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# Potential Flow Accompanied with Vortices about Rotating Impellers with Radial Vanes. -I.

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**Abstract**

Two-dimensional potential flow through an impeller having radial vanes is considered by using the complex velocity potential. The impeller consist of radial straight vanes arranged symmetrically about center of impeller, and it rotates with a uniform angular velocity about the center. It is assumed that isolated vortices are set up at points lying between the vanes. The flow is to consist of four parts, namely; (a) discharge flow issuing from the center of impeller, (b) so-called channel vortex flow, or the flow caused by the pushing action of moving blades, (c) circulatory flow about each vane, (d) flow caused by isolated vortices, whose vortex centers lie at some point between the vanes.

By consideration of combined flow caused by these four component flows, and imposing the condition of finitude of flow-velocity at two edges of each vane, the relation between the discharge, amount of circulation and strength of isolated vortex is deduced. The result is explained by a numerical example.

## I. Introduction

It is long years ago that the problem of rotating impellers with radial vanes has been studied, by using the complex velocity potential.[1] Here the author intends to treat the similar problem for the case in which there exist isolated vortices at points lying in between the radial vanes. The author is not aware whether this case has ever been treated. Nevertheless, the author has recently studied this problem, since he felt the growing tendency of use of impellers with radial vanes in some recent engineering fields.

## II. Conformal transformation for the impeller with radial vanes

In Fig. 1, the four complex planes $z$, $z_1$, $z_2$ and $z_3$ are shown.

The relation or correspondence between these four planes are given by,

$$z_1=\frac{1}{2}\left(z+\frac{1}{z}\right), \qquad z_2=p+qz_1, \qquad z_3=(z_2)^{1/n}, \tag{1}$$

where $p$ and $q$ are positive constants, and $n$ is the positive integer representing

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[1] See, for example, Hydraulische Probleme (Julius Springer), 1926.

number of vanes of impeller. From these relations, we see that the unit circle $z=e^{i\theta}$ in the $z$-plane is transformed into a straight line of length 2 in the $z_1$-plane, which again is transformed into the straight line of length $2q$, in the $z_2$-plane. Finally, this straight line $AB$ of length $2q$ in the $z_2$-plane is transformed into an array of radial blades, arranged symmetrically about the origin, in the $z_3$-plane, consisting of $n$ vanes.

Denoting by $r_1$ and $r_2$, the inside and outside radii of the impeller, we have

$$r_1=(p-q)^{1/n}$$

$$r_2=(p+q)^{1/n}$$

So that, if we put $\varepsilon=r_1/r_2$, then we shall have

$$q/p=(1-\varepsilon)^n/(1+\varepsilon)^n,$$

$$p=r_2^n\left(1+\frac{q}{p}\right).$$

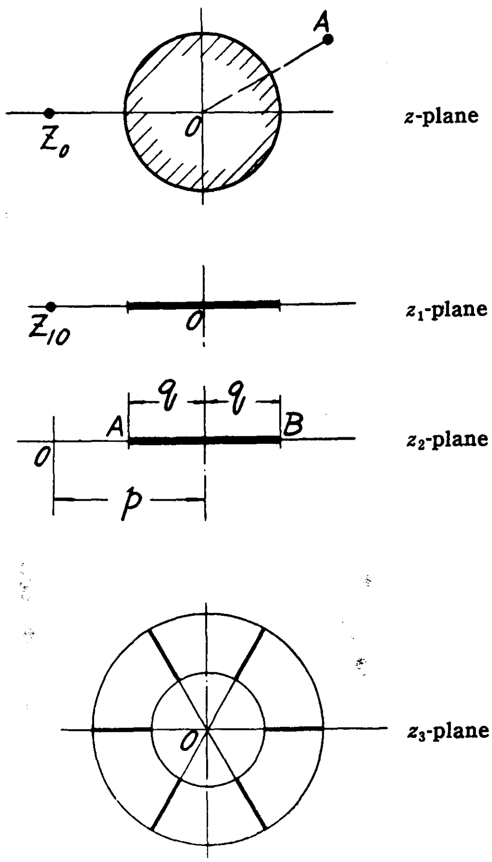


**Fig. 1.** Conformal representation of Impeller with radial vanes in $z_3$-plane from unit circle in $z$-plane.

## III. Complex velocity-potential for the flow around the impeller with radial vanes

The flow around the rotating impeller having radial vanes is here thought to be composed of the following component flows.

