Title	Potential flow accompanied with vortices about rotating impellers with radial vanesI.
Sub Title	
Author	鬼頭, 史城(Kito, Fumiki)
Publisher	慶応義塾大学藤原記念工学部
Publication year	1961
Jtitle	Proceedings of the Fujihara Memorial Faculty of Engineering Keio University Vol.14, No.52 (1961.) ,p.1(1)- 9(9)
JaLC DOI	
Abstract	Two-dimensional potential flow through an impeller having radial vanes is considered by using the complex velocity potential. The impeller consist of radial straight vanes arranged symmetrically about center of impeller, and it rotates with a uniform angular velocity about the center. It is assumed that isolated vortices are set up at points lying between the vanes. The flow is to consist of four parts, namely; (a) discharge flow issuing from the center of impeller, (b) so-called channel vortex flow, or the flow caused by the pushing action of moving blades, (c) circulatory flow about each vane, (d) flow caused by isolated vortices, whose vortex centers lie at some point between the vanes. By consideration of combined flow caused by these four component flows, and imposing the condition of finitude of flow-velocity at two edges of each vane, the relation between the discharge, amount of circulation and strength of isolated vortex is deduced. The result is explained by a numerical example.
Notes	
Genre	Departmental Bulletin Paper
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=KO50001004-00140052- 0001

慶應義塾大学学術情報リポジトリ(KOARA)に掲載されているコンテンツの著作権は、それぞれの著作者、学会または出版社/発行者に帰属し、その権利は著作権法によって 保護されています。引用にあたっては、著作権法を遵守してご利用ください。

The copyrights of content available on the KeiO Associated Repository of Academic resources (KOARA) belong to the respective authors, academic societies, or publishers/issuers, and these rights are protected by the Japanese Copyright Act. When quoting the content, please follow the Japanese copyright act.

Potential Flow Accompanied with Vortices about Rotating Impellers with Radial Vanes.-I.

(Received May. 27, 1961)

Fumiki KITO*

Abstract

Two-dimensional potential flow through an impeller having radial vanes is considered by using the complex velocity potential. The impeller consist of radial straight vanes arranged symmetrically about center of impeller, and it rotates with a uniform angular velocity about the center. It is assumed that isolated vortices are set up at points lying between the vanes. The flow is to consist of four parts, namely; (a) discharge flow issuing from the center of impeller, (b) so-called channel vortex flow, or the flow caused by the pushing action of moving blades, (c) circulatory flow about each vane, (d) flow caused by isolated vortices, whose vortex centers lie at some point between the vanes.

By consideration of combined flow caused by these four component flows, and imposing the condition of finitude of flow-velocity at two edges of each vane, the relation between the discharge, amount of circulation and strength of isolated vortex is deduced. The result is explained by a numerical example.

I. Introduction

It is long years ago that the problem of rotating impellers with radial vanes has been studied, by using the complex velocity potential.⁽¹⁾ Here the author intends to treat the similar problem for the case in which there exist isolated vortices at points lying in between the radial vanes. The author is not aware whether this case has ever been treated. Nevertheless, the author has recently studied this problem, since he felt the growing tendency of use of impellers with radial vanes in some recent engineering fields.

II. Conformal transformation for the impeller with radial vanes

In Fig. 1, the four complex planes z, z_1 , z_2 and z_3 are shown. The relation or correspondence between these four planes are given by,

$$z_1 = \frac{1}{2} \left(z + \frac{1}{z} \right), \qquad z_2 = p + q z_1, \qquad z_3 = (z_2)^{1/n}, \qquad (1)$$

where p and q are positive constants, and n is the positive integer representing

^{*}鬼頭皮城 Professor at Keio University.

⁽¹⁾ See, for example, Hydraulische Probleme (Julius Springer), 1926.

Fumiki KITO

number of vanes of impeller.









