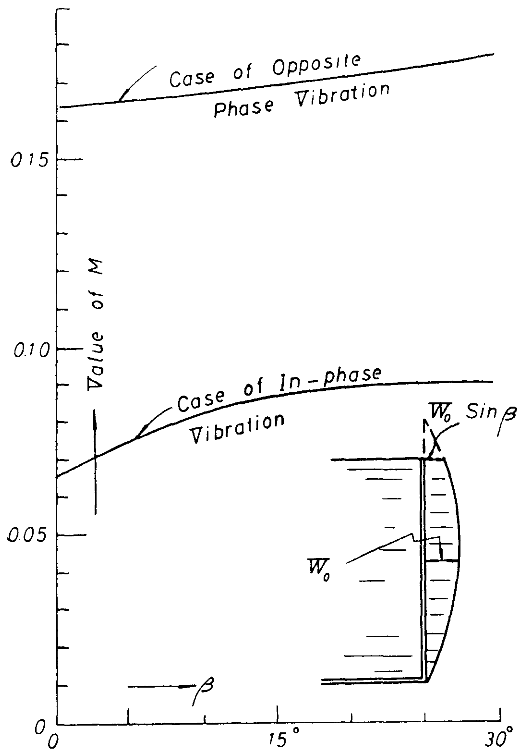

Fig. 5. Value of coefficient M of Virtual Mass, when the Top-edge Line is slightly vibrating.