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On Fluctuating Flow of Water through a Circular Pipe

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Abstract

In a straight pipe of circular section, a steady flow of water is taking place. This steady flow consist of uniform axial flow which is accompanied by a rotational motion with uniform angular velocity. At a given cross-section (say, at $z=0$), some disturbance is given to this steady flow. This disturbance gives non-uniformity of axial velocity which varies as $\sin \omega t$ with the time t. In this report, the resulting flow set up in the pipe is studied analytically, by solving hydrodynamical equation of motion.

The resultant force of water pressure acting on the inner-surface of the cylindrical pipe is estimated, and summarized as a convenient formula. The author wishes that it may throw some light on the question of estimation of "live load" of a pipe which is carrying a fluctuating flow.

I. Introduction

Let us consider the flow of water through a straight pipe of circular section. At the steady state, let us assume that the flow consist of a uniformly axial flow, accompanied with a rotational motion about the axis of the circular pipe. This rotational motion of water may conveniently be regarded to be the one with uniform angular velocity about the axis of pipe.

Moreover, let a small disturbance of flow be superimposed upon the above-mentioned steady flow, at the given cross-section of the pipe (say, at the section $z=0$). Then, this small disturbance will travel downstream through the pipe line. It is the object of the present paper to estimate the amount of this fluctuating fllow of water, thus set-up. The water is assumed to be an incompressible, non viscous fluid, and the disturbed part of the flow to be very small in comparison with the steady flow.

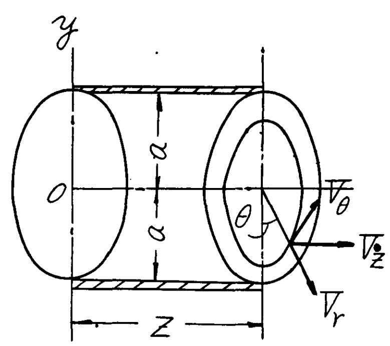

Fig. 1. Velocity components referred to cylindrical coordinates.

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When a pipe line is connected to a hydraulic machinery, such as a water-turbine or a pump, the flow throgh the pipe can not always be steady. In fact, there are many cases in which the fluctuating flow is introduced through the pipe line. This fluctuating flow through the pipe-line may exert a force which would cause the bodily-vibration of the pipe-line. The author hopes that, the present paper may give some information about the amount of force, which would cause the bodily-vibration of the pipe line.

II. The fundamental equation

Taking a system of cylindrical coodinates (r, θ, z), let the steady flow be expressed by $(V_r, V_\theta, V_z,)$, while the disturbed part of flow will be denoted by (v_r, v_θ, v_z). The pressure of water will be denoted by P, for steady flow, and by p, for the disturbed part of the flow.

First, for the steady flow, the Euler's equation of motion can be written;

$$\left.\begin{aligned}
-\frac{1}{\rho}\frac{\partial P}{\partial r} &= \frac{\partial V_r}{\partial t}+V_r\frac{\partial V_r}{\partial r}+V_\theta\frac{\partial V_r}{r\partial\theta}+V_z\frac{\partial V_r}{\partial z}-\frac{V_\theta^2}{r}\,,\\
-\frac{1}{\rho}\frac{\partial P}{r\partial\theta} &= \frac{\partial V_\theta}{\partial t}+V_r\frac{\partial V_\theta}{\partial r}+V_\theta\frac{\partial V_\theta}{r\partial\theta}+V_z\frac{\partial V_\theta}{\partial z}+\frac{V_\theta V_r}{r}\,,\\
-\frac{1}{\rho}\frac{\partial P}{\partial z} &= \frac{\partial V_z}{\partial t}+V_r\frac{\partial V_z}{\partial r}+V_\theta\frac{\partial V_z}{r\partial\theta}+V_z\frac{\partial V_z}{\partial z}\,,
\end{aligned}\right\} \tag{1}$$

and, the equation of continuity can be written;

$$\frac{1}{r}\frac{\partial(rV_r)}{\partial r}+\frac{\partial V_\theta}{r\partial\theta}+\frac{\partial V_z}{\partial z}=0\,. \tag{2}$$