From these relations, we see that the unit circle $z=e^{i\theta}$ in the z-plane is transformed into a straight line of length 2 in the z_1 -plane, which again is transformed into the straight line of length 2q, in the z_2 -plane. Finally, this straight line AB of length 2q in the z_2 -plane is transformed into an array of radial blades, arranged symmetrically about the origin, in the z_3 -plane, consisting of n vanes.

> Denoting by r_1 and r_2 , the inside and outside radii of the impeller, we have

$$r_1 = (p-q)^{1/n}$$

 $r_2 = (p+q)^{1/n}$

So that, if we put $\varepsilon = r_1/r_2$, then we shall have

$$q/p = (1-\varepsilon)^n/(1+\varepsilon)^n,$$

$$p = r_2^n \left(1+\frac{q}{p}\right).$$

Fig. 1. Conformal representation of Impeller with radial vanes in z_3 -plane from unit circle in z-plane.

III. Complex velocity-potential for the flow around the impeller with radial vanes

The flow around the rotating impeller having radial vanes is here thought to be composed of the following component flows.

(a) Discharge flow issuing from the center of impeller, as sketched in Fig. 2(a). The complex velocity potential representing this flow will be given by,

$$W_a = \frac{Q}{2\pi} \Big[(1+i\tan\gamma)\log(z-z_0) + (1-i\tan\gamma)\log\left(\frac{1}{z}-z_0\right) \Big], \qquad (2)$$

where Q is the discharge or quantity of flow. If the flow is one which converges into the center of impeller, Q is to be taken a negative value. γ is the angle which repesents the degree of rotation of the issuing (or, converging) flow. For a positive value of γ , the flow will rotate in clockwise direction. z_0 is a point in the z-plane, which is the source-point. We put $z_0 = -c$, where c is a positive real number greater than 1.

(b) Channel vortex flow, or, the flow generated by action of the rotating blades. The complex velocity potential representing this flow will be given by,

$$W_{b} = \omega i \left[\frac{a_{1}}{z} + \frac{a_{2}}{z^{2}} + \frac{a_{3}}{z^{3}} + \cdots \right],$$
(3)

where ω is the angular velocity of rotation of the impeller. a_1 , a_2 , are unknown constants (real numbers) to be determined later. ω is taken positive when the impeller rotates in clockwise direction.

(c) Circulating flow around each vane, which is given by the complex velocity potential

$$W_c = \frac{J}{2\pi i} \log z . \qquad (4)$$



Fig. 2. Four component flows through rotating impeller with radial vanes.

When J is positive, the circulation will take place in counter-clockwise direction.(d) Flow due to isolated vortices which exist behind each vanes, represented by

$$W_{d} = \frac{i\Gamma_{d}}{2\pi} \log \frac{z-A}{z-\left(\frac{1}{A}\right)},\tag{5}$$

 Γ_a being the strength of isolated vortex, is positive when the rotation takes place in clockwise direction. A is a complex number which represents the center of the isolated vortex. \overline{A} is the complex number conjugate to A.

The combined flow is given by the complex velocity potential W_s , where

$$W_{s} = W_{a} + W_{b} + W_{c} + W_{d} . ag{6}$$

IV. Determination of constants a_1, a_2, \dots for the channel vortex flow

The velocity at any point in the z_3 plane, of flow W_b is to be obtained by the equation,

$$V_{x} - iV_{y} = \frac{dW_{b}}{dz_{3}} = \frac{dW_{b}}{dz}\frac{dz}{dz_{3}} = \omega i \left[-\sum_{m=1}^{\infty} \frac{ma_{m}}{z^{m+1}}\right]\frac{dz}{dz_{3}},$$
(7)

while we have

$$\frac{1}{(dz/dz_3)} = \frac{dz_3}{dz} = \frac{dz_3}{dz_2}\frac{dz_2}{dz_1}\frac{dz_1}{dz} \qquad \frac{dz}{dz_3} = \frac{2n}{q}(z_2)^{1-1/n}\frac{z}{(z-1/z)}$$
(8)