(a) Discharge flow issuing from the center of impeller, as sketched in Fig. 2(a). The complex velocity potential representing this flow will be given by,

$$W_a=\frac{Q}{2\pi}\left[(1+i\tan\gamma)\log(z-z_0)+(1-i\tan\gamma)\log\left(\frac{1}{z}-z_0\right)\right], \qquad (2)$$

where $Q$ is the discharge or quantity of flow. If the flow is one which converges into the center of impeller, $Q$ is to be taken a negative value. $\gamma$ is the angle which repesents the degree of rotation of the issuing (or, converging) flow. For a positive value of $\gamma$, the flow will rotate in clockwise direction. $z_0$ is a point in the $z$-plane, which is the source-point. We put $z_0=-c$, where $c$ is a positive real number greater than 1.

(b) Channel vortex flow, or, the flow generated by action of the rotating blades. The complex velocity potential representing this flow will be given by,

$$W_b=\omega i\left[\frac{a_1}{z}+\frac{a_2}{z^2}+\frac{a_3}{z^3}+\cdots\cdots\right], \tag{3}$$

where $\omega$ is the angular velocity of rotation of the impeller. $a_1$, $a_2$, …… are unknown constants (real numbers) to be determined later. $\omega$ is taken positive when the impeller rotates in clockwise direction.

(c) Circulating flow around each vane, which is given by the complex velocity potential

$$W_c=\frac{J}{2\pi i}\log z\,. \tag{4}$$

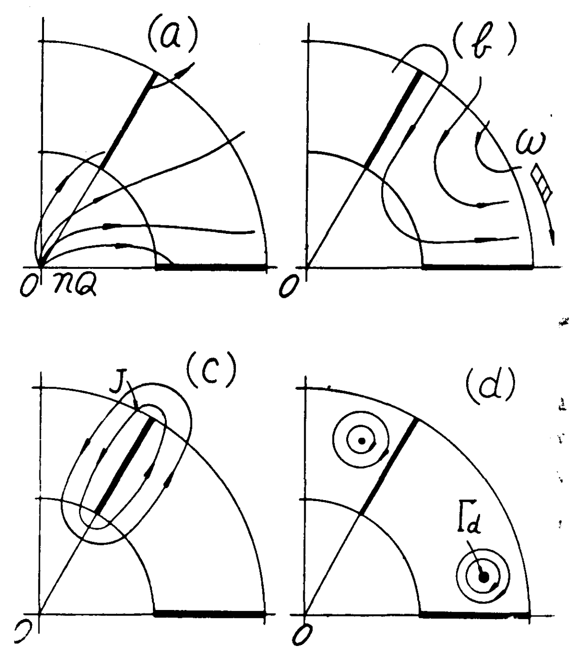


**Fig. 2. Four component flows through rotating impeller with radial vanes.**

When $J$ is positive, the circulation will take place in counter-clockwise direction.

(d) Flow due to isolated vortices which exist behind each vanes, represented by

$$W_d=\frac{i\Gamma_d}{2\pi}\log\frac{z-A}{z-\left(\frac{1}{\bar{A}}\right)}, \tag{5}$$

$\Gamma_d$ being the strength of isolated vortex, is positive when the rotation takes place in clockwise direction. $A$ is a complex number which represents the center of the isolated vortex. $\bar{A}$ is the complex number conjugate to $A$.

The combined flow is given by the complex velocity potential $W_s$, where

$$W_s=W_a+W_b+W_c+W_d\,. \tag{6}$$

## IV. Determination of constants $a_1$, $a_2$, …… for the channel vortex flow

The velocity at any point in the $z_3$ plane, of flow $W_b$ is to be obtained by the equation,

$$V_x-iV_y=\frac{dW_b}{dz_3}=\frac{dW_b}{dz}\frac{dz}{dz_3}=\omega i\left[-\sum_{m=1}^{\infty}\frac{ma_m}{z^{m+1}}\right]\frac{dz}{dz_3}, \tag{7}$$

while we have

$$\frac{1}{(dz/dz_3)}=\frac{dz_3}{dz}=\frac{dz_3}{dz_2}\frac{dz_2}{dz_1}\frac{dz_1}{dz} \qquad \frac{dz}{dz_3}=\frac{2n}{q}(z_2)^{1-1/n}\frac{z}{(z-1/z)} \tag{8}$$