Secondly, for the fluctuating flow, similar equations can be given, when we relace (V_r, V_θ, V_z) and P, by $(V_r+v_r, V_\theta+v_\theta, V_z+v_z)$ and $P+p$ in the above equations (1) and (2). Substracting corresponding members of these two systems of equations, and neglecting squares and products of v_r, v_θ and v_z, we have;

$$\left.\begin{aligned}
-\frac{1}{\rho}\frac{\partial p}{\partial r} &= \frac{\partial v_r}{\partial t}+v_r\frac{\partial V_r}{\partial r}+v_\theta\frac{\partial V_r}{r\partial\theta}+v_z\frac{\partial V_r}{\partial z}+V_r\frac{\partial v_r}{\partial r}+V_\theta\frac{\partial r}{r\partial\theta}+V_z\frac{\partial v_r}{\partial z}\\
&\quad-\frac{1}{r}(2v_\theta V_\theta)\,,\\
-\frac{1}{\rho}\frac{\partial p}{r\partial\theta} &= \frac{\partial v_\theta}{\partial t}+v_r\frac{\partial V_\theta}{\partial r}+v_\theta\frac{\partial V_\theta}{r\partial\theta}+v_z\frac{\partial V_\theta}{\partial z}+V_r\frac{\partial v_\theta}{\partial r}+V_\theta\frac{\partial v_\theta}{r\partial\theta}+V_z\frac{\partial v_\theta}{\partial z}\\
&\quad+\frac{1}{r}(V_\theta v_r+V_r v_\theta)\,,\\
-\frac{1}{\rho}\frac{\partial p}{\partial z} &= \frac{\partial v_z}{\partial t}+v_r\frac{\partial V_z}{\partial r}+v_\theta\frac{\partial V_z}{r\partial\theta}+v_z\frac{\partial V_z}{\partial z}\\
&\quad+V_r\frac{\partial v_z}{\partial r}+V_\theta\frac{\partial v_z}{r\partial\theta}+V_z\frac{\partial v_z}{\partial z}\,,
\end{aligned}\right\} \tag{3}$$

and

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r}+\frac{\partial v_\theta}{r\partial_\theta}+\frac{\partial v_z}{\partial z}=0 \ . \tag{4}$$

According to our assumption about the steady flow we have $V_r=0$, $V_\theta=\Omega r$, $V_z=V_0$, where Ω and V_0 are constants. Putting these values into (3) and (4), we have,

$$\left.\begin{aligned}
-\frac{1}{\rho}\frac{\partial p}{\partial r}&=\frac{\partial v_r}{\partial t}+\Omega r\frac{\partial v_r}{r\partial\theta}+V_0\frac{\partial v_r}{\partial z}-2\Omega v_\theta \ , \\
-\frac{1}{\rho}\frac{\partial p}{r\partial\theta}&=\frac{\partial v_\theta}{\partial t}+\Omega v_r+\Omega\frac{\partial v_\theta}{\partial\theta}+V_0\frac{\partial v_\theta}{\partial z} \\
&\quad+\frac{2}{r}\Omega r v_r \ , \\
-\frac{1}{\rho}\frac{\partial p}{\partial z}&=\frac{\partial v_z}{\partial t}+\Omega\frac{\partial v_z}{\partial\theta}+V_0\frac{\partial v_z}{\partial z} \ , \\
\frac{1}{r}\frac{\partial(rv_r)}{\partial r}&+\frac{\partial v_\theta}{r\partial\theta}+\frac{\partial v_z}{\partial z}=0 \ .
\end{aligned}\right\} \tag{5}$$

This is the fundamental equation of our problem. If we put $V_0=0$ in the above eq. (5), it reduces to the equation treated by Lord Kelvin (see, for example, Gray and Mathews, Treatise on Bessel Functions.)

So that, the solution given below will naturally be almost the same as given by Lord Kelvin, except that, here the terms with V_0 appear.

