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Contribution to the Theory of Systematic Sampling and Bedding Methods in Quality Control

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Résumé

Contribution à la théorie d'échantillonnage systématique et du procédé de "bedding" dans le damaine du control de qualité.

L'échantillonnage systématique et la méthode de "bedding" de la matière masseuse se sont développées notamment dans le domaine du controle de qualité depuis peu de temps. Surtout, comme en sidérurgie, quand on doit manier une énorme quantité de matière, il est très difficile de tirer un échantillon aléatoire ou de melanger la matière en grande échelle, et on est amené à éviter cette difficulté par plusieures méthodes. Par exemple, pour tirer un échantillon, il est recommendé d'échantillonner sur la courroie de transport, ou pour mélanger un gros lot on a introduit la méthode de "bedding". On peut donner un modèle mathématique à ce procédé d'échantillonnage systématique ou de bedding. Même en cas où la matière n'est pas homogène et que la composition en grandeur de graines se différe considérablement d'une partie à l'autre, on peut démontrer que, si l'on échatillonne systématiquement par un petit intervalle, l'échantillon résultat s'approche à un échantillon aléatoire d'un lot bien mélange. Ce modèle mathématique s'applique aussi à l'échantillonnage en grappe d'une matière bien mélangée par le procédé de "bedding." Ce type d'échantillonnage systématique se réduit mathématiquement à un cas spécial de la théorie ergodique. Jusqu'ici, la plupart des auteurs qui ont traité ce problème admettaient à priori un modèle stochastique, un processus stochastique stationnaire par exemple, pour la variation de qualité de la matière. L'auteur essie de construire une théorie d'échantillonnage du point de vue microscopique en n'admettant qu'une hypothèse assez générale sur répartition de graines de la matière.

I. Introduction

The systematic sampling and bedding methods of bulk material have recently been developed in the field of quality control, particularly in the iron and steel industry, because of their convenience in daily work and their advantages over the methods based on a randomized scheme in case a tremendous amount of bulk material is surveyed and handled.

Operation plans are mapped out on the basis of the average quality of the raw materials fed into the manufacturing processes. The discrepancy between the

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estimated and actual values of the average quality of these materials is expected to disturb to a considerable degree the uniformity of finished products. For instance, the quality of cast iron is much dependent on the average quality of the ingredients charged into the cupola. It is necessary, therefore, to work out the means of evaluating the average quality of materials handled in bulk, within relatively closed limits.

If material is gathered from several sources, its quality often differs from one source to another. So material from each source must be sampled and measured separately, or materials from various sources must be thoroughly mixed in order to achieve the satisfactory uniformity of quality throughout the whole lot.

If the purpose of sampling is to evaluate the average quality of such bulk material, it is convenient to take samples, each composed of a series of increments systematically taken from material flow on a conveyor belt. Moreover, for quality control, we must obtain information about the average quality of each sublot of raw material fed into the manufacturing processes at one time, instead of the average quality of the whole lot piled in storage.

It is not necessarily economical to take from every sublot such samples as may provide sufficiently accurate estimates about the average quality. Therefore, various types of piling, known as "layering" or "bedding", are adopted to achieve satisfactory uniformity in quality throughout the whole lot piled in storage. In one of these types, material is spread on the storage yard like a large number of parallel noodles. Each sublot of the material fed into the manufacturing processes from the storage consists of the slices vertically cut off from the noodle-like piles of material. It is obvious that each sublot of the material will be more homogeneous if we mix again the slices by the bedding method before we feed the sublot of slices into the manufacturing processes. In any case, the material can thus be mixed fairly well. And it will be homogeneous to some extent in quality when it is fed into the production lines.

In this bedding method, the average quality of a sample of adequate size, composed of the vertically cut off slices, would provide a desired accurate estimate about the average quality of every sublot. It is to be noted here that there is similarity between the physical composition of the sample taken from such material piles and that of the sample composed of a series of increments systematically taken from material flow on the conveyor belt from a ship to the storage.

There are two methods of piling, namely ordinary bedding and "switch-back" or "zig-zag" bedding. In the former method, material flow is spread on the storage yard in a parallel way in the same direction, but in the latter method in the opposite direction alternately one after another.

In both methods, the lot to be fed into the manufacturing processes consists of the slices perpendicular to the parallel lines.

We can construct a mathematical model which will represent the physical composition of the sample taken from the lot of material piled by the bedding method or that of the sample composed of a series of increments systematically taken from material flow on the conveyor belt. By establishing such a mathematical model, we can show that, if we take a systematic sample with a sufficiently small sampling interval from the flow of material, the sample thus obtained may be treated as a random sample taken from the lot which was mixed fairly well by the bedding method. The mathematical model for the systematic sampling method or bedding method, in its extreme case where the sampling interval is sufficiently small or the number of parallel noodles is sufficiently large, may be considered as an example of the application of the ergodic properties.

So far, many writers have treated this problem from the macroscopic point of view, assuming a priori that the fluctuation of the quality of material on the conveyor belt is subject to a stochastic model such as a stationary stochastic process.

In this paper, the author will try to construct a theory for the systematic sampling and bedding methods from the microscopic point of view, taking into consideration the heterogeneity of the distribution of piece sizes and contents in bulk material on the conveyor belt.

First: The author will explain this subject by illustrative exmples. Then, the problems to be solved from the mathematical point of view will be put forth.

Second: The author will establish a theoretical model, which describes the state of material flow on the conveyor belt. Thus, the mathematical formulae will be introduced for the variance of the numbers of a given category found in increments systematically taken from material flow. The time series which represents the initial state of material flow may be assumed to be considerably evened and locally randomized, if the material has been locally mixed by a mixer while the material is being carried to the storage by the conveyor belt.

Third: The author will discuss a stochastic model more complicated than the above model. Herein will be studied a lot in which the distribution of piece sizes and contents is represented by a mixture of pieces classified into a limited number of categories.

Fourth: The author will set up a theoreticall model, which describes the physical composition of the interpenetrating samples, each composed of a series of increments systematically taken from material flow. By using the results derived from this model, we can investigate into the nature of the lot mean estimate (ash content, humidity, etc.) calculated from the systematic samples of bulk material.

Fifth: The author will discuss the relations between randomness and the operration of statistical control, mainly taking into consideration the operation of statistical control for attaining uniformity in the quality of raw material. Thus, the mathematical theory of randomness, which clarifies the relations between the physical aspects of the state of control and the quantitative aspects of the data obtainable under a given state of control, will be developed.

Sixth: Using the well-known method of arbitrary functions, the author will discuss the mixing mechanism of the bedding method. This discussion will, naturally, throw much light on the stochastic theory of systematic sampling and bedding methods of bulk material. The mixing mechanism of the bedding method is somewhat like the process that is employed by the baker in making puff pastry. The mixing mechanism in making puff pastry was investigated in detail by E. Hopf and others. Their fruitful results can be utilized for the study of the mixing mechanism of the bedding method and the mathematical structure of the systematic sampling method. But some modifications must be introduced in their study of the mixing mechanism, if material is locally mixed either spontaneously in transit or by a properly designed mixer while the material is being carried to the storage. In this case, it would be better to study the mixing mechanism of the bedding operation which is applied to the locally mixed material flow on the conveyor belt.

Lastly, it should be noted that the theory developed here will also be applicable to the grinding and subdivision process of samples — an indispensable operation to obtain material suitable in size for analysis. To avoid the fine grinding of a large amount of material, grinding and subdivision operation is often carried out in several stages. Thus, in sampling bulk material we have to consider not only the errors of the original sampling operation but also those of the subsequent subdivision. Besides, we must pay attention to the fact that bulk material is mixed fairly well by the grinding process, for this helps us to apply the stochastic theory of the systematic sampling and bedding methods to the grinding and subdivision process of samples.

II. Sampling from material in motion

To take a representative sample from material in motion, such as coal or iron ore on a conveyor belt, is a problem far easier than sampling from stationary bulk. The ordinary practice is to collect increments of the material at regular intervals as it passes at a certain point: these increments may afterwards either be mixed together in a single gross sample or analyzed separately. It is to be noted here that such separate analysis is not necessarily economical in daily work.

In order to treat this subject properly from the statistical point of view, we must take out the essential points of the problem from the practical data that have so far been collected.

Fig. 1 shows the changing quality represented by the percentage ash content of each increment of coal which is sampled at regular intervals from the conveyor belt as it passes at a certain point.

A smooth sinuous curve might be drawn to represent the changing level of quality:

the points representing test values from successive increments would be scattered around this curve, not lying precisely on it owing to small-scale local variation in quality superimposed on the main trend. This small-scale local variation appears even between the contents of two increments taken in rapid succession.

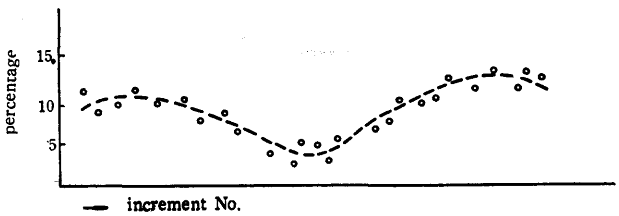

Fig. 1.

It can be readily seen that the changing level of quality is caused by the heterogeneous distributions of piece sizes and contents in bulk material carried by the conveyor belt. It appears that the main trend represented by the smooth sinuous curve and the small-scale local variation along the main trend have been brought by some kind of small-scale mixing operation (for example, by grinding operation) of bulk material while it is being carried to the storage by the conveyor belt. Thus, we may consider such material flow on the conveyor belt as a kind of locally randomized material flow.

We will discuss the systematic sampling method as well as the mixing mechanism by the bedding method for such material flow from the statistical point of view. In order to investigate into the variability of material flow on the conveyor belt, we must test each increment separately.

But in practice, a single measure of the average quality of the whole lot is usually obtained, by mixing together the systematic increments into a single gross sample, which alone is tested. If the local variation superimposed upon the main trend is small, as in this illustration, a very reliable estimate of the average quality may be obtained from the gross sample.

As was pointed out by Dr. E. S. Pearson, the most important point is that if sufficient increments are collected, the reliability of the estimate as a measure of true avarage quality of the whole lot will depend on the magnitude of the deviation from the trend curve, and not on the exent of the fluctuations of the curve itself.

The author will explain this point by a set of practical data representing the percentage ash contents in 27 pairs of increments systematically taken from a lot of coal. The sampling method adopted here is shown in Fig. 2. The 27 pairs of increments were collected from material flow on the conveyor belt at regular intervals (ten-minute intervals), where a and b correspond to the two paired increments taken in rapid succession. The data in table 1 are the percentage ash contents of the 27 pairs of increments.

In the following discussion, we will assume that the sizes of the collected increments were nearly the same, although in practice it is very difficult to take increments of exactly the same size.

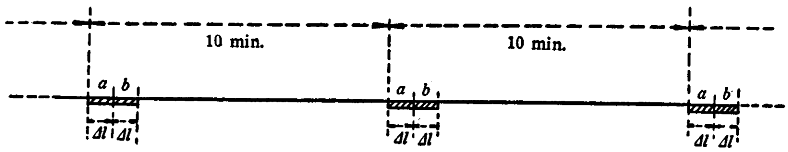

Fig. 2.

Table 1.

Increment No.	1	2	3	4	5	6	7	8	9
a	4.3	2.3	3.6	2.2	5.7	3.2	3.9	5.6	4.5
b	2.0	5.6	2.8	2.7	6.3	3.0	4.7	5.4	3.6
combined	3.15	3.95	3.20	2.45	6.00	3.10	4.30	5.50	4.05
Increment No.	10	11	12	13	14	15	16	17	18
a	6.2	8.1	8.2	7.9	11.9	9.1	8.3	7.3	5.4
b	6.3	8.6	7.8	7.2	8.9	9.1	8.1	8.3	5.5
combined	6.25	8.35	8.00	7.55	10.4	9.10	8.20	8.80	5.45
Increment No.	19	20	21	22	23	24	25	26	27
a	2.7	2.1	7.2	4.0	7.2	11.3	9.4	7.2	4.8
b	2.3	3.0	8.4	3.6	7.4	10.3	8.1	6.8	5.6
combined	2.50	2.55	7.80	3.80	7.30	10.80	8.75	7.00	5.20

First we divide the 27 pairs of increments into 18 interpenetrating samples as follows,

$\mathrm{I}'a=\{1a,\ 10a,\ 19a\}$, $\mathrm{I}'b=\{1b,\ 10b,\ 19b\}$,
$\mathrm{II}'a=\{2a,\ 11a,\ 20a\}$, $\mathrm{II}'b=\{2b,\ 11b,\ 20b\}$,
$\mathrm{III}'a=\{3a,\ 12a,\ 21a\}$, $\mathrm{III}'b=\{3b,\ 12b,\ 21b\}$,
$\mathrm{IV}'a=\{4a,\ 13a,\ 22a\}$, $\mathrm{IV}'b=\{4b,\ 13b,\ 22b\}$,
$\mathrm{V}'a=\{5a,\ 14a,\ 23a\}$, $\mathrm{V}'b=\{5b,\ 14b,\ 23b\}$,
$\mathrm{VI}'a=\{6a,\ 15a,\ 24a\}$, $\mathrm{VI}'b=\{6b,\ 15b,\ 24b\}$,
$\mathrm{VII}'a=\{7a,\ 16a,\ 25a\}$, $\mathrm{VII}'b=\{7b,\ 16b,\ 25b\}$,
$\mathrm{VIII}'a=\{8a,\ 17a,\ 26a\}$, $\mathrm{VIII}'b=\{8b,\ 17b,\ 26b\}$,
$\mathrm{IX}'a=\{9a,\ 18a,\ 27a\}$, $\mathrm{IX}'b=\{9b,\ 18b,\ 27b\}$.

The calculated percntage ash contents in the interpenetrating samples thus composed are shown in Table 2.

Table 2.

Interpenetrating sample No.	I′	II′	III′	IV′	V′	VI′	VII′	VIII′	IX′
a	4.40	4.17	6.33	4.70	8.27	7.87	7.20	6.70	4.90
b	3.53	5.73	6.33	4.50	7.53	7.47	6.97	6.83	4.90
combined interpenerating samp.	3.97	4.95	6.33	4.60	7.90	7.67	7.09	6.77	4.90

Then, we divide the 27 pairs of increments into 6 interpenetrating samples as follows,

$$\mathrm{I}''a=\{1a,\ 4a,\ 7a,\ 10a,\ 13a,\ 16a,\ 19a,\ 22a,\ 25a\},$$
$$\mathrm{I}''b=\{1b,\ 4b,\ 7b,\ 10b,\ 13b,\ 16b,\ 19b,\ 22b,\ 25b\},$$
$$\mathrm{II}''a=\{2a,\ 5a,\ 8a,\ 11a,\ 14a,\ 17a,\ 20a,\ 23a,\ 26a\},$$
$$\mathrm{II}''b=\{2b,\ 5b,\ 8b,\ 11b,\ 14b,\ 17b,\ 20b,\ 23b,\ 26b\},$$
$$\mathrm{III}''a=\{3a,\ 6a,\ 9a,\ 12a,\ 15a,\ 18a,\ 21a,\ 24a,\ 27a\},$$
$$\mathrm{III}''b=\{3b,\ 6b,\ 9b,\ 12b,\ 15b,\ 18b,\ 21b,\ 24b,\ 27b\}.$$

The calculated ash contents of the interpenetrating samples thus composed are shown in Table 4.