Or, on the surface of the vane, we have,

$$z=e^{i\theta}, \qquad z_1=\cos\theta, \qquad z_2=p+q\cos\theta\,,$$

and so, by (7) and (8), on the surface of the vane which lie along the real positive axis of $z_3$-plane;

$$V_x-iV_y=-\omega\left[\sum_{m=1}^{\infty} ma_m(\cos m\theta-i\sin m\theta)\right]$$

$$\times\left(\frac{n}{q}\right)(p+q\cos\theta)^{1-1/n}\frac{1}{\sin\theta}$$

Equating the value $V_y$ obtained from the above equation (which is the normal velocity to vane surface), to $-\omega R$, where $R$ is the radial distance from the origin, we have

$$\sum_{m=1}^{\infty} ma_m\sin m\theta=\frac{q}{n}(p+q\cos\theta)^{2/n-1}\sin\theta\,. \tag{9}$$

This equation tells us that the values of $ma_m$ ($m=1, 2, \ldots\ldots$) are to be obtained as Fourier coefficients of right-hand side of (9). This is what has already been shown[1]. Now, integrating both sides of this equation (9), we obtain

$$\sum_{m=1}^{\infty} a_m(1-\cos m\theta)=\frac{1}{2}\left[(p+q)^{2/n}-(p+q\cos\theta)^{2/n}\right]. \tag{10}$$

Also, putting $\theta=\pi$ into this equation (10), we have

$$a_1+a_3+a_5+\ldots\ldots=\frac{1}{4}\left[(p+q)^{2/n}-(p-q)^{2/n}\right]. \tag{11}$$

Further, multiplying by $\cos m\theta$ both sides of equation (10), and integrating with respect to $\theta$, from $\theta=0$ to $\theta=2\pi$, we have

$$a_m=-\frac{1}{\pi}\int_0^{2\pi}\frac{f(\theta)}{\sin\left(\frac{1}{2}\theta\right)}\cos m\theta\,\sin\frac{\theta}{2}\,d\theta\,, \tag{12}$$

where we put for shortness,

$$f(\theta)=\frac{1}{2}\left[(p+q)^{2/n}-(p+q\cos\theta)^{2/n}\right].$$

Now, from the formula

$$2\cos m\theta\,\sin\frac{\theta}{2}=\sin\left(m+1-\frac{1}{2}\theta\right)-\sin\left(m-\frac{1}{2}\right)\theta\,,$$

we have

$$2\sum_{m=1}^{M}\cos m\theta\,\sin\frac{\theta}{2}=\sin\left(M+1-\frac{1}{2}\right)\theta-\sin\frac{1}{2}\theta$$

And so, by summing up both sides of equation (12), we have,

$$\sum_{m=1}^{\infty}a_m=\frac{1}{4\pi}\int_0^{2\pi}[(p+q)^{2/n}-(p+q\cos\theta)^{2/n}]\,d\theta\,. \tag{13}$$

Or, by putting

$$\sum_{m=1}^{\infty}a_m=r_2^2M,\qquad (a_1+a_3+a_5+\ldots\ldots)=r_2^2N \tag{14}$$

we have,

$$M=\frac{1}{4\pi}\frac{1}{\left(1+\frac{q}{p}\right)^{2/n}}\int_0^{2\pi}\left[\left(1+\frac{q}{p}\right)^{2/n}-\left(1+\frac{q}{p}\cos\theta\right)^{2/n}\right]d\theta\,, \tag{15}$$

$$N=\frac{1}{4}\frac{1}{\left(1+\frac{q}{p}\right)^{2/n}}\left[\left(1+\frac{q}{p}\right)^{2/n}-\left(1-\frac{q}{p}\right)^{2/n}\right]. \tag{16}$$

## V. Determination of conatants $\Gamma_d$ and $J$.

For the velocity of flow $V_x+iV_y$, on the $z_3$-plane, for the combined flow $W_s=W_a+W_b+W_c+W_d$, we have

$$V_x-iV_y=\frac{dW_s}{dz_3}=\frac{dW_s}{dz}\frac{dz}{dz_3}=\Big[(1+i\tan\gamma)\frac{Q}{2\pi}\frac{1}{z-z_0}$$
$$+(1-i\tan\gamma)\frac{Q}{2\pi}\left(-\frac{1}{z}-\frac{z_0}{1-z_0z}\right)$$
$$-\omega i\sum_{m=1}^{\infty}\frac{a_m}{z^{m+1}}+\frac{J}{2\pi i}\frac{1}{z}+\frac{i\Gamma_d}{2\pi}\left(\frac{1}{z-A}-\frac{1}{z-1/\bar{A}}\right)\Big]\frac{dz}{dz_3}\,. \tag{17}$$