III. The solution of our problem

We see, by actual substitution, that our equation (5) can be satisfied by putting

$$\left.\begin{aligned}
v_r&=R\sin(mz+n\theta-\omega t) \ , \\
v_\theta&=\Theta\cos(mz+n\theta-\omega t) \ , \\
v_z&=Z\cos(mz+n\theta-\omega t) \ , \\
p&=\Pi\cos(mz+n\theta-\omega t) \ ,
\end{aligned}\right\} \tag{6}$$

where R, Θ, Z and Π are functions of r. m and ω are (positive or negative) constants. n is a positive integer. Upon substitution, we have the following system of equations for R, Θ, Z and Π.

$$\left.\begin{aligned}
-\frac{1}{\rho}\frac{d\Pi}{dr}&=(mV_0+n\Omega-\omega)R-2\Omega\Theta \ , \\
\frac{1}{\rho}\frac{n}{r}\Pi&=(mV_0+n\Omega-\omega)(-\Theta)+2\Omega R \ , \\
\frac{1}{\rho}m\Pi&=(mV_0+n\Omega-\omega)(-Z) \ , \\
\frac{1}{r}\frac{d(rR)}{dr}&-\frac{n}{r}\Theta-mZ=0 \ ,
\end{aligned}\right\} \tag{7}$$

From these equations we deduce,

$$\left.\begin{aligned} R &= \frac{N}{m\,(N^2-4\Omega^2)}\left[2n\Omega\frac{Z}{r}+N\frac{dZ}{dr}\right] , \\ \Omega &= \frac{N}{m\,(N^2-4\Omega^2)}\left[nN\frac{Z}{r}+2\Omega\frac{dZ}{dr}\right] , \\ \frac{1}{\rho}\Pi &= -\frac{N}{m}Z , \end{aligned}\right\} \quad (8)$$

where we put, for shortness, $N=n\Omega+mV_0-\omega$.

Putting the values of (8) into the last equation of (7), we have

$$\frac{d^2Z}{dr^2}+\frac{1}{r}\frac{dZ}{dr}+\left(k^2-\frac{n^2}{r^2}\right)Z=0 , \quad (9)$$

where we put

$$k^2=\frac{m^2}{N^2}(4\Omega^2-N^2) . \quad (10)$$

The value of k^2 may have positive or negative values. Here we shall examine the case in which k^2 has a positive value. In that case, the general solution of the equation (9) is given by;

$$Z=AJ_n(kr)+BY_n(kr) ,$$

where A and B are arbitrary constants, J_n and Y_n represent Bessel functions of first and second kind, of order n.

When water occupies the whole inner domain $0\leqq r\leqq a$ of the pipe, there can exist no singularity at $r=0$, so we must have $B=0$. Thus, the solution of our problem is given by

$$\left.\begin{aligned} Z&=AJ_n(kr) , \\ \Pi&=-\rho(N/m)Z , \end{aligned}\right\} \quad (11)$$

Next, at the boundary surface $r=a$ of the cylindrical pipe, we must have $v_r=0$, or $R=0$. From the eq. (8) we see that we must have

$$\frac{2n\Omega}{a}J_n(ka)+NkJ_n'(ka)=0 ,$$

or

$$\frac{kaJ_n'(ka)}{J_n(ka)}=-\frac{2n\Omega}{N} . \quad (12)$$

Since we have, by (10),

$$N=2m\Omega/\sqrt{k^2+m^2} ,$$

the eq. (12) can be rewritten as follows;

$$\frac{\xi J_n'(\xi)}{J_n(\xi)}=-\frac{n}{(ma)}\sqrt{\xi^2+(ma)^2} . \quad (13)$$

This equation (13), regarded as an equation for ξ, have an infinite number of real roots. For each one of the root $\xi=\lambda$, we have the relation

$$\frac{\omega}{\Omega}=\left[1-\frac{2\,(ma)}{\sqrt{\lambda^2+(ma)^2}}\right]+(ma)\frac{V_0}{\Omega a}\,. \tag{14}$$

From this equation, we see that, when the angular frequency ω of the disturbed flow is given, the value of ma (or m) must be so chosen as to satisfy the relation (14). Thus, there are an infinite number of solutions of the form (11), corresponding to each pair of values of $(\lambda_i,\ m_i)$ $(i=1,\ 2,\ 3......)$.

For a given value of angular frequency ω, and given distribution of v_z at a given instant (say $t=0$), and tor a given plane (say $z=0$), we can obtain the solution for disturbed flow at any point inside the pipe, by making linear combination of special solutions for $(\lambda_i,\ m_i)$.

VI. Detailed study for the case of $n=1$

When we evaluate the amount of resultant force, exerted by the water pressure p acting on the surface $r=a$ of circular pipe, we find it null except for $n=1$. For the use in practical problem, of estimating amount of resultant force which tend to vibrate the pipe as a whole body, we shall examine in more detail the case of $n=1$.