Table 4.

Interpenetrating sample No.	I″	II″	III″
a	5.43	6.38	6.37
b	5.00	6.70	6.30
combined	5.22	6.54	6.34

Lastly we divide the 27 pairs of increments into 2 interpenetrating samples as follows,

$$\mathrm{I}'''a=\{1a,\ 2a,\ 3a,\ \ldots\ldots,\ 27a\},$$
$$\mathrm{I}'''b=\{1b,\ 2b,\ 3b,\ \ldots\ldots,\ 27b\}.$$

The calculated ash contents of the interpenetrating samples thus composed are 6.06 for $\mathrm{I}'''a$, 6.00 for $\mathrm{I}'''b$, and 6.03 for the combined interpenetrating sample of $\mathrm{I}'''a$, $\mathrm{I}'''b$.

We will make the analysis of variance for these data to measure the degree of variability between the interpenetrating samples composed of these systematic increments.

The first step is to test the significance of the "between pairs" variance compared with the "within pairs" variance. Here, the "within pairs" variance may be considered as an approximate measure of variability between the increments taken from the thoroughly mixed lot.

This is presented in Table 5.

Table 5.

	d. f.	*s. s.*	*m. s.* $=u^2$
Between pairs	26	327.66	12.60
Within pairs	27	18.51	0.68
Totals	53	346.17	

The second step is to divide the "between pairs" variance into "between combined interpenetrating samples" variance and "within combined interpenetrating samples" variance.

The result of such analysis is presented in Table 6, in case we treat the combined interpenetrating samples I′, II′, III′, …… IV′.

Table 6.

	d. f.	*s. s.*	*m. s.* $=u^2$
Between combined interpenetrating samples	8	99.76	12.50
Within combined interpenetrating samples	18	227.70	12.65
Totals	26	327.66	12.60

The result of the analysis is presented in Table 7, in case we treat the combined interpenetrating samples I″, II″, III″.

Table 7.

	d. f.	*s. s.*	*m. s.* $=u^2$
Between combined interpenetrating samples	2	17.82	8.91
Within combined interpenetrating samples	24	309.84	12.91
Totals	26	327.66	12.60

By such analysis of variance, we can investigate into how the variance between the contents of the interpenetrating samples decrease as the number of increments in each sample increase, and we can determine how many increments should be taken in order to assure a certain degree of reliability for the estimate calculated from the sample composed of such systematic increments.

If the original lot of material has been thoroughly mixed before we take the systematic increments, then the variance between contents of the interpenetrating samples composed of systematic increments will decrease in inversely proportional to the number of increments in such samples. This is shown by the dotted curve in Fig 3. Together with this dotted curve, the estimated decreasing tendency of the variability between the contents of the interpenetrating samples is shown by the solid curve.

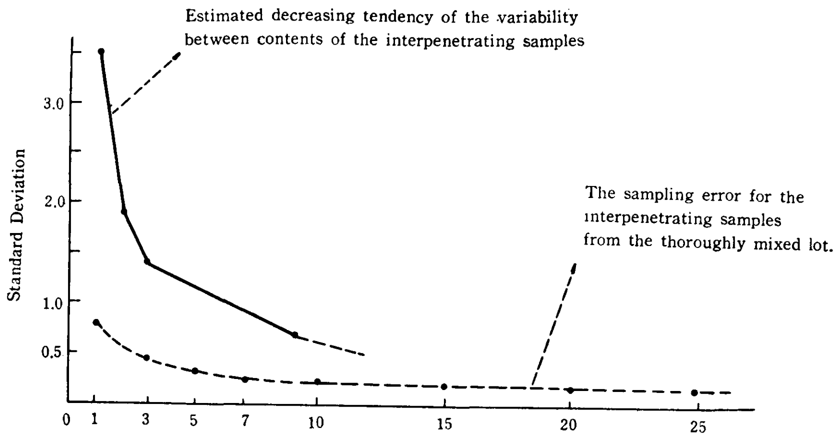

Fig. 3.

From the above analysis of practical data, it can easily be supposed that the variability in ash content of samples composed of systematic increments (with each increment's weight as w) can be represented as

$$\sigma_0^2 = \sigma_N^2 + \frac{\sigma_i^2}{Nw},$$

where σ_i^2 is the variance for a sample of unit weight from a thoroughly mixed lot, and σ_N^2 represents the variation due to the systematic sampling method which is applied to the main trend.

The first term of the right-hand side of the above formula seems to converge to zero rapidly as N is increased, particularly when the main trend in quality level is represented by a smooth curve.

From the above analysis, it can easily be supposed that σ_N^2 decreases to zero rapidly compared with the magnitude of $\frac{\sigma_i^2}{Nw}$ as N is increased, where the magnitude σ_i^2 is nearly independent of N.

For sufficiently large N, we may write

$$\sigma_0^2 = \frac{\sigma_i^2}{Nw}$$

which shows the variability in ash content of a sample of size Nw taken from a thoroughly mixed lot. The term $\sigma_0^2 = \sigma_N^2 + \frac{\sigma_i^2}{Nw}$ corresponds to the solid curve in Fig. 3. And the term $\frac{\sigma_i^2}{Nw}$ corresponds to the dotted curve in Fig. 3. It is to be noted here that the solid curve has a tendency to approach asymptotically to the dotted curve as N is increased.

Now, how many systematic increments should be taken so that a sample composed of such systematic increments might be regarded as a random sample taken from the homogeneous material flow? In the systematic sampling of bulk material it is often advantageous to take two or more interpenetrating samples composed of systematic increments, as was illustrated by the above example. If the error variance for the discrepancies between the contents of the interpenetrating samples is significantly large compared with the "within paires" variance, we cannot regard such samples as random samples taken from the thoroughly mixed lot. So we cannot treat the pooled mean of the percentage contents of the interpenetrating samples as if it were a mean of the percentage contents of samples taken from the homogeneous material flow. In such a case, we cannot estimate directly the variance of the pooled mean of the percentage contents of the interpenetrating samples. We must estemate it indirectly by an extrapolation method, making use of the above-mentioned decreasing tendency of the variance between the contents of the interpenetrating samples. In order to solve this problem, it is desirable to establish a statistical theory for pooling data along the lines of the works developed by Dr. T. A. Bancroft, Dr. T. Kitagawa, and others.

We have sofar discussed the systematic sampling method in the field of quality control and pointed out many valuable problems to be solved from the mathematical point of view. But these problems have been found not only in the field of quality control, but also in other fields such as timber and social surveys. It has been pointed out in many examples of various fields that systematic sampling, when used on the type of material for which it is suitable, is likely to have an error variance which is somewhat less than random sampling with one unit per stratum. We will discuss these points in detail in the following sections.

III. Theoretical models of sampling at regular intervals from a conveyor belt

Let us suppose that conditions are as described in the examples in the previous section. The equation to the trend curve, which may be discontinuous, is usually represented by

$$y=f(t)$$

The unit for t may be taken as an interval in time, supposed constant, between the collection of k successive increments.

To begin with, we will discuss the sampling problems along the line of the theoretical models, which have been developed by many authors from the macroscopic point of view. The values of the characteristic under consideration as obtained from these increments by chemical analysis or otherwise will be written x_1, x_2, x_3, ..., x_t, ..., x_k. Further, the difference between x_t and the trend curve will be denoted by z_t, so that

$$x_t=y_t+z_t=f(t)+z_t$$

In many cases of practical sampling it is considered only necessary to obtain an averaged measure of quality for the whole lot. This average measure will really be represented by the total area under the trend curve, or by the integral

$$I=\frac{1}{k}\int_0^k f(t)\,dt$$

The average measure obtained from the k increments will be,

$$X_k=\frac{1}{k}(x_1+x_2+\ \ldots\ldots\ +x_k)=\frac{1}{k}(y_1+y_2+\ \ldots\ldots\ +y_k)+\frac{1}{k}(z_1+z_2+\ \ldots\ldots\ +z_k)$$

The first expression on the right-hand side of the above equation is based on the sum of equidistant ordinates of the trend curve, that is to say, provides a first approximation to the integral I. The second term is the mean value of deviations from the trend curve, and its probable difference from zero will be measured by a standard error σ_z^2/k, assuming that $z_1, z_2, \ldots\ldots, z_k$ are the realized values of k independent random valiables $z_1, z_2, \ldots\ldots, z_k$, all having the same distribution $F_z(x)$ with the mean value 0 and the standard deviation σ_z.

It follows from the above model that the reliability of X_k as an estimate of I will depend on two factors ......

(i) Whether, taking into consideration the fluctuations and discontinuities in the trend curve, sufficient ordinates are being used to provide an adequate quadrature of the area under this curve?

(ii) What is the magnitude of deviations from the trend curve due to the more local variations in quality?

The above-mentioned theoretical aspects of sampling from material flow are suggested by Dr. E. S. Pearson. On sampling from processes depending upon a continuous parameter, many papers have been published by Dr. T. Kitagawa, Dr. A. E. Jones, Dr. G. H. Jowett, and others.

Especially, Dr. T. Kitagawa treated the sampling problems in the case of the non-stationary stochastic process in more generalized form. In any case, it would be better to assume the following mathematical model.

$$x_t=f(t)+z_t\ ,$$

where z_t represents a stationary stochastic process.

The results obtained from the mathematical model established for one-dimensional material flow will easily be extended to the 3-dimensional case representing the actual state of bulk material. As was pointed out by Dr. E. S. Pearson, if increments are drawn from different points chosen at regular space intervals in stationary bulk material, the problem may be regarded as one of cubature of the volume under a 3-dimensional surface in a 4-dimensional surface in a 4-dimensional space. But, for simplicity, we will treat mainly the mathematical model representing the one dimensional material flow in the following sections. The author will discuss the sampling problems from a slightly different angle in this paper.

IV. Theoretical models of the bedding operation for material flow (non-stochastic models)

We will first derive the mathematical formulae for the time series which represents the quality level in the generated state of material flow by tne ordinary bedding method. Let $\lambda'(t)$ denote the original (non-stochastic) time series which represents the quality level in the initial state of material flow, $\lambda'(t)$ is the first order derivative of a non-decreasing absolutely continuous function $\lambda(t)$ defined on the closed interval $[-\pi, \pi]$.

Then, the time series $r_k(t)$ which represents the quality level in the state of material flow generated by κ-fold ordinary bedding operation can be expressed as follows,

$$r_k(t)=\frac{1}{k}\sum_{i=1}^{k}\lambda'\left\{-\pi+\left(i-\frac{1}{2}\right)h+\frac{t}{k}\right\}, \tag{1}$$

where $kh=2\pi$, $-\pi\leqq t\leqq\pi$.

In practice, we may assume that $\lambda'(t)$ is a square summable function defined on the closed interval $[-\pi, \pi]$.

Then the series

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt) \tag{2}$$

is the Fourier series of the function $\lambda'(t)$, where

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos nt\ \lambda'(t)\,dt, \qquad b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\sin nt\ \lambda'(t)\,dt. \tag{3}$$

In this case, the time series $r_k(t)$ defined on the closed interval $[-\pi, \pi]$ which represents the quality level in the state of material flow generated by k-fold ordinary bedding operation can be expressed as follows,

$$\begin{aligned} r_k(t)&=\frac{1}{k}\sum_{i=1}^{k}\lambda'\left\{-\pi+\left(i-\frac{1}{2}\right)\frac{2\pi}{k}+\frac{t}{k}\right\} \\ &\sim\frac{1}{k}\sum_{i=1}^{k}\left[\frac{1}{2}a_0+\sum_{n=1}^{\infty}(-1)^n\left\{a_n\cos n\left(\frac{2i-1}{k}\pi+\frac{t}{k}\right)+b_n\sin n\left(\frac{2i-1}{k}\pi+\frac{t}{k}\right)\right\}\right] \\ &\sim\frac{1}{k}\sum_{i=1}^{k}\left\{\frac{1}{2}A_{i0}+\sum_{m=1}^{\infty}(A_{im}\cos mt+B_{im}\sin mt)\right\}, \end{aligned} \tag{4}$$

where

$$A_{i0}=a_0+\sum_{n=1}^{\infty}\frac{k}{n\pi}\left\{a_n\left(\sin i\frac{2n\pi}{k}-\sin\overline{i-1}\frac{2n\pi}{k}\right)+b_n\left(\cos\overline{i-1}\frac{2n\pi}{k}-\cos i\frac{2n\pi}{k}\right)\right\},$$

$$A_{im}=\sum_{n=1}^{\infty}\frac{(-1)^m\frac{n}{k}}{\left\{\left(\frac{n}{k}\right)^2-m^2\right\}\pi}\left\{a_n\left(\sin i\frac{2n\pi}{k}-\sin\overline{i-1}\frac{2n\pi}{k}\right)+b_n\left(\cos\overline{i-1}\frac{2n\pi}{k}-\cos i\frac{2n\pi}{k}\right)\right\},$$

$$B_{im}=\sum_{n=1}^{\infty}\frac{(-1)^m m}{\left\{\left(\frac{n}{k}\right)^2-m^2\right\}\pi}\left\{-a_n\left(\cos\overline{i-1}\frac{2n\pi}{k}-\cos i\frac{2n\pi}{k}\right)+b_n\left(\sin i\frac{2n\pi}{k}-\sin\overline{i-1}\frac{2n\pi}{k}\right)\right\} \quad (5)$$

It is to be noted here that we put

$$\frac{\left(\frac{n}{k}\right)}{\left\{\left(\frac{n}{k}\right)^2-m^2\right\}\pi}\left(\sin i\frac{2n\pi}{k}-\sin\overline{i-1}\frac{2n\pi}{k}\right)=(-1)^m,$$

$$\frac{\frac{n}{k}}{\left\{\left(\frac{n}{k}\right)^2-m^2\right\}\pi}\left(\cos\overline{i-1}\frac{2n\pi}{k}-\cos i\frac{2n\pi}{k}\right)=0, \quad (6)$$

when $n=mk$.

From the relations (4), (5) and (6), we can easily derive the next relation,

$$r_k(t)\sim\frac{1}{2}a_0+\sum_{m=1}^{\infty}(a_{km}\cos mt+b_{km}\sin mt) \quad (7)$$

Now we will derive the formulae for the variance of the original time series $\lambda'(t)$.

Since $\lambda'(t)$ is a square summable function defined on the closed interval $[-\pi, \pi]$, it is obvious that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\lambda'(t)-\frac{1}{2\pi}\int_{-\pi}^{\pi}\lambda'(t)\,dt\right\}^2dt=\frac{1}{2}\sum_{n=1}^{\infty}(a_n{}^2+b_n{}^2) \quad (8)$$

Next we will derive the formulae for the variance of the generated time series $r_k(t)$.

Since $r_k(t)$ is also a square summable function defined on the closed interval $[-\pi, \pi]$, we can easily derive the next relation

$$\sigma^2_{0\cdot b(k)}=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{r_k(t)-\frac{1}{2\pi}\int_{-\pi}^{\pi}r_k(t)\,dt\right\}^2dt=\frac{1}{2}\sum_{m=1}^{\infty}(a^2{}_{km}+b^2{}_{km}), \quad (9)$$

where the suffix $0\cdot b(k)$ means a k-fold ordinary bedding operation.