As we see from the equation (8), the value of $dz/dz_3$ becomes infinitely large for $z=e^{i\theta}$, and $\theta=0$ and $\theta=\pi$, that is at both ends of each blade. In order that the velocity of flow $V_x+iV_y$ at two edges, on the $z_3$-plane, be of finite values, the values of $dW_s/dz$ must also vanish at these two ends. Namely, we put

$$\frac{Q}{2\pi}(i\tan\gamma)\frac{1}{1+c}-\omega i\sum_{m=1}^{\infty}a_m+\frac{i\Gamma_d}{2\pi}\frac{-(a^2-1)}{1+a^2-2a\cos\alpha}+\frac{J}{2\pi i}=0\,. \tag{18}$$

$$\frac{Q}{2\pi}(i\tan\gamma)\frac{1}{1-c}-\omega i\sum_{m=1}^{\infty}a_m\cos m\pi$$
$$+\frac{i\Gamma_d}{2\pi}\frac{-(a^2-1)}{1+a^2-2a\cos(\pi-\alpha)}+\frac{J}{2\pi i}=0\,. \tag{19}$$

From these equations, we find that,

$$\frac{\Gamma_d}{2\pi}\frac{4a\cos\alpha\,(a^2-1)}{(1+a^2)^2-4a^2\cos^2\alpha}=-\left(\frac{Q}{2\pi}\right)\left(\frac{2c}{c^2-1}\right)+2\omega(a_1+a_3+a_5+\ldots\ldots) \qquad (20)$$

$$\frac{J}{2\pi}=-\left(\frac{Q}{2\pi}\right)\left(\frac{2c}{c^2-1}\right)\frac{a^2+1+2a\cos\alpha}{4a\cos\alpha}+\left(\frac{Q}{2\pi}\right)\left(\frac{1}{1+c}\right)\tan\gamma$$

$$+2\omega(a_1+a_3+a_5\ldots\ldots)\frac{a^2+1+2a\cos\alpha}{4a\cos\alpha}-\omega(a_1+a_2+a_3\ldots\ldots), \qquad (21)$$

which enables us to determine the values of $\Gamma_d$ and $J$.

## VI. Numerical example

As a numerical example, let us take up the case of $n=6$, and $r_1/r_2=\varepsilon=1/2$ In this case, we have

$$\frac{q}{p}=\left[1-\left(\frac{1}{2}\right)^6\right]\Big/\left[1+\left(\frac{1}{2}\right)^6\right]=0.9687\ .$$

Firstly, the center of isolated vortices at the $z_3$-plane, which corresponds to the point $A=ae^{i\alpha}$, will be given by calculation of the formula,

$$\left(\frac{z_3}{r_2}\right)^n=\frac{1+\frac{1}{2}\frac{q}{p}\left(ae^{i\alpha}+\frac{1}{a}e^{-i\alpha}\right)}{1+\frac{q}{p}}\,. \qquad (22)$$

Actual values, obtained from (22), for the case of $n=6$ and $\varepsilon=1/2$, are shown in curves in Fig. 3.

Let us take, as a possible case, $a=1.50$, $\alpha=135°$. In this case, we have,

$$\frac{a^2+1+2a\cos\alpha}{4a\cos\alpha}=-0.266$$

$$\frac{(1+a^2)^2-4a^2\cos^2\alpha}{(a^2-1)\cdot 4a\cos\alpha}=-1.18$$

Next, the graph of the curve

$$Y=\left(1+\frac{q}{p}\right)^{2/n}-\left(1+\frac{q}{p}\cos\theta\right)^{2/n}$$

being as shown in Fig. 4, we find,

$$\int_0^{2\pi}\left[\left(1+\frac{q}{p}\right)^{2/n}-\left(1+\frac{q}{p}\cos\theta\right)^{2/n}\right]\cdot d\theta=2.38$$

And,

$$M=\frac{2.38}{4\pi\times1.253}=0.151\,,$$

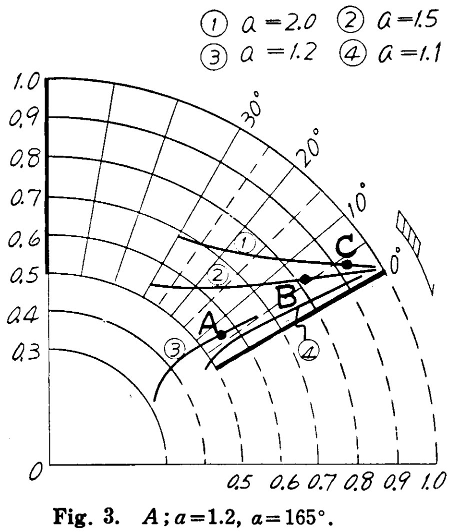


Fig. 3. $A$; $a=1.2$, $\alpha=165°$.
$B$; $a=1.5$, $\alpha=135°$.
$C$; $a=2.0$, $\alpha=80°$.
Position of Center of Isolated Vortex for Different Values of $a$ and $\alpha$ ($n=6$).