For $n=1$, the eq. (13) can be written,

$$\frac{\xi J_1'(\xi)}{J_1(\xi)}=\frac{\xi J_0(\xi)}{J_1(\xi)}-1=-\frac{1}{(ma)}\sqrt{\xi^2+(ma)^2}\,. \tag{15}$$

The roots λ_i of the eq. (15) can be obtained by finding the points of intersection of two plane curves

$$y=\frac{\xi J_1'(\xi)}{J_1(\xi)} \qquad \text{and} \qquad y=-\frac{\sqrt{\xi^2+(ma)^2}}{(ma)}\,.$$

This is shown in Fig 2, which shows only the lowest roots.
The result is summarized in Fig. 3, which shows us the relation between values of λ and (ma), which satisfy the eq. (13). In Fig. 4, the relation between (ω/Ω) and (ma), as given by the eq. (14) is shown, it being understood that the value of λ is connected to (ma) by the curve of Fig. 3.

For the disturbed flow given by (11), or

$$Z=AJ_1(kr)\,,\qquad \Pi=-\rho\,(N/m)\,Z\,, \tag{11'}$$

let the maximum value of Z be V_a. Then we have

$$A=V_a/J_1(ka)=V_a/J_1(\lambda)$$

and the unknown constant A is determined.

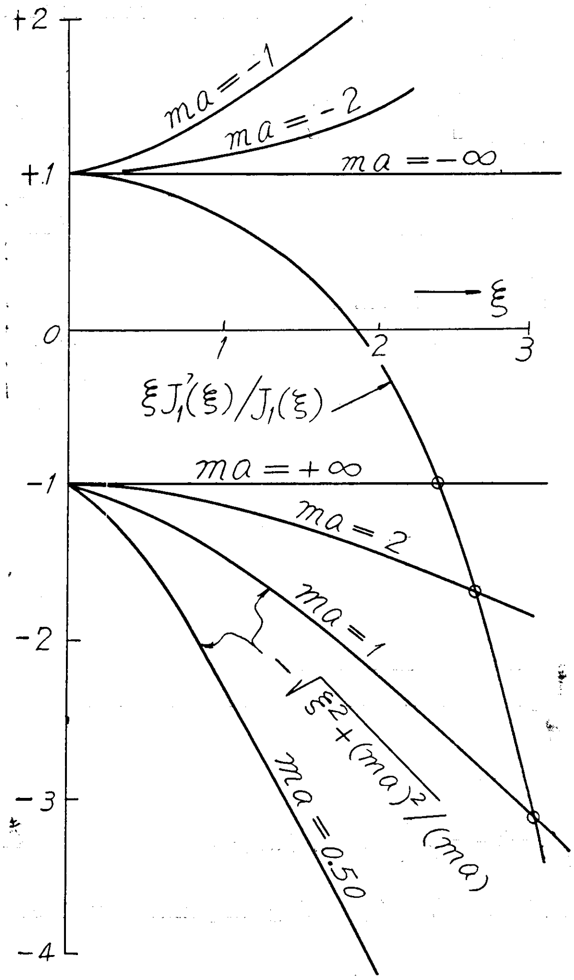

Fig. 2. Graph for finding the roots of eq. $\xi J_1'(\xi) / J_1(\xi) = -\sqrt{\xi^2+(ma)^2}/(ma)$.

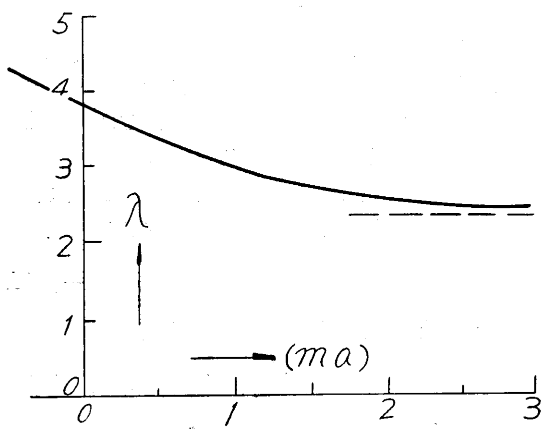

Fig. 3. The relation between λ and (ma).