Thus we can compare the variance of the generated time series $r_k(t)$ with that of the original time series $\lambda'(t)$.

It is obvious that $\sigma^2_{0\cdot b(k)}=\frac{1}{2}\sum_{m=1}^{\infty}(a_{km}{}^2+b_{km}{}^2)$ $(k=1, 2, \ldots\ldots)$ is a monotone decreasing sequence and tends to 0 as $k\to\infty$.

And if the Fourier coefficients a_n, b_n decrease rapidly as n is increased, the efficiency of the bedding operation will be very large even for small values of k.

For the switch-back bedding operation, we obtain the next formula,

$$\sigma^2_{s\cdot b(2k)} = \frac{1}{2}\sum_{m=1}^{\infty} a^2_{2km} \tag{10}$$

where the suffix $s \cdot b(2k)$ means a $2k$-fold switch-back bedding operation.

From the relation (10), it will be obvious that the efficiency of the switch-back bedding operation for a time series which has a monotone decreasing or increasing smooth trend is generally large even for small values of k, since the Fourier coefficients a_n of such time series has a tendency to decrease rapidly as n is increased. In any case, we can estimate how the magnitude $\frac{1}{2}\sum_{m=1}^{\infty}(a^2{}_{km}+b^2{}_{km})$ or $\frac{1}{2}\sum_{m=1}^{\infty} a^2_{2km}$ decreases as k is increased, from the properties of the Fourier coefficients of the periodic function $\lambda'(t)$. If the periodic function $\lambda'(t)$ is continuously differentiable up to the k-th order and the $(k+1)$-th derivative $\lambda^{(k+1)}(t)$ satisfies the Dirichlet's conditions, then the Fourier coefficients a_n, b_n of $\lambda'(t)$ satisfy the next inequalities,

$$|a_n| \leqq \frac{M}{n^{k+1}}\,; \quad |b_n| \leqq \frac{M}{n^{k+1}}, \tag{11}$$

where M is a positive numbers.

If we put $\lambda'(t) = \alpha t$ for the interval $[-\pi, \pi]$, then the Fourier coefficients a_n, b_n of $\lambda'(t)$ are calculated as follows,

$$a_n = 0, \qquad \text{for all } n;$$

$$b_n = \frac{2(-1)^{n-1}}{n}\alpha.$$

Hence the magnitude $\frac{1}{2}\sum_{m=1}^{\infty}(a^2{}_{km}+b^2{}_{km})$ is given by $\frac{1}{2}\frac{4}{k^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\alpha^2 = \frac{\pi^2}{3k^2}\alpha^2$, and we can easily estimate the decreasing tendency of $\sigma^2_{0\cdot b(k)}$ as k is increased.

If we put $\lambda'(t) = \alpha t^2$ for the interval $[-\pi, \pi]$, then the Fourier coefficients a_n, b_n of $\lambda'(t)$ are calculated as follows,

$$a_0 = \frac{2}{3}\pi^2\alpha,$$

$$a_n = (-1)^n \frac{4}{n^2}\alpha, \text{ for } n = 1, 2, \ldots\ldots$$

$$b_n = 0, \text{ for all } n.$$

Hence the magnitude $\frac{1}{2}\sum_{m=1}^{\infty}(a^2{}_{km}+b^2{}_{km})$ is given by $\frac{1}{2}\frac{16}{k^4}\sum_{m=1}^{\infty}\frac{1}{m^4}\alpha^2 = \frac{(2\pi)^4}{180k^4}\alpha^2$, and we can easily estimate the decreasing tendency of $\sigma^2_{0\cdot b(k)}$ as k is increased.

If $\lambda'(t)$ is considerably smooth and may be developed in Maclaurin's series up to

the higher order term in the interval $[-\pi, \pi]$,
$\lambda'\left\{-\pi+\left(i-\frac{1}{2}\right)h+\frac{t}{k}\right\}$ can be written as the following formula,

$$\lambda'\left\{-\pi+\left(i-\frac{1}{2}\right)h+\frac{t}{k}\right\} \doteqdot \sum_{p=0}^{\infty} \frac{B_p\left(\frac{t}{k}\right)}{p\,!}\left\{\frac{\lambda^{(p)}\{-\pi+ih\}-\lambda^{(p)}\{-\pi+(i-1)h\}}{h}\right\} \quad (12)$$

where

$$B_p\left(\frac{t}{k}\right)=b_0\left(\frac{t}{k}\right)^p+\binom{n}{1}b_1\left(\frac{t}{k}\right)^{p-1}h+\binom{n}{2}b_2\left(\frac{t}{k}\right)^{p-2}h^2+\ldots\ldots+b_ph^p.$$

Here $\lambda^{(p)}(t)$ denotes the p-th derivative of $\lambda(t)$, and $b_0, b_1, b_2, \ldots\ldots$ are the Bernoulli numbers.

Hence

$$B_0\left(\frac{t}{k}\right)=1,\ B_1\left(\frac{t}{k}\right)=\frac{t}{k}-\frac{h}{2},\quad B_2\left(\frac{t}{k}\right)=\left(\frac{t}{k}\right)^2-\left(\frac{t}{k}\right)h+\frac{h^2}{6},$$

$$B_3\left(\frac{t}{k}\right)=\left(\frac{t}{k}\right)^3-\frac{3}{2}\left(\frac{t}{k}\right)^2h+\frac{1}{2}\left(\frac{t}{k}\right)h^2,\ \ldots\ldots$$

From (1) and (12), we can easily derive the next result,

$$r_k(t) \doteqdot \sum_{p=0}^{n}\frac{B_p\left(\frac{t}{k}\right)}{p\,!}\left\{\frac{\lambda^{(p)}(\pi)-\lambda^{(p)}(-\pi)}{2\pi}\right\} \quad (13)$$

or

$$r_k(t)-\frac{1}{2\pi}\int_{-\pi}^{\pi}\lambda'(t)\,dt=\sum_{p=1}^{n}\frac{B_p\left(\frac{t}{k}\right)}{p\,!}\left\{\frac{\lambda^{(p)}(\pi)-\lambda^{(p)}(-\pi)}{2\pi}\right\}, \quad (14)$$

Here we can write

$$\overline{r_k}=\frac{1}{2\pi}\int_{-\pi}^{\pi}r_k(t)\,dt=\frac{1}{2\pi}\int_{-\pi}^{\pi}\lambda'(t)\,dt=\bar{\lambda}'.$$

Then the formula for the variance of the generated time series $r_k(t)$ will be given as follows,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\{r_k(t)-\overline{r_k}\}^2\,dt \doteqdot \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\sum_{p=1}^{n}\frac{B_p\left(\frac{t}{k}\right)}{p\,!}\ \frac{\lambda^{(p)}(\pi)-\lambda^{(p)}(-\pi)}{2\pi}\right\}^2dt. \quad (15)$$

From (15), we can easily derive the next formula for sufficiently large value of k, assuming that the effects of the higher order terms are negligible,

$$\sigma^2_{0\cdot b(k)} \doteqdot \frac{1}{12}\left\{\frac{\lambda'(\pi)-\lambda'(-\pi)}{2\pi}\right\}^2\left(\frac{2\pi}{k}\right)^2+\frac{1}{720}\left\{\frac{\lambda''(\pi)-\lambda''(-\pi)}{2\pi}\right\}^2\left(\frac{2\pi}{k}\right)^4$$
$$-\frac{1}{360}\left\{\frac{\lambda'(\pi)-\lambda'(-\pi)}{2\pi}\ \frac{\lambda''(\pi)-\lambda''(-\pi)}{2\pi}\right\}\left(\frac{2\pi}{k}\right)^4 \quad (16)$$

For the case of switch-bedding operation, we can obtain easily the next formula for sufficiently large value of k, assuming that the effects of the higher order terms are negligible,

$$\sigma^2_{s\cdot b(2k)} \doteqdot \frac{1}{720}\left\{\frac{\lambda''(\pi)-\lambda''(-\pi)}{2\pi}\right\}^2 \left(\frac{2\pi}{k}\right)^4 \tag{17}$$

The variance given by $\sigma^2_{0\cdot b(k)}$ may also be considered as the variance of the mean of a random sample composed of k values at k points systematically taken from the material flow on the conveyer belt by random start method.

Similarly the variance given by $\sigma^2_{s\cdot b(2k)}$ may be considered also as the variance of the mean of a random sample composed of $2k$ values at $2k$ points (zig-zag) systematically taken from material flow on the conveyor belt by a random start method.

V. Theoretical models of the bedding operation for material flow (A lot composed of a given category)

First we will derive mathematical formulae for the variance of the numbers of pieces of a given category found in the increments of a fixed size Δl taken from the material flow, the state of which is described by a Poisson process. Even if the original time series which represents the initial state of material flow cannot be assumed to be a well-known stochastic process, we have a sufficient reason to assume that the time series which represent the state of materi^l flow going out through a properly designed mixer will be a stochastic process with a considerably smooth mean value function. If we discuss the accumulated number function of of pieces of a given category found in the material flow on the conveyor belt, such time series may be assumed to be a realization of a Poisson process approximately.

Here it is supposed that for every pair $t<s$, $X(s, \omega)-X(t, \omega)$ is integral valued, which

$$P_r\{X(s,\omega)-X(t,\omega)=\nu\} = \frac{e^{-\{\lambda(s)-\lambda(t)\}}\{\lambda(s)-\lambda(t)\}^\nu}{\nu\,!}, \quad (\nu=0,\ 1,\ 2,\ \cdots\cdots) \tag{18}$$

where $\lambda(t)$ is a non-decreasing absolutely continuous function defined on the interval $[-\pi, \pi]$.

Now we put

$$Y(t, \Delta l, \omega)=\begin{cases} X(t+\Delta l, \omega)-X(t, \omega), & \text{when } -\pi \leqq t \leqq \pi-\Delta l \\ X(\pi, \omega)-X(t, \omega) & \\ \quad +X(\Delta l-2\pi+t, \omega)-X(-\pi, \omega), & \text{when } \pi-\Delta l \leqq t \leqq \pi . \end{cases} \tag{19}$$

$$\bar{Y}(\Delta l, \omega)=\underset{t}{E}\{Y(t, \Delta l, \omega)\}=\frac{1}{2\pi}\int_{-\pi}^{\pi} Y(t, \Delta l, \omega)\,dt \tag{20}$$

$$\{S(\Delta l, \omega)\}^2 = \underset{t}{E}\{Y(t, \Delta l, \omega) - \tilde{Y}(\Delta l, \omega)\}^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\{Y(t, \Delta l, \omega) - \tilde{Y}(\Delta l, \omega)\}^2\, dt \qquad (21)$$

Then $\tilde{Y}(\Delta l, \omega)$ and $\{S(\Delta l, \omega)\}^2$ may be considered, respectively, as the mean and variance with respect to t of the numbers of particles found in the increments of a fixed size Δl taken from the material flow, the state of which is described by a Poisson process.

Next we put

$$V(\Delta l, \tau, \omega) = \underset{t}{E}\,[\{Y_p(t+\tau, \Delta l, \omega) - \tilde{Y}(\Delta l, \omega)\}\{Y(t, \Delta l, \omega) - \tilde{Y}(\Delta l, \omega)\}]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\{Y_p(t+\tau, \Delta l, \omega) - \tilde{Y}(\Delta l, \omega)\}\{Y(t, \Delta l, \omega) - \tilde{Y}(\Delta l, \omega)\}\, dt\,, \qquad (22)$$

where $2\pi \geqq \tau > 0$, and

$$Y_p(t+\tau, \Delta l, \omega) = \begin{cases} Y(t+\tau, \Delta l, \omega), & \text{when } \pi \geqq t+\tau \geqq -\pi\,, \\ Y(t+\tau-2\pi, \Delta l, \omega), & \text{when } 3\pi \geqq t+\tau \geqq \pi\,. \end{cases}$$

Then $V(\Delta l, \tau, \omega)$ may be considered as the circular serial covariance between the numbers of particles founds in the two increments of a fixed size Δl (with a constant distance τ) taken from the above material flow.

Hence we can easily derive the following results,

$$\underset{\omega}{E}\{\tilde{Y}(\Delta l, \omega)\} = \frac{\Delta l}{2\pi}\{\lambda(\pi) - \lambda(-\pi)\} = \bar{\lambda}'\,\Delta l\;;$$

$$\underset{\omega}{E}\{S(\Delta l, \omega)\}^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\{(\lambda_p(t+\Delta l) - \lambda_p(t)) - \bar{\lambda}'\,\Delta l\}^2 dt + \left(1 - \frac{\Delta l}{2\pi}\right)\bar{\lambda}'\,\Delta l\;; \qquad (24)$$

$$\underset{\omega}{E}\{V(\Delta l, \tau, \omega)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\{(\lambda_p(t+\tau+\Delta l) - \lambda_p(t+\tau)) - \bar{\lambda}'\,\Delta l\}$$

$$\{(\lambda_p(t+\Delta l) - \lambda_p(t)) - \bar{\lambda}'\,\Delta l\}\, dt + \varphi(\tau, \Delta l) - \frac{\Delta l}{2\pi}\,\bar{\lambda}'\,\Delta l\,; \qquad (25)$$

where

$$\bar{\lambda}' = \frac{1}{2\pi}\{\lambda(\pi) - \lambda(-\pi)\}\,;$$

$$\lambda_p(t) = \begin{cases} \lambda(t), & \text{when } -\pi \leqq t \leqq \pi\,, \\ \lambda(\pi) + \lambda(t-2\pi) - \lambda(-\pi), & \text{when } \pi \leqq t \leqq 3\pi\,; \end{cases}$$

and

$$\varphi(\tau, \Delta l) = \begin{cases} 0, & \text{when } |\tau| > \Delta l\,; \\ (1 - |\tau|/\Delta l), & \text{when } \Delta l \geqq |\tau| > 0\,. \end{cases}$$

If we put $\lambda^D(t)=\dfrac{\lambda(t+\Delta l)-\lambda(t)}{\Delta t}$, then

$$\underset{\omega}{E}\{S(\Delta l,\omega)\}^2=\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\{\lambda^D(t)-\bar{\lambda}'\}^2dt\right](\Delta l)^2+\left(1-\frac{\Delta l}{2\pi}\right)\bar{\lambda}'\Delta l \tag{26}$$

$$\underset{\omega}{E}\{V(\Delta l,\tau,\omega)\}=\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\{\lambda_p{}^D(t+\tau)-\bar{\lambda}'\}\{\lambda_p{}^D(t)-\bar{\lambda}'\}dt\right](\Delta l)^2$$
$$+\bar{\lambda}'\Delta l\,\varphi(\tau,\Delta l)-\frac{\Delta l}{2\pi}\bar{\lambda}'\Delta l. \tag{27}$$

(i) If the time series represents the accumulated number function of pieces of a given category found in the material flow from a uniform lot thoroughly mixed, which is assumed to be a realization of a temporally homogeneous Poisson process, then

$$\underset{\omega}{E}\,\underset{t}{E}\{Y(t,\Delta l,\omega)\}=\underset{\omega}{E}\{\tilde{Y}(\Delta l,\omega)\}=\bar{\lambda}'\Delta l, \tag{28}$$

$$\underset{\omega}{E}\{S(\Delta l,\omega)\}^2=\left(1-\frac{\Delta l}{2\pi}\right)\bar{\lambda}'\Delta l, \tag{29}$$

$$\underset{\omega}{E}\{V(\Delta l,\tau,\omega)\}=\bar{\lambda}'\Delta l\,\varphi(\tau,\Delta l)-\frac{\Delta l}{2\pi}\bar{\lambda}'\Delta l. \tag{30}$$

Hence the expected variance of the particles found in a sample composed of n non-overlapping increments of size $\Delta l/n$ taken systematically will be given by the next formula,

$$\underset{\omega}{E}\left\{S\left(n\times\frac{\Delta l}{n},\ \omega\right)\right\}^2=\left(1-\frac{\Delta l}{2\pi}\right)\bar{\lambda}'\Delta l, \tag{31}$$

which is equal to the expected variance $\underset{\omega}{E}\{S(\Delta l,\omega)\}^2$ of the number of particles found in a sample composed of an increments of size Δl taken from the material flow.