Or, on the surface of the vane, we have,

 $z = e^{i\theta}, \quad z_1 = \cos\theta, \quad z_2 = p + q\cos\theta,$

and so, by (7) and (8), on the surface of the vane which lie along the real positive axis of z_3 -plane;

$$V_x - iV_y = -\omega \left[\sum_{m=1}^{\infty} ma_m (\cos m\theta - i \sin m\theta)\right]$$
$$\times \left(\frac{n}{q}\right) (p + q \cos \theta)^{1 - 1/n} \frac{1}{\sin \theta}$$

Equating the value V_{ν} obtained from the above equation (which is the normal velocity to vane surface), to $-\omega R$, where R is the radial distance from the origin, we have

$$\sum_{m=1}^{\infty} m a_m \sin m\theta = \frac{q}{n} \left(p + q \cos \theta \right)^{2/n-1} \sin \theta .$$
 (9)

This equation tells us that the values of ma_m (m=1, 2,) are to be obtained as Fourier coefficients of right-hand side of (9). This is what has already been shown⁽¹⁾. Now, integrating both sides of this equation (9), we obtain

$$\sum_{m=1}^{\infty} a_m \left(1 - \cos m\theta \right) = \frac{1}{2} \left[(p+q)^{2/n} - (p+q\cos\theta)^{2/n} \right].$$
(10)

Also, putting $\theta = \pi$ into this equation (10), we have

$$a_1 + a_3 + a_5 + \dots = \frac{1}{4} \left[(p+q)^{2/n} - (p-q)^{2/n} \right]. \tag{11}$$

Further, multiplying by $\cos m\theta$ both sides of equation (10), and integrating with respect to θ , from $\theta = 0$ to $\theta = 2\pi$, we have

$$a_{m} = -\frac{1}{\pi} \int_{0}^{2\pi} \frac{f(\theta)}{\sin\left(\frac{1}{2}\theta\right)} \cos m\theta \, \sin\frac{\theta}{2} \, d\theta \,, \qquad (12)$$

where we put for shortness,

$$f(\theta) = \frac{1}{2} [(p+q)^{2/n} - (p+q\cos\theta)^{2/n}].$$

Now, from the formula

$$2\cos m\theta \,\sin\frac{\theta}{2} = \sin\left(m+1-\frac{1}{2}\,\theta\right) - \sin\left(m-\frac{1}{2}\right)\theta\,,$$

we have

4

Flow about Radial Impellers

$$2\sum_{m=1}^{M} \cos m\theta \, \sin \frac{\theta}{2} = \sin \left(M + 1 - \frac{1}{2}\right)\theta - \sin \frac{1}{2}\theta$$

And so, by summing up both sides of equation (12), we have,

$$\sum_{m=1}^{\infty} a_m = \frac{1}{4\pi} \int_0^{2\pi} \left[(p+q)^{2/n} - (p+q\cos\theta)^{2/n} \right] d\theta \,. \tag{13}$$

Or, by putting

$$\sum_{m=1}^{\infty} a_m = r_2^2 M, \qquad (a_1 + a_3 + a_5 + \dots) = r_2^2 N \qquad (14)$$

we have,

$$M = \frac{1}{4\pi} \frac{1}{\left(1 + \frac{q}{p}\right)^{2/n}} \int_{0}^{2\pi} \left[\left(1 + \frac{q}{p}\right)^{2/n} - \left(1 + \frac{q}{p}\cos\theta\right)^{2/n} \right] d\theta , \qquad (15)$$

$$N = \frac{1}{4} \frac{1}{\left(1 + \frac{q}{p}\right)^{2/n}} \left[\left(1 + \frac{q}{p}\right)^{2/n} - \left(1 - \frac{q}{p}\right)^{2/n} \right].$$
(16)