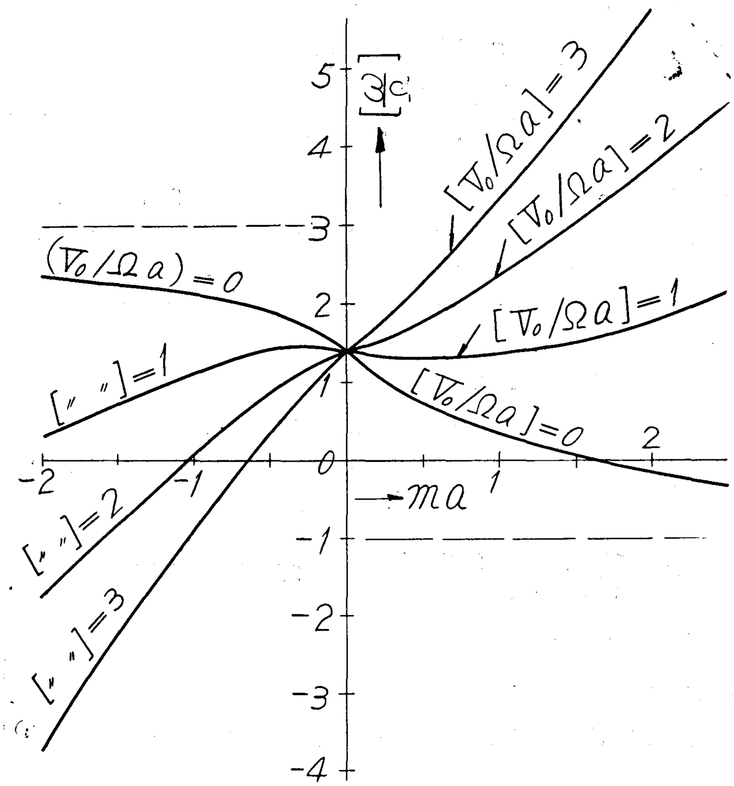

Fig. 4. The relation between (ma) and (ω/Ω).

Next, let the value of angular frequency ω of disturbed flow be given. Then, by Fig. 4 (values of $V_0/\Omega a$ and ω/Ω being known), corresponding value of (ma) can be found out. The wave-length L of fluctuating flow is then obtained by

$$L=2\pi a/(ma)$$

(a) The resultant force exerted by water on the pipe-line

Combining equs. (6) and (11′) we have

$$p=-\rho\left(\frac{N}{m}\right)V_a\frac{J_1(kr)}{J_1(\lambda)}\cos(mz+\theta-\omega t)\ ,$$

and so the pressure on the pipe wall $(r=a)$ is

$$p=-\rho\left(\frac{N}{m}\right)V_a\cos(mz+\theta-\omega t)\ .$$

Tne resultant force of this water pressure, acting on length $\frac{1}{2}L$ (half the wave-length) of pipe-line is found to be

$$F=\rho(N/m)V_a\times(\pi a)\times\left(\frac{2}{\pi}\times\frac{L}{2}\right)=\rho(N/m)V_a(aL)\cos\omega t\ .$$

Or, putting the value of N into it, we find finally,

$$F = \frac{r}{g} \frac{2(\Omega a)V_a}{\sqrt{\lambda^2+(ma)^2}} (aL) \cos \omega t \ , \tag{16}$$

where γ is the specific weight of water. (Ωa) is the peripheral velocity of the rotating water.

As a numerical example, let us consider a pipe line 3 m in dia. ($a=1.50$ m), through which water is flowing with uniform axial velocity of 5 m/sec. The water is also making a rotaing motion such that $\Omega a=3$ m/sec. To this steady flow a disturbance is given, for which $V_a=1$ m/sec., having the frequency such that $\omega/\Omega=2$ (this corresponds to about 1.5/cycle/sec). For this case, we have, by Fig. 4, $(ma)=0.70$ and so, by Fig. 3, $\lambda=3.20$. Also we have

$$L=2\pi \times 1.50 \div 0.70 = 13.4 \text{ m} \ .$$

Putting these values into the formula (16), we have, for the amplitude of the force F,

$$F_a = \frac{1000}{9.8} \times \frac{2\times 3\times 1}{\sqrt{(3.2)^1+(0.70)^2}} = 3780 \text{ kg} = 3.78 \text{ ton}$$

per 6.7 m length of the pipe line.