As will be easily calculated, the results (28), (29), (30), (31) obtained for a temporally homogenous Poisson process equally hold for a system of finite particles, each position of which is assumed to be a realization of a uniformly distributed (independent) random variable in the interval $[-\pi,\pi]$. It is to be noted here that we put $\bar{\lambda}'=M/2\pi$, where M is the number of particles.

(ii) Let $Z(t,\omega)$ be a generated process from the original process $X(t,\omega)$ with $\lambda(t)$ by 2κ-fold ordinary or switch-back bedding operation.

Then we can easily prove that such generated process is also a Poisson process with $\lambda^*(t)$, where $\lambda^*(t)$ has been derived from the original function $\lambda(t)$ by ordinary or switch-back bedding operation as has been shown in section 4.

For the original process $X(t, \omega)$ with $\lambda(t)$, we have

$$\underset{\omega\ t}{EE}\{Y(t, \Delta l, \omega)\} = \bar{\lambda}'\ \Delta l = \left(\frac{1}{2}a_0\right)\Delta l\ , \tag{32}$$

$$\underset{\omega}{E}\{S(\Delta l, \omega)\}^2 = \left(1-\frac{\Delta l}{2\pi}\right)\left(\frac{1}{2}a_0\right)\Delta l + \frac{1}{2}\left[\sum_{n=1}^{\infty}(a_n^2+b_n^2)\ \xi^2(n\Delta l)\right](\Delta l)^2 \tag{33}$$

$$\underset{\omega}{E}\{V(\Delta l, \tau, \omega)\} = \left(\frac{1}{2}a_0\right)\Delta l\ \ \varphi(\tau, \Delta l) - \frac{\Delta l}{2\pi}\left(\frac{1}{2}a_0\right)\Delta l$$
$$+\frac{1}{2}\left[\sum_{n=1}^{\infty}(a_n^2+b_n^2)(\cos n\tau)\ \xi^2(n\Delta l)\right](\Delta l)^2\ ; \tag{34}$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos nt\ \lambda'(t)\,dt\ , \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin nt\ \lambda'(t)\,dt\ ,$$

$$\xi^2(x) = (\sin x)^2\left\{1+4\left(\sin\frac{x}{2}\right)^2\right\}\Big/x^2$$

For the process generated by $2k$-fold ordinary bedding operation, we have

$$\underset{\omega\ t}{EE}\{Y(t, \Delta l, \omega)\} = \bar{\lambda}'\Delta l = \left(\frac{1}{2}a_0\right)\Delta l\ , \tag{35}$$

$$\underset{\omega}{E}\{S_{0\cdot b(2k)}(\Delta l, \omega)\}^2 = \left(1-\frac{\Delta l}{2\pi}\right)\left(\frac{1}{2}a_0\right)\Delta l + \frac{1}{2}\left[\sum_{m=1}^{\infty}(a_{2mk}^2+b_{2mk}^2)\ \xi^2(m\Delta l)\right](\Delta l)^2 \tag{36}$$

$$\underset{\omega}{E}\{V_{0\cdot b(2k)}(\tau, \Delta l, \omega)\} = \left(\frac{1}{2}a_0\right)\Delta l\ \ \varphi(\tau, \Delta l) - \frac{\Delta l}{2\pi}\left(\frac{1}{2}a_0\right)\Delta l$$
$$+\frac{1}{2}\left[\sum_{m=1}^{\infty}(a_{2mk}^2+b_{2mk}^2)(\cos m\tau)\ \xi^2(m\Delta l)\right](\Delta l)^2\ . \tag{37}$$

For the process generated by $2k$-fold switch-back bedding operation, we have

$$\underset{\omega\ t}{EE}\{Y(t, \Delta l, \omega)\} = \bar{\lambda}'\ \Delta l = \left(\frac{1}{2}a_0\right)\Delta l\ , \tag{38}$$

$$\underset{\omega}{E}\{S_{s\cdot b(2k)}(\Delta l, \omega)\}^2 = \left(1-\frac{\Delta l}{2\pi}\right)\left(\frac{1}{2}a_0\right)\Delta l$$
$$+\frac{1}{2}\left[\sum_{m=1}^{\infty}a_{2mk}^2\ \xi^2(m\Delta l)\right](\Delta l)^2\ . \tag{39}$$

$$\underset{\omega}{E}\{V_{s\cdot b(2k)}(\Delta l, \tau, \omega)\} = \left(\frac{1}{2}a_0\right)\Delta l\ \ \varphi(\tau,\Delta l) - \frac{\Delta l}{2\pi}\left(\frac{1}{2}a_0\right)\Delta l$$
$$+\frac{1}{2}\left[\sum_{m=1}^{\infty}a_{2mk}^2(\cos m\tau)\ \xi^2(m\Delta l)\right](\Delta l)^2\ . \tag{40}$$

From the results obtained above, we can compare the expected variance of the generated process with that of the original process. In any case, we can conclude that as $k\to\infty$,

$$\underset{\omega}{E}\{S_{b(k)}(\Delta l,\omega)\}^{2}\to\left(1-\frac{\Delta l}{2\pi}\right)\bar{\lambda}'\Delta l\,,$$

$$\underset{\omega}{E}\{V_{b(k)}(\tau,\Delta l,\omega)\}\to\left\{\varphi(\tau,\Delta l)-\frac{\Delta l}{2\pi}\right\}\bar{\lambda}'\Delta l$$

which are respectively the expected variance and expected circular serial covariance of $Y(t, \Delta l, \omega)$ from a temporally homogenous Poisson process treated in case (i).

Thus we can discuss the efficiency of mixing operation by the bedding method or the precision of the estimation obtained by a systematic or zig-zag sampling method for a Poisson process with mean value function $\lambda(t)$.

If we divide a time series into two components, i. e. a systematic component and a stochastic component, then we can conclude that the effect of bedding operation for reducing the variability of the fluctuation of a time series is considerably large for the systematic component, but not remarkable for the stochastic component. As was pointed out in the previous section, if the periodic function $\lambda'(t)$ statisfies the Dirichet's conditions, then the Fourier coefficients a_n, b_n of $\lambda'(t)$ satisfy the next inequalities,

$$|a_n|\leqq\frac{M}{n}\ ;\qquad |b_n|\leqq\frac{M}{n}$$

where M is a positive numbers. From these inequalities, we can easily estimate the decreasing tendency of $\sigma^2_{0\cdot b(k)}$ and $\sigma^2_{s\cdot b(2k)}$ for sufficiently large value of k.

VI. Theoretical models of bedding operation for material flow (a lot composed of mixture of pieces in a limited number of categories)

We will discuss here a lot in which the distribution of piece sizes and contents is represented by a mixture of pieces classified in a limited number of categories. All pieces from a given category will be considered as having a constant content, C_j (percentage), and piece weight, m_j, characteristic of that category.

Now we assume that the accumulated number function of pieces of a given category found in the material flow on the conveyor belt may be considered as a realization of a Poisson process $X_j(t,\omega)$ with $\lambda_j(t)$ $(j=1, 2, \ldots\ldots, M)$.

Besides, we assume that $X_1(t,\omega)$, $X_2(t,\omega)$, ..., $X_m(t,\omega)$ are mutually independent Poisson processes.

Now let $Z_j(t,\omega)$ be a generated process from the original process $X_j(t,\omega)$ by k-fold ordinary or $2k$-fold switch-back bedding operation.

Then we can easily say that such generated process is also a Poisson process with $\lambda_j^*(t)$, where $\lambda_j^*(t)$ has been derived from the original function $\lambda_j(t)$ by bedding operation in section 4.

And it is easily seen that $Z_1(t, \omega), Z_2(t, \omega), \ldots\ldots, Z_m(t, \omega)$ are also mutually independent Poisson processes.

Besides, we have

$$\underset{\omega}{E}\{Z_j(s, \omega)-Z_j(t, \omega)\}=\lambda_j^*(s)-\lambda_j^*(t)$$

$$\underset{\omega}{E}\{Z_j(s, \omega)-Z_j(t, \omega)\}^2=\{\lambda_j^*(s)-\lambda_j^*(t)\}^2+\{\lambda_j^*(s)-\lambda_j^*(t)\}\ .$$

Thus we can easily derive the relations such as (32), (33), (34), (35), (36), (37), (38), (39), (40) for the generated Poisson process $Z_j(t, \omega)$ with $\lambda_j^*(t)$, $(j=1, 2, \ldots\ldots, M)$.

Now we will estimate the influence of pieces of various sizes and contents on the efficiency of mixing operation by bedding method or the variability of the content in a sample composed of a series of increments systematically taken from the material flow on the conveyor belt.

The content in the whole lot is given by

$$\zeta(\omega)=\frac{\sum_{j=1}^{M} C_j m_j\{Z_j(\pi, \omega)-Z_j(-\pi, \omega)\}}{\sum_{j=1}^{M} m_j\{Z_j(\pi, \omega)-Z_j(-\pi, \omega)\}} \tag{41}$$

The content in an increment taken at the interval $(t, t+\Delta l)$ is given by

$$\zeta(t, \Delta l, \omega)=\frac{\sum_{j=1}^{M} C_j m_j Z_j(t, \Delta l, \omega)}{\sum_{j=1}^{M} m_j Z_j(t, \Delta l, \omega)} \tag{42}$$

when $\sum_{j=1}^{M} m_j Z_j(t, \Delta l, \omega) \neq 0$.

It is to be noted here that the material flow must be kept at a constant level as far as possible, i. e.

$$\sum_{j=1}^{M} m_j\{\lambda_j^*(t+\Delta l)-\lambda_j^*(t)\} \fallingdotseq \frac{\Delta l}{2\pi}\sum_{j=1}^{M} m_j\{\lambda_j^*(\pi)-\lambda_j^*(-\pi)\}=\{\sum_{j=1}^{M} m_j \bar{\lambda}_j'\}(\Delta l)\ . \tag{43}$$

Here $\zeta(\omega)$ may be rewritten as follows,

$$\zeta(\omega)=\frac{\frac{1}{2\pi}\int_{-\pi}^{\pi} \zeta(t, \Delta t, \omega)\{\sum_{j=1}^{M} m_j Z_j(t, \Delta l, \omega)\}dt}{\frac{1}{2\pi}\int_{-\pi}^{\pi}\{\sum_{j=1}^{M} m_j Z_j(t, \Delta l, \omega)\}dt} \tag{44}$$

Now we put

$$\bar{\zeta}(\Delta l, \omega)=\underset{t}{E}\{\zeta(t, \Delta l, \omega)\}=\frac{1}{2\pi}\int_{-\pi}^{\pi}\zeta(t, \Delta l, \omega)\,dt \tag{45}$$

Then, we have the next approximation formula as the bias of the estimate $\bar{\zeta}(\Delta l, \omega)$ from $\zeta(\omega)$,

$$\underset{\omega}{E}\{\tilde{\zeta}(\Delta l, \omega)-\tilde{\zeta}(\omega)\}$$

$$\fallingdotseq \bar{C}\left(1-\frac{\Delta l}{2\pi}\right)\left\{\frac{\sum_{j=1}^{M} m_j^2 \bar{\lambda}'(\Delta l)}{(\sum_{j=1}^{M} m_j \bar{\lambda}_j')^2 (\Delta l)^2} - \frac{\sum_{j=1}^{M} C_j m_j^2 \bar{\lambda}_j'(\Delta l)}{(\sum_{j=1}^{M} C_j m_j \bar{\lambda}_j')(\sum_{j=1}^{M} m_j \bar{\lambda}_j')(\Delta l)^2}\right\}$$

$$+\bar{C}\left\{\frac{\frac{1}{2\pi}\int_{-\pi}^{\pi}\{\sum_{j=1}^{M} m_j(\lambda_j^{*D}(t)-\bar{\lambda}_j')\}^2 dt}{(\sum_{j=1}^{M} m_j \bar{\lambda}_j')^2}\right.$$

$$\left.-\frac{\frac{1}{2\pi}\int_{-\pi}^{\pi}\{\sum_{j=1}^{M} C_j m_j(\lambda_j^{*D}(t)-\bar{\lambda}_j')\sum_{j=1}^{M} m_j(\lambda_j^{*D}(t)-\bar{\lambda}_j')\} dt}{(\sum_{j=1}^{M} C_j m_j \bar{\lambda}_j')(\sum_{j=1}^{M} m_j \bar{\lambda}_j')}\right\}, \tag{46}$$

assuming that the effects of the higher order terms are negligibly small; where

$$\bar{C}=\frac{\sum_{j=1}^{M} C_j m_j\{\lambda_j(\pi)-\lambda_j(-\pi)\}}{\sum_{j=1}^{M} m_j\{\lambda_j(\pi)-\lambda_j(-\pi)\}}=\frac{\sum_{j=1}^{M} C_j m_j \bar{\lambda}_j'}{\sum_{j=1}^{M} m_j \bar{\lambda}_j'}. \tag{47}$$

As will be easily seen from (46), the bias of estimate $\tilde{\zeta}(\Delta l, \omega)$ is considerably small if the material flow is kept at a constant level; i. e.

$$E\{\tilde{\zeta}(\Delta l, \omega)-\tilde{\zeta}(\omega)\}$$

$$\fallingdotseq \bar{C}\left(1-\frac{\Delta l}{2\pi}\right)\left\{\frac{\sum_{j=1}^{M} m_j^2 \bar{\lambda}_j' \Delta l}{(\sum_{j=1}^{M} m_j \bar{\lambda}_j')^2 (\Delta l)^2} - \frac{\sum_{j=1}^{M} C_j m_j^2 \bar{\lambda}_j' \Delta l}{(\sum_{j=1}^{M} C_j m_j \bar{\lambda}_j')(\sum_{j=1}^{M} m_j \bar{\lambda}_j')(\Delta l)^2}\right\} \tag{48}$$

Here we put

$$\{S(\Delta l, \omega)\}^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}\{\zeta(t, \Delta l, \omega)-\tilde{\zeta}(\omega)\}^2 dt \tag{49}$$

as an approximate variance of the $\zeta(t, \Delta l, \omega)$.