V. Determination of conatants Γ_d and J.

For the velocity of flow $V_x + iV_y$, on the z_3 -plane, for the combined flow $W_s = W_a + W_b + W_c + W_d$, we have

$$V_{x} - i V_{y} = \frac{dW_{s}}{dz_{3}} = \frac{dW_{s}}{dz} \frac{dz}{dz_{3}} = \left[(1 + i \tan \gamma) \frac{Q}{2\pi} \frac{1}{z - z_{0}} + (1 - i \tan \gamma) \frac{Q}{2\pi} \left(-\frac{1}{z} - \frac{z_{0}}{1 - z_{0} z} \right) - \omega i \sum_{m=1}^{\infty} \frac{a_{m}}{z^{m+1}} + \frac{J}{2\pi i} \frac{1}{z} + \frac{i \Gamma_{d}}{2\pi} \left(\frac{1}{z - A} - \frac{1}{z - 1/\overline{A}} \right) \right] \frac{dz}{dz_{3}}.$$
 (17)

As we see from the equation (8), the value of dz/dz_3 becomes infinitely large for $z=e^{i\theta}$, and $\theta=0$ and $\theta=\pi$, that is at both ends of each blade. In order that the velocity of flow V_x+iV_y at two edges, on the z_3 -plane, be of finite values, the values of dW_s/dz must also vanish at these two ends. Namely, we put

$$\frac{Q}{2\pi}(i\tan\gamma)\frac{1}{1+c} - \omega i \sum_{m=1}^{\infty} a_m + \frac{i\Gamma_d}{2\pi} \frac{-(a^2-1)}{1+a^2-2a\cos\alpha} + \frac{J}{2\pi i} = 0.$$
(18)
$$\frac{Q}{2\pi}(i\tan\gamma)\frac{1}{1-c} - \omega i \sum_{m=1}^{\infty} a_m\cos m\pi + \frac{i\Gamma_d}{2\pi} \frac{-(a^2-1)}{1+a^2-2a\cos(\pi-\alpha)} + \frac{J}{2\pi i} = 0.$$
(19)

Fumiki KITO

From these equations, we find that,

6

$$\frac{\Gamma_{a}}{2\pi} \frac{4a \cos\alpha (a^{2}-1)}{(1+a^{2})^{2}-4a^{2} \cos^{2}\alpha} = -\left(\frac{Q}{2\pi}\right) \left(\frac{2c}{c^{2}-1}\right) + 2\omega (a_{1}+a_{3}+a_{5}+\dots)$$
(20)
$$\frac{J}{2\pi} = -\left(\frac{Q}{2\pi}\right) \left(\frac{2c}{c^{2}-1}\right) \frac{a^{2}+1+2a \cos\alpha}{4a \cos\alpha} + \left(\frac{Q}{2\pi}\right) \left(\frac{1}{1+c}\right) \tan\gamma$$
$$+ 2\omega (a_{1}+a_{3}+a_{5}\dots) \frac{a^{2}+1+2a \cos\alpha}{4a \cos\alpha} - \omega (a_{1}+a_{2}+a_{3}\dots),$$
(21)

which enables us to determine the values of Γ_d and J.

VI. Numerical example

As a numerical example, let us take up the case of n=6, and $r_1/r_2 = \varepsilon = 1/2$ In this case, we have

$$\frac{q}{p} = \left[1 - \left(\frac{1}{2}\right)^{6}\right] / \left[1 + \left(\frac{1}{2}\right)^{6}\right] = 0.9687$$

Firstly, the center of isolated vortices at the z_3 -plane, which corresponds to the point $A = ae^{i\alpha}$, will be given by calculation of the formula,

$$\left(\frac{z_3}{r_2}\right)^n = \frac{1 + \frac{1}{2} \frac{q}{p} \left(a e^{i\omega} + \frac{1}{a} e^{-i\alpha}\right)}{1 + \frac{q}{p}}.$$
 (22)

Actual values, obtained from (22), for the case of n=6 and $\varepsilon = 1/2$, are shown in curves in Fig. 3.