(b) The center of mass-flow

Referring to Fig. 6, the center of mass-flow, at any cross-section of the pipe,

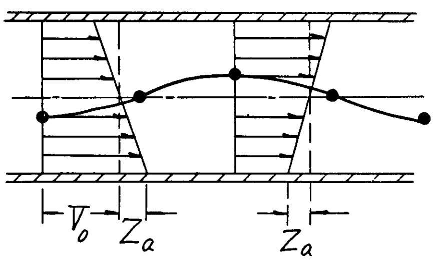

Fig. 5. Illustrating the fluctuating flow through a circular pipe.

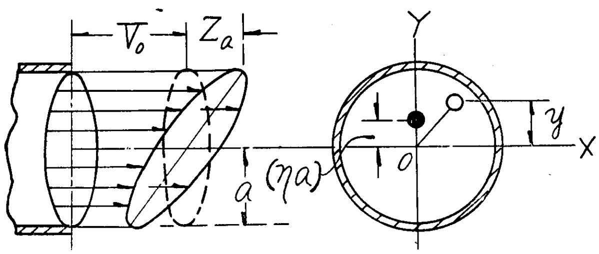

Fig. 6. Showing the center of mass-flow.

is given by

$$\eta a = M \div \left\{ \frac{\gamma}{g} (\pi a^2 V_o) \right\} ,$$

where ηa is the distance from the XX-axis of the center of mass-flow, and M is defined by

$$M = \frac{\gamma}{g} \int_0^a dr \int_0^{2\pi} d\theta \, [V_o + v_z] \, yr \, dr \, d\theta \ .$$

Putting the value of v_z as given by (6) and (11), into the above expression for M, we obtain,

$$\eta = \frac{Z_a}{V_o} \left[\frac{J_2(\lambda)}{\lambda J_1(\lambda)} \right] \sin (mz + \omega t) \ .$$

(c) Calculation of reaction force by "change of momentum"

Imagine that the total mass of water flowing through any cross-section of the pipe, is concentrated at the center of mass-flow, and is in motion like a mass-particle. In making this motion, reaction-force will be exerted in a direction transverse to the axis of pipe. The amount F_R of this force is estimated as follows;

The gradient α, of the path of the mass-particle is

$$\alpha = \frac{\partial}{\partial z} [a\eta \sin (mz - \omega t)] = ma\eta \cos (mz - \omega t) \ ,$$

Whence we have

$$F_R = \left(\frac{\gamma}{g} \pi a^2 V_o \right) \Big| \, \alpha \, \Big|_{z=0}^{z=L/2} = \left(\frac{\gamma}{g} \pi a^2 V_o \right) (2ma\eta V_o) \cos \omega t$$

$$= \frac{2\gamma}{g} \left[\frac{(ma)^2}{2} \right] \left[\frac{J_2(\lambda)}{\lambda J_1(\lambda)} \right] [Z_a V_o] \ . \qquad (17)$$

Thus, writing the forces F and F_R in the form,

$$F = \frac{2\gamma}{g} [(\Omega a) Z_a] (aL) \cdot K \cdot \cos \omega t \ ,$$

$$F_R = \frac{2\gamma}{g} [(\Omega a) Z_a] (aL) \cdot K_R \cdot \cos \omega t \ ,$$

we shall have

$$K = \frac{1}{\sqrt{\lambda^2 + (ma)^2}} \ ,$$

$$K_R = \frac{(ma)^2}{2} \left[\frac{J_2(\lambda)}{\lambda J_1(\lambda)} \right] \left[\frac{V_o}{\Omega a} \right] \ .$$

It may be interesting to compare the values of two coefficients K and K_R. In table 1, this comparison is made for the case in which $V_0/(\Omega a) = 2$.

Table 1.

$ma=$	0.5	1.0	2.0
$K=$	0.255	0.316	0.297
$K_R=$	0.536	0.486	1.56

From this table, we infer that K_R is (in general) much larger than K. For the case of $ma=2.0$, K_R is almost five times as large as K. But, in this case, half-wave-length $\frac{1}{2}L$ is only equal to $(\pi/4)\times$ diameter of the pipe. Such a case of extremely small wave-length may probably not occur in actual pipe-line, which is connected to hydraulic machinery.