Then we have

$$\underset{\omega}{E}\{S(\Delta l, \omega)\}^2 \fallingdotseq \bar{C}^2\left(1-\frac{\Delta l}{2\pi}\right)\left\{\frac{\sum_{j=1}^{M} C_j^2 m_j^2 \bar{\lambda}_j' \Delta l}{(\sum_{j=1}^{M} C_j m_j \bar{\lambda}_j')^2 (\Delta l)^2} + \frac{\sum_{j=1}^{M} m_j^2 \bar{\lambda}_j' \Delta l}{(\sum_{j=1}^{M} m_j \bar{\lambda}_j')^2 (\Delta l)^2}\right.$$

$$\left.-2\frac{\sum_{j=1}^{M} C_j m_j^2 \bar{\lambda}_j' \Delta l}{(\sum_{j=1}^{M} C_j m_j \bar{\lambda}_j')(\sum_{j=1}^{M} m_j \bar{\lambda}_j')(\Delta l)^2}\right\} + \bar{C}^2\left\{\frac{\frac{1}{2\pi}\int_{-\pi}^{\pi}\{\sum_{j=1}^{M} C_j m_j(\lambda_j^{*D}(t)-\bar{\lambda}_j')\}^2 dt}{(\sum_{j=1}^{M} C_j m_j \lambda_j')^2}\right.$$

$$+\frac{\frac{1}{2\pi}\int_{-\pi}^{\pi}\{\sum_{j=1}^{M} m_j(\lambda_j^{*D}(t)-\bar{\lambda}_j')\}^2 dt}{(\sum_{j=1}^{M} m_j \bar{\lambda}_j')^2}$$

$$\left.-2\frac{\frac{1}{2\pi}\int_{-\pi}^{\pi}\{\sum_{j=1}^{M} C_j m_j(\lambda_j^{*D}(t)-\bar{\lambda}_j')\}\{\sum_{j=1}^{M} m_j(\lambda_j^{*D}(t)-\bar{\lambda}_j')\} dt}{(\sum_{j=1}^{M} C_j m_j \bar{\lambda}_j')(\sum_{j=1}^{M} m_j \bar{\lambda}_j')}\right\} \tag{50}$$

assuming that the effects of the higher order terms are negligibly small.

Next we put

$$V(\tau, \Delta l, \omega)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\{\zeta_p(t+\tau, \Delta l, \omega)-\bar{\zeta}(\omega)\}\{\zeta(t, \Delta l, \omega)-\bar{\zeta}(\omega)\}dt ,$$

as an approximate circular serial covariance between the contents

$\zeta_p(t+\tau, \Delta l, \omega)$ and $\zeta(t, \Delta l, \omega)$, where

$$\zeta_p(t)=\begin{cases}\zeta(t) , & \text{when } -\pi\leqq t\leqq\pi ,\\ \zeta(t-2\pi) , & \text{when } \pi\leqq t\leqq 3\pi .\end{cases}$$

Then we have

$$\begin{aligned}
&\underset{\omega}{E}\{V(\tau, \Delta l, \omega)\}\\
&\doteqdot \bar{C}^2\left[\left\{\varphi(\tau, \Delta l)-\frac{\Delta l}{2\pi}\right\}\left\{\frac{\sum_{j=1}^{M} C_j^2 m_j^2 \bar{\lambda}_j'(\Delta l)}{(\sum_{j=1}^{M} C_j m_j \bar{\lambda}_j' \Delta l)^2}+\frac{\sum_{j=1}^{M} m_j^2 \bar{\lambda}_j'(\Delta l)}{(\sum_{j=1}^{M} m_j \bar{\lambda}_j' \Delta l)^2}\right.\right.\\
&\left.\left.-2\frac{\sum_{j=1}^{M} C_j m_j^2 (\Delta l)}{(\sum_{j=1}^{M} C_j m_j \bar{\lambda}_j' \Delta l)(\sum_{j=1}^{M} m_j \bar{\lambda}_j' \Delta l)}\right\}\right]\\
&+\bar{C}^2\left[\frac{\frac{1}{2\pi}\int_{-\pi}^{\pi}\{\sum_{j=1}^{M} C_j m_j(\lambda_{jp}^{*D}(t+\tau)-\bar{\lambda}_j')\}\{\sum_{j=1}^{M} C_j m_j(\lambda_j^{*D}(t)-\bar{\lambda}_j')\}dt}{\{\sum_{j=1}^{M} C_j m_j \bar{\lambda}_j'\}^2}\right.\\
&+\frac{\frac{1}{2\pi}\int_{-\pi}^{\pi}\{\sum_{j=1}^{M} m_j(\lambda_{jp}^{*D}(t+\tau)-\bar{\lambda}_j')\}\{\sum_{j=1}^{M} m_j(\lambda_j^{*D}(t)-\bar{\lambda}_j')\}dt}{\{\sum_{j=1}^{M} m_j \bar{\lambda}_j'\}^2}\\
&-\frac{\frac{1}{2\pi}\int_{-\pi}^{\pi}\{\sum_{j=1}^{M} C_j m_j(\lambda_{jp}^{*D}(t+\tau)-\bar{\lambda}_j')\}\{\sum_{j=1}^{M} m_j(\lambda_j^{*D}(t)-\bar{\lambda}_j')\}dt}{\{\sum_{j=1}^{M} C_j m_j \bar{\lambda}_j'\}\{\sum_{j=1}^{M} m_j \bar{\lambda}_j'\}}\\
&\left.-\frac{\frac{1}{2\pi}\int_{-\pi}^{\pi}\{\sum_{j=1}^{M} C_j m_j(\lambda_j^{*D}(t)-\bar{\lambda}_j')\}\{\sum_{j=1}^{M} m_j\{\lambda_{jp}^{*D}(t+\tau)-\bar{\lambda}_j'\}dt}{\{\sum_{j=1}^{M} C_j m_j \bar{\lambda}_j'\}\{\sum_{j=1}^{M} m_j \bar{\lambda}_j'\}}\right], \qquad (51)
\end{aligned}$$

assuming that the effects of the higher order terms are negligibly small. If the material flow is kept at a constant level, we have

$$\begin{aligned}
&\underset{\omega}{E}\{S(\Delta l, \omega)\}^2\doteqdot \bar{C}^2\left[\left(1-\frac{\Delta l}{2\pi}\right)\left\{\frac{\sum_{j=1}^{M} C_j^2 m_j^2 \bar{\lambda}_j'\Delta l}{(\sum_{j=1}^{M} C_j m_j \bar{\lambda}_j' \Delta l)^2}+\frac{\sum_{j=1}^{M} m_j^2 \bar{\lambda}_j'\Delta l}{(\sum_{j=1}^{M} m_j \bar{\lambda}_j' \Delta l)^2}\right.\right.\\
&\left.\left.-2\frac{\sum_{j=1}^{M} C_j m_j^2 \bar{\lambda}_j'\Delta l}{(\sum_{j=1}^{M} C_j m_j \bar{\lambda}_j' \Delta l)(\sum_{j=1}^{M} m_j \bar{\lambda}_j' \Delta l)}\right\}+\frac{\frac{1}{2\pi}\int_{-\pi}^{\pi}\{\sum_{j=1}^{M} C_j m_j(\lambda_j^{*D}(t)-\bar{\lambda}_j')\}^2 dt}{(\sum_{j=1}^{M} C_j m_j \bar{\lambda}_j')^2}\right], \qquad (52)
\end{aligned}$$

$$\underset{\omega}{E}\{V(\tau, \Delta l, \omega)\} \doteqdot \bar{C}^2 \left[\left\{ \varphi(\tau, \Delta l) - \frac{\Delta l}{2\pi} \right\} \left\{ \frac{\sum_{j=1}^{M} C_j^2 m_j^2 \bar{\lambda}_j' \Delta l}{(\sum_{j=1}^{M} C_j m_j \bar{\lambda}_j \Delta l)^2} + \frac{\sum_{j=1}^{M} m_j^2 \bar{\lambda}_j' \Delta l}{(\sum_{j=1}^{M} m_j \bar{\lambda}_j' \Delta l)^2} \right.\right.$$

$$\left. -2 \frac{\sum_{j=1}^{M} C_j m_j^2 \bar{\lambda}_j' \Delta l}{(\sum_{j=1}^{M} C_j m_j \bar{\lambda}_j' \Delta l)(\sum_{j=1}^{M} m_j \bar{\lambda}_j' \Delta l)} \right\}$$

$$\left. + \frac{\frac{1}{2\pi} \int_{-\pi}^{\pi} \{\sum_{j=1}^{M} C_j m_j (\lambda_{jp}^{*D}(t+\tau) - \bar{\lambda}_j') \sum_{j=1}^{M} C_j m_j (\lambda_j^{*D}(t) - \bar{\lambda}_j')\} dt}{(\sum_{j=1}^{M} C_j m_j \bar{\lambda}_j')^2} \right] \tag{53}$$

Here we put $\bar{\lambda}_j'\Delta l = n_j$, which is the expected number of pieces of category j found in an increment of size Δl. Then we have

$$\underset{\omega}{E}\{\xi(\Delta l, \omega) - \xi(\omega)\} \doteqdot \bar{C} \left[\left(1 - \frac{\Delta l}{2\pi}\right) \left\{ \frac{\sum_{j=1}^{M} m_j^2 n_j}{(\sum_{j=1}^{M} m_j n_j)^2} - \frac{\sum_{j=1}^{M} C_j m_j^2 n_j}{(\sum_{j=1}^{M} m_j n_j)(\sum_{j=1}^{M} C_j m_j n_j)} \right\} \right], \tag{54}$$

$$\underset{\omega}{E}\{S(\Delta l, \omega)\}^2 \doteqdot \bar{C}^2 \left[\left(1 - \frac{\Delta l}{2\pi}\right) \left\{ \frac{\sum_{j=1}^{M} C_j^2 m_j^2 n_j}{(\sum_{j=1}^{M} C_j m_j n_j)^2} + \frac{\sum_{j=1}^{M} m_j^2 n_j}{(\sum_{j=1}^{M} m_j n_j)^2} \right.\right.$$

$$\left.\left. -2 \frac{\sum_{j=1}^{M} C_j m_j^2 n_j}{(\sum_{j=1}^{M} C_j m_j n_j)(\sum_{j=1}^{M} m_j n_j)} \right\} + \frac{\frac{1}{2\pi} \int_{-\pi}^{\pi} \{\sum_{j=1}^{M} C_j m_j (\lambda_j^{*D}(t) - \bar{\lambda}_j')\}^2 dt}{\{\sum_{j=1}^{M} C_j m_j \bar{\lambda}_j'\}^2} \right], \tag{55}$$

$$\underset{\omega}{E}\{V(\tau, \Delta l, \omega)\} \doteqdot \bar{C}^2 \left[\left\{ \varphi(\tau, \Delta l) - \frac{\Delta l}{2\pi} \right\} \left\{ \frac{\sum_{j=1}^{M} C_j^2 m_j^2 n_j}{(\sum_{j=1}^{M} C_j m_j n_j)^2} + \frac{\sum_{j=1}^{M} m_j^2 n_j}{(\sum_{j=1}^{M} m_j n_j)^2} \right.\right.$$

$$\left. -2 \frac{\sum_{j=1}^{M} C_j m_j^2 n_j}{(\sum_{j=1}^{M} C_j m_j n_j)(\sum_{j=1}^{M} m_j n_j)} \right\}$$

$$\left. + \frac{\frac{1}{2\pi} \int_{-\pi}^{\pi} \{\sum_{j=1}^{M} C_j m_j (\lambda_{jp}^{*D}(t+\tau) - \bar{\lambda}_j') \sum_{j=1}^{M} C_j m_j (\lambda_j^{*D}(t) - \bar{\lambda}_j')\} dt}{(\sum_{j=1}^{M} C_j m_j \bar{\lambda}_j')^2} \right]. \tag{56}$$

We will discuss here the effects of the systematic component on the total variance by using Fourier series expansion of $\lambda_j'(t)$.

For the generated process by $2k$-fold ordinary bedding operation, the systematic component of $\underset{\omega}{E}\{S(\Delta l, \omega)\}^2$ may be written as follows,

$$\bar{C}^2 \frac{\frac{1}{2\pi} \int_{-\pi}^{\pi} \{\sum_{j=1}^{M} C_j m_j (\lambda_j^{*D}(t) - \bar{\lambda}_j')\}^2 dt}{\{\sum_{j=1}^{M} C_j m_j \bar{\lambda}_j'\}^2} = \bar{C}^2 \frac{\left[\frac{1}{2} \sum_{n=1}^{\infty} (\alpha_{2nk}^2 + \beta_{2nk}^2) \xi^2(n\Delta l)\right] (\Delta l)^2}{\left\{\frac{1}{2}\alpha_0\right\}^2}, \tag{57}$$

and the systematic component of $E\{V(\tau, \Delta l, \omega)\}^2$ may be written as follows,

$$\bar{C}^2 \frac{\frac{1}{2\pi}\int_{-\pi}^{\pi}\{\sum_{j=1}^{M} C_j m_j (\lambda_{jp}^{*D}(t+\tau)-\bar{\lambda}_j') \sum_{j=1}^{M} C_j m_j (\lambda_j^{*D}(t)-\lambda_j')\}\,dt}{\{\sum_{j=1}^{M} C_j m_j \bar{\lambda}_j'\}^2}, \tag{58}$$

$$=\bar{C}^2 \frac{\left[\frac{1}{2}\sum_{n=1}^{\infty}(\alpha_{2nk}^2+\beta_{2nk}^2)(\cos n\tau)\,\xi^2(n\Delta l)\right](\Delta l)^2}{\left\{\frac{1}{2}\alpha_0\right\}^2}, \tag{59}$$

where

$$\alpha_n=\sum_{j=1}^{M} C_j m_j a_{j,n}, \qquad \beta_n=\sum_{j=1}^{M} C_j m_j b_{j,n}.$$

Similarly, for the generated process by $2k$-fold switch-back bedding operation, the systematic component of $\underset{\omega}{E}\{S(\Delta l, \omega)\}^2$ may be written as follows,

$$\bar{C}^2 \frac{\left[\frac{1}{2}\sum_{n=1}^{M}\alpha_{2nk}^2\,\xi^2(n\Delta l)\right](\Delta l)^2}{\left\{\frac{1}{2}\alpha_0\right\}^2}, \tag{60}$$

and the systematic component of $\underset{\omega}{E}\{V(\tau, \Delta l, \omega)\}^2$ may be written as follows,

$$\bar{C}^2 \frac{\left[\frac{1}{2}\sum_{n=1}^{\infty}\alpha_{2nk}^2(\cos n\tau)\,\xi^2(n\Delta l)\right](\Delta l)^2}{\left\{\frac{1}{2}\alpha_0\right\}^2}. \tag{61}$$

Here we may regard α_n, β_n as the Fourier coefficients of $f(t)=\sum_{j=1}^{M} C_j m_j \lambda_j'(t)$.