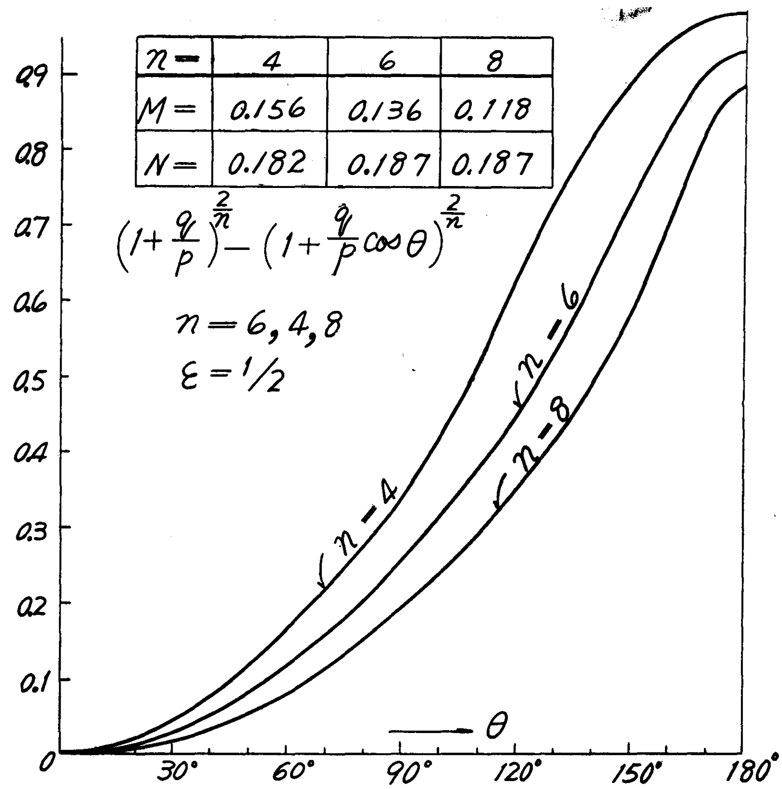


Fig. 4. Graph of the Curve

$$y=\left(1+\frac{p}{q}\right)^{2/n}-\left(1-\frac{p}{q}\cos\theta\right)^{2/n}$$

$$N=\frac{0.939}{4\times1.253}=0.188\,.$$

The value of $c$, being given by

$$\frac{1}{2}\left(c+\frac{1}{c}\right)=\frac{q}{p}\,,$$

we find, for our case, $c=1.25$ and $2c/c^2-1=4.45$.

$$F_2=\left(\frac{Q}{2\pi}\right)\frac{1}{\omega r_2^2}\,.$$

Noting that $Q$ is the discharge per each vane of impeller, the total discharge is seen to be equal to $nQ$. Also, we have $(nQ)/(2\pi r_2)=v_{m2}$, where $v_{m2}$ is the mean radial velocity of flow at the circumference of radius $r_2$. From these considerations we obtain,

$$F_2=\frac{v_{m2}}{nu_2}\,. \tag{23}$$

Thus, the values of $\Gamma_a$ and $J$, for the present case, are given by,

$$\left(\frac{\Gamma_d}{2\pi}\right)\frac{1}{\omega r_2^2}=[-4.45\,F_2+0.188\times 2](-1.18)=5.25\,F_2-0.444$$

$$\left(\frac{J}{2\pi}\right)\frac{1}{\omega r_2^2}=-4.45\,F_2\times(-0.266)+\frac{1}{2.25}\tan\gamma\,F_2+2\times 0.188$$

$$\times(-0.266)-0.151=(0.118+0.445\tan\gamma)F_2-0.251$$

The values of $\Gamma_d$ and $J$ given by these equations are shown as graph in Fig. 5.