Let us take, as a possible case, a=1.50, $\alpha=135^{\circ}$. In this case, we have,

$$\frac{a^2 + 1 + 2a \cos \alpha}{4a \cos \alpha} = -0.266$$
$$\frac{(1 + a^2)^2 - 4a^2 \cos^2 \alpha}{(a^2 - 1) \cdot 4a \cos \alpha} = -1.18$$

Next, the graph of the curve

$$Y = \left(1 + \frac{q}{p}\right)^{2/n} - \left(1 + \frac{q}{p}\cos\theta\right)^{2/n}$$

being as shown in Fig. 4, we find,

$$\int_{0}^{2\pi} \left[\left(1 + \frac{q}{p} \right)^{2/n} - \left(1 + \frac{q}{p} \cos \theta \right)^{2/n} \right].$$
$$d\theta = 2.38$$

And,

$$M = rac{2.38}{4\pi imes 1.253} = 0.151$$
 ,



Position of Center of Isolated Vortex for Different Values of a and a (n=6).

Flow about Radial Impellers



$$N = \frac{0.939}{4 \times 1.253} = 0.188 \, .$$

The value of c, being given by

$$\frac{1}{2}\left(c+\frac{1}{c}\right)=\frac{q}{p}$$

we find, for our case, c=1.25 and $2c/c^2-1=4.45$.

$$F_2 = \left(\frac{Q}{2\pi}\right) \frac{1}{\omega r_2^2}.$$

Noting that Q is the discharge per each vane of impeller, the total discharge is seen to be equal to nQ. Also, we have $(nQ)/(2\pi r_2) = v_{m_2}$, where v_{m_2} is the mean radial velocity of flow at the circumference of radius r_2 . From these considerations we obtain,

$$F_2 = \frac{v_{m2}}{nu_2}.$$

Thus, the values of Γ_{d} and J, for the present case, are given by,

Fumiki KITO

$$\left(\frac{\Gamma_{d}}{2\pi}\right)\frac{1}{\omega r_{2}^{2}} = \left[-4.45 F_{2} + 0.188 \times 2\right](-1.18) = 5.25 F_{2} - 0.444$$

$$\left(\frac{J}{2\pi}\right)\frac{1}{\omega r_{2}^{2}} = -4.45 F_{2} \times (-0.266) + \frac{1}{2.25} \tan \gamma F_{2} + 2 \times 0.188$$

$$\times (-0.266) - 0.151 = (0.118 + 0.445 \tan \gamma) F_{2} - 0.251$$

The values of Γ_d and J given by these equations are shown as graph in Fig. 5.



Fig. 5A. Chart for Γ_a and J. $(n=6, \epsilon=1/2, a=1.2, a=165^\circ)$



Fig. 5B. Chart for Γ_a and J. $(n=6, \epsilon=1/2, a=1.5, a=135^\circ)$

Flow abuot Radial Impellers

From this graph we see that when F_2 is as small as 0.05, both Γ_d and J have negative values, while when F_2 is as large as 0.10, Γ_d may become positive.

VII. Concluding remarks

In the above treatment, it was possible to examine the nature of flow through rotating impeller with radial vanes, the flow being assumed to consist of four component flows W_a , W_b , W_c and W_a . It is to be noted that the isolated vortices are not in equilibrium, but will tend to move. Thus the flow is not a steady one, but is only a representation of state of flow at some instant. It would be a very interesting thema to study the motion of isolated vortices. Also, it may be interesting to study the state of flow when there exist more than one isolated vortices in water region lying between the two adjacent vanes. These questions are left for future studies. The amount of torque required to keep the impeller in steady rotation with the angular velocity ω , will be discussed in the Report II of the same title as the present paper.