It should be noted here that $g(t)=\sum_{j=1}^{M} m_j \lambda_j'(t)$ is kept nearly at a constant level, so that we may regard as follows,

$$\frac{\sum_{j=1}^{M} m_j a_{j,n}}{\sum_{j=1}^{M} m_j a_{j,0}} \fallingdotseq 0, \qquad \frac{\sum_{j=1}^{M} m_j b_{j,n}}{\sum_{j=1}^{M} m_j a_{j,0}} \fallingdotseq 0 \qquad (n=1, 2, 3, \cdots\cdots)$$

We can easily derive the next inequalities,

$$\alpha_n^2=(\sum_{j=1}^{M} C_j m_j a_{j,n})^2 \leqq \{\sum_{j=1}^{M} C_j^2 m_j^2 \sum_{j=1}^{M} a_{j,n}^2\}\left\{1-\frac{(\sum_{j=1}^{M} C_j m_j^2)^2}{(\sum_{j=1}^{M} C_j^2 m_j^2)(\sum_{j=1}^{M} m_j^2)}\right\}, \tag{62}$$

$$\beta_n^2=(\sum_{j=1}^{M} C_j m_j b_{j,n})^2 \leqq \{\sum_{j=1}^{M} C_j^2 m_j^2 \sum_{j=1}^{M} b_{j,n}^2\}\left\{1-\frac{(\sum_{j=1}^{M} C_j m_j^2)^2}{(\sum_{j=1}^{M} C_j^2 m_j^2)(\sum_{j=1}^{M} m_j^2)}\right\}, \tag{63}$$

under the assumption that the relation $\sum_{j=1}^{M} m_j a_{j,n}=0$, $\sum_{j=1}^{M} m_j b_{j,n}=0$ hold.

If $\dfrac{(\sum_{j=1}^{M} C_j m_j^2)^2}{(\sum_{j=1}^{M} C_j^2 m_j^2)(\sum_{j=1}^{M} m_j^2)}$ can be taken nearly 1, then $\alpha_n=\sum_{j=1}^{M} C_j m_j a_{j,n}$ and $\beta_n=\sum_{j=1}^{M} C_j m_j b_{j,n}$ will be very small compared with $\alpha_0=\sum_{j=1}^{M} C_j m_j a_{j,0}$.

In any case, the effect of variability of the systematic component on the total variance will be negligibly small compared with that of the stochastic component if we take k sufficiently large.

After all, it will be easily seen that,

$$\underset{\omega}{E}\{S(\Delta l,\omega)\}^2 \rightarrow \bar{C}^2\left(1-\frac{\Delta l}{2\pi}\right)\left\{\frac{\sum_{j=1}^{M} C_j^2 m_j^2 n_j}{(\sum_{j=1}^{M} C_j m_j n_j)^2}+\frac{\sum_{j=1}^{M} m_j^2 n_j}{(\sum_{j=1}^{M} m_j n_j)^2} -2\frac{\sum_{j=1}^{M} C_j m_j^2 n_j}{(\sum_{j=1}^{M} C_j m_j n_j)(\sum_{j=1}^{M} m_j n_j)}\right\}, \tag{64}$$

$$\underset{\omega}{E}\{V(\tau,\Delta l,\omega)\} \rightarrow \bar{C}^2\left\{\varphi(\tau,\Delta l)-\frac{\Delta l}{2\pi}\right\}\left\{\frac{\sum_{j=1}^{M} C_j^2 m_j^2 n_j}{(\sum_{j=1}^{M} C_j m_j n_j)^2}+\frac{\sum_{j=1}^{M} m_j^2 n_j}{(\sum_{j=1}^{M} m_j n_j)^2} -2\frac{\sum_{j=1}^{M} C_j m_j^2 n_j}{(\sum_{j=1}^{M} C_j m_j n_j)(\sum_{j=1}^{M} m_j n_j)}\right\}, \tag{65}$$

if a lot of material is thoroughly mixed by bedding operation with properly designed mixer. We have also

$$\underset{\omega}{E}\{\zeta(\Delta l,\omega)-\zeta(\omega)\} \rightarrow \bar{C}\left(1-\frac{\Delta l}{2\pi}\right)\left\{\frac{\sum_{j=1}^{M} m_j^2 n_j}{(\sum_{j=1}^{M} m_j n_j)^2}-\frac{\sum_{j=1}^{M} C_j m_j^2 n_j}{(\sum_{j=1}^{M} C_j m_j n_j)(\sum_{j=1}^{M} m_j n_j)}\right\}, \tag{66}$$

as the bias of the estimate $\zeta(\Delta l,\omega)$ from $\zeta(\omega)$.

Now the increment of size Δl considered here may be replaced by a random sample composed of (sufficiently large) $2k$ increments of size $\Delta l/2k$ systematically taken from the material flow on the conveyor belt by random start method.

Besides, if the expected number of pieces in a sample of size Δl is sufficiently large ($\Delta l/2\pi$ is sufficiently small) and the effect of each piece is individually negligible to the total variation of the sample, then $\zeta(t,\Delta l,\omega)-\zeta(\omega)$ may be considered as a random variable with normal distribution $N\{m^*, (\sigma^*)^2\}$ approximately, by applying the central limit theorem; where

$$m^* \doteqdot \bar{C}\left\{\frac{\sum_{j=1}^{M} m_j^2 n_j}{(\sum_{j=1}^{M} m_j n_j)^2} - \frac{\sum_{j=1}^{M} C_j m_j^2 n_j}{(\sum_{j=1}^{M} C_j m_j n_j)(\sum_{j=1}^{M} m_j n_j)}\right\}, \tag{67}$$

$$(\sigma^*)^2 \doteqdot \bar{C}^2\left\{\frac{\sum_{j=1}^{M} C_j^2 m_j^2 n_j}{(\sum_{j=1}^{M} C_j m_j n_j)^2} + \frac{\sum_{j=1}^{M} m_j^2 n_j}{(\sum_{j=1}^{M} m_j n_j)^2} - 2\frac{\sum_{j=1}^{M} C_j m_j^2 n_j}{(\sum_{j=1}^{M} C_j m_j n_j)(\sum_{j=1}^{M} m_j n_j)}\right\}. \tag{68}$$

Now if we take two samples $(t, t+\Delta l)$ and $(s, s+\Delta l)$, then $\zeta(t, \Delta l, \omega) - \bar{\zeta}(\omega)$ and $\zeta(s, \Delta l, \omega) - \bar{\zeta}(\omega)$ may be considered as two random variables with normal distribution $N\{m^*, (\sigma^*)^2\}$; the expected circular serial correlation coefficient between them is given by $\varphi(|s-t|, \Delta l) - \Delta l/2\pi$ approximately.

From the above discussion we may say that a lot of bulk material, whatever the initial state of material may be, will be considerably homogenous by k-fold bedding operation with properly designed mixer (if we take k sufficiently large) during the material is being transported by conveyor belt.

Finally, we can easily derive the following simplified approximation formula by summarizing the relations (55), (57) and (60);

$$\underset{\omega}{E}\{S(\Delta l, \omega)\}^2 \doteqdot \sigma_k^2 + \frac{\sigma_i^2}{kw}, \tag{69}$$

where

$$\sigma_k^2 = \begin{cases} \bar{C}^2 \dfrac{\left[\frac{1}{2}\sum_{n=1}^{\infty}(\alpha_{2nk}^2+\beta_{2nk}^2)\xi^2(n\Delta l)\right](\Delta l)^2}{\left\{\frac{1}{2}\alpha_0\right\}^2} \\ \text{for the ordinary bedding operation}, \\ \bar{C}^2 \dfrac{\left[\frac{1}{2}\sum_{n=1}^{\infty}\alpha_{2nk}^2\xi^2(n\Delta l)\right](\Delta l)^2}{\left\{\frac{1}{2}\alpha_0\right\}^2} \\ \text{for the switch-back bedding operation}, \end{cases}$$

$$\sigma_i^2 = \bar{C}^2\left\{\frac{\sum_{j=1}^{M} C_j^2 m_j^2 n_{j,0}}{(\sum_{j=1}^{M} C_j m_j n_{j,0})^2} + \frac{\sum_{j=1}^{M} m_j^2 n_{j,0}}{(\sum_{j=1}^{M} m_j n_{j,0})^2} - 2\frac{\sum_{j=1}^{M} C_j m_j^2 n_{j,0})}{(\sum_{j=1}^{M} C_j m_j n_{j,0})(\sum_{j=1}^{M} m_j n_{j,0})}\right\}$$

is the variance for a sample of unit weight $w_0 = \sum_{j=1}^{M} m_j n_{j,0}$ taken from the thoroughly mixed lot, and w means the weight of each increment in sample composed of k systematic increments.

The relation (69) was suggested in section 2, when we treated the practical data on sampling from material in motion. In Fig. 3, it was remarked that the solid curve has a tendency to approach asymptotically to the dotted curve as the number of the increments in samples is increased. We can explain this fact as follows.

As was discussed before, the systematic component σ_k^2 of the approximate variance $\underset{\omega}{E}\{S(\Delta l, \omega)\}$ has a tendency to decrease to zero in the order $0\left(\frac{1}{k^{2m+2}}\right)$ for sufficiently large values of k, if the periodic function $f(t)$ is continuously dfferentiable up to the $(m-1)$-th order and the m-th derivative $f^{(m)}(t)$ satisfies the Dirichlet's conditions. In practice, it may be natural to assume that $f(t)$ satisfies only the Dirichlet's conditions. Then σ_k^2 has a tendency to decrease to zero in the order $0\left(\frac{1}{k^2}\right)$ for sufficiently large values of k. In any case, the systematic component σ_k^2 decreases rapidly to zero compard with the stochastic component $\frac{\sigma_i^2}{kw}$, which decreases to zero in the order $0\left(\frac{1}{k}\right)$ for fixed w. It is to be noted here that the function $f(t)$ will be considerably smooth if the material is locally mixed by properly designed mixer during its transportation. Hence it is easily seen that the mixing operation by bedding method will be considerably effective, when we apply this method to the locally randomized material flow which have a main trend curve having been smoothed by a mixing operation.

VII. Theoretical models of interpenetrating samples composed of systematic increments taken from material flow

In the illustrative example of section 2, we treated the interpenetrating samples composed of systematic increments taken from the material flow on a conveyor belt. The mathematical models will be established for such interpenetrating samples. We will treat r pairs of interpenetrating samples, each of which is composed of k increments systematically taken from the material flow by random start method. For such r pairs of interpenetrating samples, we consider $2r$ random variables as follows,

$$\begin{aligned}&\zeta(t, \Delta l, \omega),\ \zeta(t+\Delta l, \Delta l, \omega),\ \zeta\left(t+\frac{2\pi}{r}, \Delta l, \omega\right),\ \zeta\left(t+\frac{2\pi}{r}+\Delta l, \Delta l, \omega\right),\\ &\zeta\left(t+2\frac{2\pi}{r}, \Delta l, \omega\right),\ \zeta\left(t+2\frac{2\pi}{r}+\Delta l, \Delta l, \omega\right), \cdots\cdots\\ &\cdots\cdots, \zeta\left(t+\frac{r-1}{r}2\pi, \Delta l, \omega\right),\ \zeta\left(t+\frac{r-1}{r}2\pi+\Delta l, \Delta l, \omega\right). \qquad (70)\end{aligned}$$

Here we put

$$\xi(t, \Delta l, \omega)=\zeta(t+\Delta l, \Delta l, \omega)-\zeta(t, \Delta l, \omega) . \qquad (71)$$

Then, for sufficiently large value of κ, $\xi(t, \Delta l, \omega)$ may be considered approximately as a random variable with normal distribution $N\{0, (\sigma^{**})^2\}$, where

$$(\sigma^{**})^2=2\bar{C}^2\left\{\frac{\sum_{j=1}^{M} C_j^2 m_j^2 n_j}{(\sum_{j=1}^{M} C_j m_j n_j)^2}+\frac{\sum_{j=1}^{M} m_j^2 n_j}{(\sum_{j=1}^{M} m_j n_j)^2}-2\frac{\sum_{j=1}^{M} C_j m_j^2 n_j}{(\sum_{j=1}^{M} C_j m_j n_j)(\sum_{j=1}^{M} m_j n_j)}\right\}$$

$$+\bar{C}^2\frac{\left[\sum_{n=1}^{\infty}(\alpha_{2nk}^2+\beta_{2nk}^2)\{1-\cos(n\Delta l)\}\xi^2(n\Delta l)\right]}{\left\{\frac{1}{2}\alpha_0\right\}^2}(\Delta l)^2 \tag{72}$$

Hereafter, we shall put

$$2\bar{C}\left\{\frac{\sum_{j=1}^{M} C_j^2 m_j^2 n_j}{(\sum_{j=1}^{M} C_j m_j n_j)^2}+\frac{\sum_{j=1}^{M} m_j^2 n_j}{(\sum_{j=1}^{M} m_j n_j)^2}-2\frac{\sum_{j=1}^{M} C_j m_j^2 n_j}{(\sum_{j=1}^{M} C_j m_j n_j)(\sum_{j=1}^{M} m_j n_j)}\right\}=W\,. \tag{73}$$

The expected circular serial covariance between $\xi\left(t+p\frac{2\pi}{r},\ \Delta l,\ \omega\right)$ and $\xi\left(t+q\frac{2\pi}{r},\ \Delta l,\ \omega\right)$ is calculated to be

$$\bar{C}^2\frac{\left[\sum_{n=1}^{\infty}(\alpha_{2kn}^2+\beta_{2kn}^2)\cos n\,(p-q)\frac{2\pi}{r}\{1-\cos(n\Delta l)\}\ \xi^2(n\Delta l)\right]}{\left\{\frac{1}{2}\alpha_0\right\}^2}(\Delta l)^2\,. \tag{74}$$

If the incremenl size Δl is taken sufficiently small compared with 2π, and $f(t)$ is smooth to some extent, then the systematic componenis of (72) and (74) will be negligibly small compared with the stochastic component in equation (72) even though k is not so large. Hence we may write as follows,

$$(\sigma^{**})^2 \doteqdot W\,, \tag{75}$$

which is approximately equal to $2(\sigma^*)^2$ and will be an estimate of the variance of a sample of size Δl taken from the same lot in its thoroughly mixed state.