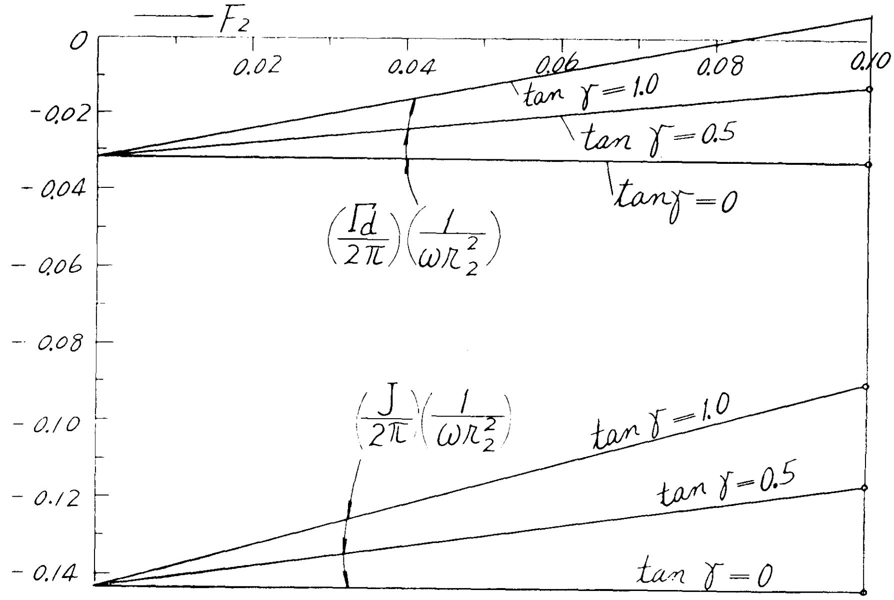


Fig. 5A. Chart for $\Gamma_d$ and $J$. ($n=6$, $\varepsilon=1/2$, $a=1.2$, $\alpha=165°$)

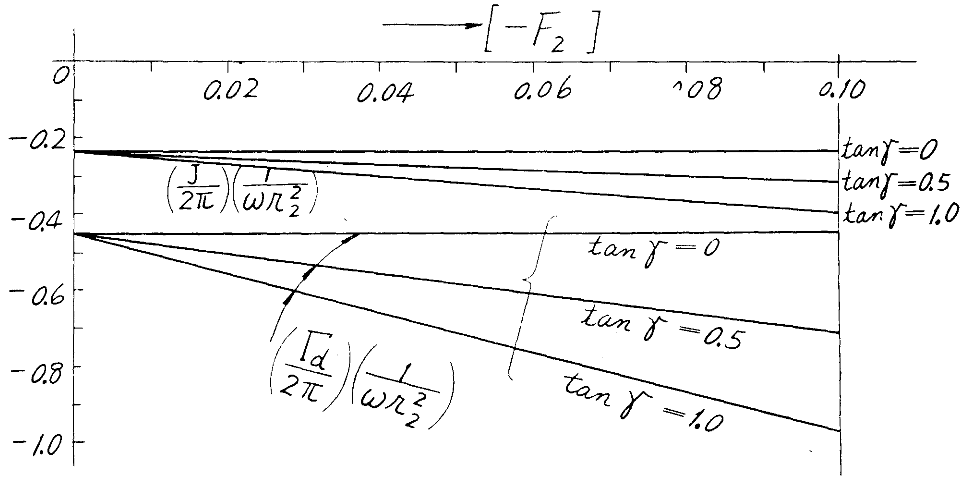


Fig. 5B. Chart for $\Gamma_d$ and $J$. ($n=6$, $\varepsilon=1/2$, $a=1.5$, $\alpha=135°$)

From this graph we see that when $F_2$ is as small as 0.05, both $\Gamma_d$ and $J$ have negative values, while when $F_2$ is as large as 0.10, $\Gamma_d$ may become positive.

## VII. Concluding remarks

In the above treatment, it was possible to examine the nature of flow through rotating impeller with radial vanes, the flow being assumed to consist of four component flows $W_a$, $W_b$, $W_c$ and $W_d$. It is to be noted that the isolated vortices are not in equilibrium, but will tend to move. Thus the flow is not a steady one, but is only a representation of state of flow at some instant. It would be a very interesting thema to study the motion of isolated vortices. Also, it may be interesting to study the state of flow when there exist more than one isolated vortices in water region lying between the two adjacent vanes. These questions are left for future studies. The amount of torque required to keep the impeller in steady rotation with the angular velocity $\omega$, will be discussed in the Report II of the same title as the present paper.