Thus $\xi(t,\ \Delta l,\ \omega)$, $\xi\left(t+\frac{2\pi}{r},\ \Delta l,\ \omega\right)$, $\xi\left(t+2\frac{2\pi}{2},\ \Delta l,\ \omega\right)$, ... , $\xi\left(t+\frac{r-1}{r}2\pi,\ \Delta l,\ \omega\right)$

may be considered approximately as the normal multivariate distribution, with mean value 0 and variance-covariance matrix

$$\begin{pmatrix} 2(\sigma^*)^2 & & & 0 \\ & 2(\sigma^*)^2 & & \\ & & \ddots & \\ 0 & & & 2(\sigma^*)^2 \end{pmatrix} \tag{76}$$

Next we put

$$\eta(t,\ \Delta l,\ \omega)=\zeta(t+\Delta l,\ \Delta l,\ \omega)+\zeta(t,\ \Delta l,\ \omega)-2\bar{\zeta}(\omega) \tag{77}$$

Then $\eta(t, \varDelta l, \omega)$, $\eta\left(t+\frac{2\pi}{r}, \varDelta l, \omega\right)$, $\eta\left(t+2\frac{2\pi}{r}, \varDelta l, \omega\right)$, ... , $\eta\left(t+\frac{r-1}{r}2\pi, \varDelta l, \omega\right)$ may be considered as r random variables with mean $m^{***}=2m^{*}$ and variance-covariance matrix

$$\begin{pmatrix} v_0^{***} & v_1^{***} & \cdots & \cdots & v_{r-1}^{***} \\ v_1^{***} & v_0^{***} & v_1^{***} & & \vdots \\ \vdots & v_1^{***} & v_0^{***} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & v_1^{***} \\ v_{r-1}^{***} & \cdots & \cdots & v_1^{***} & v_0^{***} \end{pmatrix} \tag{78}$$

where

$$v_0^{***}=\left\{1-\frac{\varDelta l}{\pi}\right\}W+\bar{C}^2\frac{\left[\sum_{n=1}^{\infty}(\alpha_{2kn}^2+\beta_{2kn}^2)\{1+\cos(n\varDelta l)\}\xi^2(n\varDelta l)\right]}{\left(\frac{1}{2}\alpha_0\right)^2}(\varDelta l)^2 \tag{79}$$

$$v_p^{***}=-\frac{\varDelta l}{\pi}W+\bar{C}^2\frac{\left[\sum_{n=1}^{\infty}(\alpha_{2kn}^2+\beta_{2kn}^2)\cos\left\{pn\frac{2\pi}{r}\right\}\{1+\cos(n\varDelta l)\}\xi^2(n\varDelta l)\right]}{\left(\frac{1}{2}\alpha_0\right)^2}(\varDelta l)^2 \tag{80}$$

Next, we put

$$\{S_\eta(\varDelta l, \omega)\}^2=\frac{1}{r-1}\left[\{\eta(t, \varDelta l, \omega)-\bar{\eta}(\varDelta l, \omega)\}^2+\left\{\eta\left(t+\frac{2\pi}{r}, \varDelta l, \omega\right)-\bar{\eta}(\varDelta l, \omega)\right\}^2+\right.$$
$$\left.\cdots\cdots\cdots+\left\{\eta\left(t+\frac{r-1}{r}, \varDelta l, \omega\right)-\bar{\eta}(\varDelta l, \omega)\right\}\right]^2 \tag{81}$$

where

$$\bar{\eta}(\varDelta l, \omega)=\frac{\left\{\eta(t, \varDelta l, \omega)+\eta\left(t+\frac{2\pi}{r}, \varDelta l, \omega\right)+\cdots\cdots+\eta\left(t+\frac{r-1}{r}2\pi, \varDelta l, \omega\right)\right\}}{r} \tag{82}$$

Then

$$\underset{\omega}{E}\{S_\eta(\varDelta l, \omega)\}^2=v_0^{***}-\frac{2\{(r-1)v_1^{***}+(r-2)v_2^{***}+\cdots\cdots+v_{r-1}^{***}\}}{r(r-1)}$$

$$=W+\bar{C}^2\frac{\sum_{n=1}^{\infty}(\alpha_{2kn}^2+\beta_{2kn}^2)\{1+\cos(n\varDelta l)\}\xi^2(n\varDelta l)}{\left(\frac{1}{2}\alpha_0\right)^2}(\varDelta l)^2$$

$$-\bar{C}^2\frac{\left[\sum_{n=1}^{\infty}(\alpha_{2kn}^2+\beta_{2kn}^2)\sum_{p=1}^{r-1}(r-p)\cos\left\{pn\frac{2\pi}{r}\right\}\{1+\cos(n\varDelta l)\}\xi^2(n\varDelta l)\right]}{r(r-1)\left(\frac{1}{2}\alpha_0\right)^2}(\varDelta l)^2 \tag{83}$$

The expected circular serial covariance between $\xi\left(t+p\frac{2\pi}{r},\ \Delta l,\ \omega\right)$ and $\eta\left(t+q\frac{2\pi}{r},\ \Delta l,\ \omega\right)$ is calculated to be

$$\left[\varphi\left\{(p-q)\frac{2\pi}{r}+\Delta l,\ \Delta l\right\}-\varphi\left\{(p-q)\frac{2\pi}{r}-\Delta l,\ \Delta l\right\}\right]W$$

$$-\bar{C}^2\frac{\left[\sum_{n=1}^{\infty}(\alpha_{2kn}^2+\beta_{2kn}^2)\sin n(p-q)\frac{2\pi}{r}\sin(n\Delta l)\,\xi^2(n\Delta l)\right]}{\left\{\frac{1}{2}\alpha_0\right\}^2}(\Delta l)^2\,. \qquad (84)$$

From the variance-covariance matrix (76) we may assume that $\xi(t,\ \Delta l,\ \omega)$, $\xi\left(t+\frac{2\pi}{r},\ \Delta l,\ \omega\right)$, $\xi\left(t+2\frac{2\pi}{r},\ \Delta l,\ \omega\right)$, ..., $\xi\left(t+\frac{r-1}{r}2\pi,\ \Delta l,\ \omega\right)$ are r mutually independent random variables with normal distribution $N\{0,\ 2(\sigma^*)^2\}$ if we take Δl sufficiently small compared with 2π. Moreover, from the relation (84) we may assume that $\xi(t,\ \Delta l,\ \omega)$ and $\eta(s,\ \Delta l,\ \omega)$, $(-\pi\leqq t\leqq\pi,\ -\pi\leqq s\leqq\pi)$ are mutually independent random variables for a sufficiently small increment size Δl.

Thus we can make clear the relations between the "within pairs" variance and the "between pairs" variance in the analysis of variance of the sample data, which is composed of r pairs of measurements.

It is to be noted that the "within pairs" variance calculated here may be treated as if it were the variance between the interpenetrating samples taken from the lot in its thoroughly mixed state.

By the analysis of variance thus made, we can compare the degree of homogeneity of the material the state of which has been attained by the $2k$-fold ordinary bedding operation with the homogeneity of the material in its thoroughly mixed state. Morerover, we can measure the precision of the estimate calculated from such sample data. When the lot in question is mixed fairly well by a bedding operation, the magnitude of the systematic component in (83) will be negligibly small compared with that of the stochastic component in the same equation. Hence the value of F calculated from the analysis of variance data may be considered to be distributed approximately according to the Snedecor's F-distribution with the corresponding degrees of freedom.

From the results of significance tests we will have one of the following two results. (a) If we may assume that the r pairs of the interpenetrating samples are randomly drawn from a normal population, then we can obtain the more reliable estimate of the lot mean by taking a pooled mean of the r pairs of measurements. The corresponding confidence limits will be easily calculated in usual way. (b) If we cannot assume that r pairs of the interpenetrating samples are randomly drawn from a normal population, then we must derive the confidence limits for the pooled mean of r pairs of measurements from the decreasing tendency curve of the variability between the contents in interpenetrating samples, by devising some kind of extrapolation method. Such decreasing tendency curve of the variability may be divided into two parts, i. e. the systematic part and the stochastic part.

As was pointed out before, the systematic part will be given by the following approximation formula,

$$\sigma^2_{\mathfrak{u}\cdot\mathfrak{d}(k)} \doteqdot c\,\frac{1}{k^2} \qquad (c \text{ is a positive constant})$$

for sufficiently large value of k, if the $\lambda'(t)$ representing the main trend in quality level of the original material flow satisfies a relation such as (12).

And the stochastic part will be given by the following approximation formula,

$$\sigma^2_{\mathfrak{u}\cdot\mathfrak{d}(k)} \doteqdot d\,\frac{1}{k} \qquad (d \text{ is a positive constant}).$$

We must estimate the confidence limits for the pooled mean of r pairs of measurements, taking into consideration those decreasing tendency curves of the two variance components.

The author has discussed so far the theory of systematic sampling and bedding methods of bulk material, establishing a mathematical model from the microscopic point of view. However, it is to be noted that there is a gap to be filled up between a theoretical model and practical data. We have treated the problem of taking samples from bulk material, the method of grinding and subdivision of samples to obtain material suitable for analysis, the necessity of introducing the bedding method to attain uniformity in the quality of bulk material, the relations between the systematic sampling and bedding methods, and the like. But the author has discussed these problems, mainly taking into consideration the physical composition of bulk material.

In order to treat the practical data more successfully, we must also pay attention to the chemical aspects of the treatment of bulk material. The grinding and subdivision of samples to obtain material suitable in size for analysis involves what is in reality a series of further sampling operations, and the error involved in these steps will be additional to the error incurred in the original sampling. Besides, the error due to the chemical changes of bulk material may be added to these errors during the grinding and subdivision operations. For example, the chemical composition of bulk material such as a sulphide may be changed during the grinding and subdivision operation of samples, owing to the increased area of the exposured surface to the air of particles in bulk material. To present such bulk material from the chemical changes during the grinding and subdivision processes, chemical engineers have devised many preventive measures and investigated into the effects of such preventive measures by planning the design of experiments. We have many valuable reports discussing the effects of such preventive measures, using the analysis of variance method. We can see also many articles discussing the measurement error due to the differences in conditions in the chemical analysis from the statistical point of view.

Some modifications must be introduced into our mathematical model, if we must pay attention to the additional error due to the differences in conditions in the chemical analysis of bulk material at the same time. But to achieve this purpose, we must set up a theoretical model which is too complicated for mathematical analysis. So the author has discussed the sampling problems of bulk material, narrowing them down to the study of the physical composition of bulk material. In order to treat the practical data successfully, we must investigate into our subject from various standpoints, putting together the results derived from various methods of study.

VIII. Randomness and operation of statistical control

The concept of the operation of statistical control was suggested for the first time by W. A. Shewhart in his excellent book, "Statistical Method from the View-point of Quality Control." He completely investigated into the nature of randomness, in connection with the operation of statistical control. It is very difficult to summarize his deep thought in this limited space. The idea of quality control developed by him has many fruitful results and cannot be fully expressed by any single idea.

Therefore, the author does not intend to summarize Dr. Shewhart's thought in this paper. The author would like to discuss the role of the operation of statistical control in connection with the randomization problems in the systematic sampling and bedding methods of bulk material. In order to be able to apply the theory of probability to these sampling problems, it is usually assumed that the method of sampling is random or the state of the population is randomness in itself. But the systematic sampling cannot be treated as a completely randomized operation, though some kind of randomization operation such as a random start method is introduced into the systematic sampling method. On the other hand, the state of the population such as bulk material cannot be regarded as randomness in itself, without introducing some kind of mixing operation. In other words, the quality of bulk material is not generally in the state of statistical control, without introducing the operation of statistical control.

In the customary application of statistical theory, we assume that we are dealing with a physical state that gives samples showing the characteristics of randomness. According to Dr. Shewhart, "Control studies have shown that such physical states of statistical control are indeed rare natural occurences, at least in physics and engineering, and furthermore that they cannot usually be brought about without the operation of statistical control, wherein comparatively large numbers of preliminary data are taken in the process of detecting and removing assignable causes of variability. Besides, the statistician has learned by experience that the random effects do not just happen, even by careful planning."

It is to be noted here that the statistician's prediction will not be valid, if certain assumptions about the cause system are not justified. Particularly, if the inferences is to be made purely with the help of the distribution theories of statistics, the experiments that provide the evidence for the inference must arise from a state of staistical control.

As was pointed out before, the population such as bulk material is not usually in a state of statistical control in itself, and the method of the systematic sampling also cannot be regarded as a completely randomized operation. However, we have known by experience that the relation between the method of systematic sampling and the characteristic of the population under consideration will gradually be brought to such a state as to ensure randomness, as the number of increments in a sample is increased. In other words, there is a mixing mechanism in the systematic sampling of bulk material, in spite of the fact that both the sampling method and the state of bulk material are not completely random but "partially random". As was stated in the introduction, there is similarity between the physical composition of the sample taken from material piled by the bedding method and that of the sample composed of a series of increments systematically taken from material flow on a conveyor. With this similarity in mind, we have studied the theory of systematic sampling.

But we have assumed that the time series representing the initial state of material flow has been already smoothed to some extent and locally randomized, the distribution of the particles of each category in the material flow being subject to an existing stochastic model such as a Poisson process. Important and natural as those methods of approach are for our subject, they cannot give insight into the true origin of laws of probability. Although the gap between a theoretical model and practical data has been filled up to a considerable degree, our method of study has been based on an existing stochastic model. So we must make searching inquiry into the mixing mechanism of the bedding method, without introducing the existing stochastic model into our theoretical structure.

As is well known, if the bedding method is applied to the bulk material, a considerable amount of mixing would result and some degree of uniformity of quality would be achieved, whatever the initial sate of material may be. This has been proved under the slightly restricted conditions in the previous sections.

The bedding method may be regarded as a special type of the operation of statistical control. We must pay attention here to the fact that there are various kinds of operation of statistical control such as drawing a sample with a replacement from a bowl, repeating an observation under the same essential conditions, going as far as one can go in the process of controlling quality by finding and removing causes of variability, and the like.

Among the various kinds of operation of statistical control, the bowl experiment is an idealized experiment representing the physical state of statistical control.

According to Dr. Shewhart, the concept of a physical state of statistical control as illustrated by the example of drawing chips from a bowl appears to be much the same as the concept of doing something "physically random". We have known by experience that in case of the bowl experiment, probability theory is usually applicable, and the differences between samples drawn under such conditions are predictable in a probability sense.

Dr. Shewhart says, "My own experience indicates that this situation does not hold, in general, for fluctuations in measurements arising under conditions merely judged to remain essentially the same".

The engineer does not deal with the ideal experiment such as drawing chips from a bowl, but he deals with measurements of one kind or another. The engineer makes efforts to attain a physical state of control of such measurements, trying to control all of the causes of variability until he has attained a state where the conditions remain "essentially the same". In the early stages of any attempt at control of a quality characteristic, assignable causes are always present even though the production operation has been repeated under presumably the same essential conditions. As these assignable causes are found and eliminated, the variation in quality gradually approaches a state of statistical control.

If we can attain such a state of statistical control, how can we know when the production process is in such a state of control? The concept of a state of statistical control is a basis for describing the engineering goal of uniform quality, and the operation of statistical control is a means of approaching this goal. Thus there remains the problem of judging how closely we have approached the goal. In order to solve this problem, we must grasp objectively the characteristics common to the physical state of statistical control such as sampling from a bowl (a typical model of randomness) and to any physical state of statistical control of some production process.

For this purpose, we must try to find out some abstract way of describing the causal relation between the physical aspects of a given state of control and the quantitative aspects of the data obtainable under such a state of control.

There are various kinds of experiments which are supposed to be in the physical state of statistical control, such as the repeated turning of a roulette wheel, the tossing of a coin or die, the drawing of chips from a bowl, the act of repeating an observation under the same essential condition, the act of going as far as one can go in the process of controlling quality by finding and removing causes of variability, and the like. We must investigate into the characteristics common to various kinds of physical state of statistical control.

We are familiar with certain erperimental phenomena called "random" which are connected with the repeated turning of a roulette wheel, the tossing of a coin or die, Buffon's needle experiment, and the like. It is very important for us to explain theoretically these experimental phenomena. Poincare made valuable contributions in this direction for the first time. Since he gave those fundamental remarks, a new branch of the theory of probability — the method of arbitrary functions — has been developed. Suppose that a roulette is spun a large number of times. The frequency with which the wheel comes to lie within a small sector $d\varphi$ will be represented by a certain function $f(\varphi)$. The large number of alternatively red and black sectors then makes the frequencies of red and black approximately equal, independent of $f(\varphi)$. By a more detailed study of the roulette we found, moreover, that after sufficiently rapid spinning even the individual sectors appear with nearly equal frequency.

By using the method of arbitrary functions, many beautiful reuslts have been derived by Hadamard, Hostinsky, v. Mises, and others. In 1918, the Polish physicist M. V. Smoluchowsky pointed out how accurately the concept of chance may be defined and how naturally the fundamental laws of probability may be derived, once frequency phenomena are recognized as produced by strictly causal mechanism. In 1934, the German mathematician E. Hopf discussed the true origin of the laws of probability more systematically in his paper, " On Causality, statistics and probability." In 1952, the Russian mathematician A. J. Khintchin discussed this problem from the standpoint of " materialism ", by using the method of arbitrary functions.

The author wants to discuss in the following sections the mixing mechanism of the bedding method of bulk material, by using the method of arbitrary functions. However, it is to be noted that we are treating only a special type of the operation of statistical control suggested by Dr. Shewhart. There are many problems to be solved in his idea of the operation of statistical control. In order to solve these problems mathematically, we must introduce into our method of study various mathematical tools, such as the ergodic theory, theory of stochastic processes, the mathematical theory of feedback control and the like.

IX. Method of arbitrary functions and mixing mechanism of bedding method

Let us suppose that a coin is dropped from a certain height above the floor. Its final position on the floor may be considered as a definite function of the initial phase (position and velocity). Slight changes of the initial phase will bring about a quite different result. We can consider here two possible events. Namely, the phase space will be divided into two parts H (head up), T (tail up) which have about the same measure within all (not too small) regions.

If the coin is dropped repeatedly a large number of times and if we describe the different initial phases by a continuous distribution, the relative frequencies of H and T will be nearly equal, independent of the function representing the initial conditions. Thus we can expect that the relative frequency will be approximately equal within most sequences.

In general, we consider a conservative mechanism, start repeatedly with certain phase P, and observe how often, after a lapse of time t, the point $P_t = T_t(P)$ comes to lie into a definite part A of the phase space G.

Suppose that we make a continuous instead of countable number of experiments, described by a distribution function $f(P)$.
Then,

$$\int_B f(P)\,dv = \int_G f(P)\,\varphi_B(P)\,dv = (f, \varphi_B) \tag{85}$$

denotes the number of times with which we start within region B, φ_B being the indicator of B,

$$\varphi_B(P) = \begin{cases} 1 & \text{when} \quad P \in B\,, \\ 0 & \text{when} \quad P \in G - B\,. \end{cases} \tag{86}$$

Here $f(P)$ is supposed to be nowhere negative and summable over G, the total number of all experiments being finite and positive. The relative number of experiments tor which, after a lapse of time t, the event A occurs, is

$$\frac{(f, \varphi_{A-t})}{(f, 1)}, \tag{87}$$

where A_{-t} denotes the set of all points of G which come to lie into A after a lapse of time t.
From the invariance property of the measure (conservative mechanism), the fraction (87) equals

$$\frac{(f_{-t}, \varphi_A)}{(f, 1)}, \quad \text{where } f_t(P) = f\{T_t(P)\}\,. \tag{88}$$

In the case of the (conservative) roulette problem, A being the event "red sectors", we should expect this quotient to tend towards one half as $t \to \infty$ independent of the way of turning the roulette wheel, i. e. independent of the distribution $f(P)$ of the initial phases.

According to E. Hopf, we have the next definition.

Definition. An event A is **statistically regular** with respect to a given conservative mechanism if, for any non-negative and summable function $f(P)$, the quotient (87) tends towards the same limit $W(A)$ as $t \to \infty$,

$$(f, \varphi_{A-t}) \to W(A)(f, 1)\,. \tag{89}$$

$W(A)$ is called the relative frequency of the event A. When f is indicator φ_B of a point set B, the statistical regularity of A implies that

$$\frac{\mathfrak{M}(BA_{-t})}{\mathfrak{M}(B)} = \frac{\mathfrak{M}(B_t\, A)}{\mathfrak{M}(B)} \to W(A) \tag{90}$$

as $t \to \infty$, independent of B.

Conversely, if (90) holds for any set B of positive measure, the event A is statistically regular in the above sense.

(It is to be noted here that the results obtained under the assumption of the conservative mechanism can be extended to the case of the dissipative mechanism.)

Now we will study the mixing mechanism of the bedding method of bulk material from the theoretical point mentioned above. We will first examine the usual method known as "layering" or "bedding". In this type of pile, the coal or iron ore is spread in thin layers, probably only 4 or 5 in. thick (in the case of coal), and over as great an area as possible. In using coal or iron ore from such a pile vertical slices would be taken from the top of the pile down to the ground.

This mixing mechanism of the bedding method is somewhat like the process that is employed by the baker in making puff pastry. The mixing operation employed by the baker is a repetition of a single operation T. We will formulate this operation as follows.

Let Q: $0 \leqq x < 1$, $0 \leqq y < 1$ be the unit square in the (x, y)-plane.
The transformation

$$T_1: \ x' = kx, \quad y' = \frac{1}{k} y, \quad k \text{ being a given integer} \geqq 2,$$

mappes Q on the rectangle

$$0 \leqq x' < k, \quad 0 \leqq y' < \frac{1}{k}.$$

This rectangle may be cut into k rectangles

$$\nu - 1 \leqq x' < \nu, \quad 0 \leqq y' < \frac{1}{k}: \quad \nu = 1, 2, \ldots\ldots, k. \tag{91}$$

The second transformation T_2 consists in shifting these k rectangles parrallel to themselves (the operation of turning about the angle π may be also adopted-this corresponds to the switch-back bedding operation) until they fill up Q again. The discontinuous transformation $T = T_2 T_1$ transforms Q into itself in a one to one manner. T preserves the ordinary plane measure

$$\mathfrak{M}(A) = \int_A dx\, dy, \quad \mathfrak{M}\{T(A)\} = \mathfrak{M}(A).$$

In making puff pastry, the baker adopts the following mixing method : a lump of butter is wrapped up into a dough: then the whole mass is rolled out and folded together. The operation is repeated several times, rolling out (T_1) and folding together (T_2), thus mixing dough and butter. Hence both dough and butter will be distributed in very thin layers. We can learn by intuition that Q will be mixed completely if we repeat the above operation infinitely.

The above mixture property means that

$$\lim_{n=\infty} \mathfrak{M}\{AT^n(B)\} = \mathfrak{M}(A)\,\mathfrak{M}(B) \tag{92}$$

holds for any two measurable point sets A, B, where $A \subset G$, $B \subset G$.

Hence we must prove that the relation (92) holds, in order to give a proof of the mixture property. The author will not repeat this proof, because we can see it in the E. Hopf's paper.

It is interesting for us to compare this mixture property of puff pastry with the mixing mechanism of the bedding operation of bulk material. But we must pay attention to the fact that we can not treat the tremendous amount of bulk material like the puff pastry. Moreover, it can not always be justified economically to repeat the operation such as $T=T_1 T_2$ a large number of times. In order to mix the vast amount of bulk material fairly well, it is necessary to elaborate a plan carrying out our purpose effectively.

We must have an eye upon the fact that bulk material in transit is easy to deal with and can be locally mixed on a small scale by a properly designed mixer. If such locally randomized material flow is spread on the storage yard like a large number of parallel noodles, then each sublot composed of the slices vertically cut off from the noodle like piles of material will be homogeneous to some extent. Besides, such a sublot will be more homogeneous, if we mix again the slices in tne same way as above before we feed the sublot composed of the slices into the manufacturing process. If we apply such a bedding method to the bulk material in transit two or three times, the material will be considerably homogeneous.

It should be noted here that the material is locally randomized not only by the properly designed mixer, but also by the grinding operation or the other spontaneous mixing mechanisms in transit. So we may consider that the bedding operation will bring about a considerable degree of homogeneity to bulk material, even if not mixed intentionally by a mixer.

We will discuss here the mixing mechanism of bedding operation by using the method of arbitrary functions. To begin with, we will direct our attention only to the pieces of a given category (such as piece size and content) found in material flow on the conveyor belt. We can express the positions of these particles in the material flow on the conveyor belt as a set of points on a finite interval such as $[0, 2\pi]$. Let the coordinates of such points be $x_1, x_2, \cdots\cdots, x_k$. If the original material flow on the first conveyor belt is carried to the next conveyor belt through a properly designed mixer, then the coordinates of the points $x_1, x_2, \ldots\ldots, x_k$ will be shifted respectively to the coordinates $y_1, y_2, \ldots\ldots, y_k$ on the interval $[0, 2\pi]$ corresponding to the positions of the particles in the material flow on the second conveyor belt. The displacements of the positions of the particles may be expressed as follows,

$$T(\mathfrak{X}) = \mathfrak{Y}$$

where $\mathfrak{X}=(x_1, x_2, \ldots\ldots, x_k)$, $\mathfrak{Y}=(y_1, y_2, \ldots\ldots, y_k)$ are two points in k-dimmensional space.

Let $f(v_1, v_2, \ldots\ldots, v_k)$ be a distribution function of $\mathfrak{Y}$ in k-dimmensional space, where

$$\int_G f(v_1, v_2, \ldots\ldots, v_k)\, dv_1\, dv_2 \ldots\ldots dv_k = 1\ .$$

The distribution function $f(v_1, v_2, \ldots\ldots, v_k)$ may be regarded as a kind of arbitrary function, which is determined by the initial positions and the displacements of the particles in the material flow in transit. The mathematical form of this arbitrary function is essentially influenced by the efficiency of the mixer. If we adopt a well designed mixer, this function may be expressed by a well known probability density function such as normal distribution function. But in general, it is not natural to express this arbitrary function by a well known probability density function. In this case, we must introduce some generalized assumptions into the mathematical form of the arbitrary function corresponding to the degree of mixing, which need a comparatively large number of parallel noodles to attain a certain degree of uniformity of bulk material by the bedding operation. It is to be noted here that we can reduce considerably the number of parallel noodles to attain the same degree of uniformity of bulk material, if we adopt a well designed mixer together with the bedding method.

Now we will discuss the mixing mechanism of the bedding method applied to the locally randomized material flow. We can imagine the following mixing process, compared to the mixing process of puff pastry adopted by the baker.

The whole lot of bulk material may be considered to be spread out on the ground like a long noodle by a conveyor belt, while the material being locally mixed by a mixer. Then the long noodle may be considered to be cut into m noodles of equal length. And these m noodles may be considered to be shifted parallel to themselves and piled up in the storage yard (the operation of turning about the angle π may be also adopted—This corresponds to the switch-back bedding operation). Then we use the sublot composed of the slices vertically cut off from the noodle like piles of material.

Now, we put

$$g(y_1, y_2, \ldots\ldots, y_k) = 1 \qquad \text{if} \quad 0 \leqq y_l \leqq \xi_l \ (l = 1, 2, \ldots\ldots, k),$$
$$\text{where} \quad 0 \leqq \xi_l \leqq 2\pi\ , \tag{93}$$
$$= 0 \qquad \text{otherwise}.$$

The Fourier coefficients of the k-dimmensional box function $g(y_1, y_2, \ldots\ldots, y_k)$ are calculated as folows,

$$C_{n_1, n_2, \cdots, n_k} = \frac{1}{(2\pi)^k} \int_0^{2\pi} \cdots \int_0^{2\pi} g(y_1, y_2, \cdots, y_k)\, e^{-i(n_1y_1 + n_2y_2 + \cdots + n_ky_k)} dy_1 dy_2 \cdots dy_k$$
$$= C_{n_1} C_{n_2} \cdots C_{n_k} \tag{94}$$

where

$$C_{n_l} = \frac{1}{2\pi} \int_0^{\xi_l} e^{-in_ly_l} dy_l = \frac{1 - e^{-in_l\xi_l}}{2\pi i n_l} \tag{95}$$

Hence we may write the Fourier series, with respect to this system, in the following form,

$$h(v_1, v_2, \ldots\ldots, v_k)$$

$$=\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\ldots\ldots\sum_{n_k=-\infty}^{\infty} C_{n_1} C_{n_2}\ldots\ldots C_{n_k}\, e^{i(n_1v_1+n_2v_2+\cdots\cdots+n_kv_k)} \tag{96}$$

Then the distribution function $F(\xi_1, \xi_2, \ldots\ldots, \xi_k)$ which represents a kind of probability distribution of the positions of the particles found in the material flow generated by the m-fold ordinary bedding operation can be expressed (by using the property of boundedly convergence of the above Fourier series) as follows,

$$F(\xi_1, \xi_2, \ldots\ldots, \xi_k)=P_r(y_1\leqq\xi_1, y_2\leqq\xi_2, \ldots\ldots, y_k\leqq\xi_k)$$

$$=\underbrace{\int_{-\infty}^{\infty}\ldots\ldots\int_{-\infty}^{\infty}}_{(k)}\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\ldots\ldots\sum_{n_k=-\infty}^{\infty} C_{n_1}\, C_{n_2}\ldots\ldots C_{n_k}$$

$$\times e^{i(n_1v_1+n_2v_2+\cdots+n_kv_k)\,m} f(v_1, v_2, \ldots\ldots, v_k)\, dv_1dv_2\ldots\ldots dv_k$$

$$=\sum_{n_1=-\infty}^{\infty}\ldots\ldots\sum_{n_k=-\infty}^{\infty} C_{n_1}\, C_{n_2}\ldots\ldots C_{n_k}$$

$$\times\underbrace{\int_{-\infty}^{\infty}\ldots\ldots\int_{-\infty}^{\infty}}_{(k)} e^{(n_1v_1+n_2v_2+\cdots+n_kv_k)\,m} f(v_1, v_2, \ldots\ldots, v_k)\, dv_1dv_2\ldots\ldots dv_k\,, \tag{97}$$

where

$$0\leqq\xi_l\leqq2\pi.$$

From the above relation we can easily derive the next result,

$$F(\xi_1, \xi_2, \ldots\ldots, \xi_k)$$

$$=\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\ldots\ldots\sum_{n_k=-\infty}^{\infty} C_{n_1}C_{n_2}\ldots\ldots C_{n_k}\, e^{i\sum_{l=1}^{k} x_l n_l m}\, \varphi(n_1m, n_2m, \ldots\ldots, n_km)$$

$$=\left\{\frac{\xi_1\xi_2\ldots\ldots\xi_k}{(2\pi)^k}\right\}+\sum_{n_1}{}'\sum_{n_2}{}'\ldots\ldots\sum_{n_k}{}' C_{n_1}\, C_{n_2}\ldots\ldots C_{n_k}\, e^{i\sum_{l=1}^{k} x_l n_l m}\, \varphi(n_1m, n_2m, \ldots\ldots, n_km), \tag{98}$$

where

$$\varphi(n_1m, n_2m, \ldots\ldots, n_km)$$

$$=\underbrace{\int_{-\infty}^{\infty}\ldots\ldots\int_{-\infty}^{\infty}}_{(k)} e^{i\{n_1(v_1-x_1)+n_2(v_2-x_2)+\cdots+n_k(v_k-x_k)\}\,m} f(v_1, v_2, \ldots\ldots, v_k)\, dv_1dv_2\ldots\ldots dv_k \tag{99}$$

and $\sum_{n_k}'$ means $\sum_{n_k=-\infty}^{-1} + \sum_{n_k=1}^{\infty}$ symbolically.

It is to be noted here that $\varphi(n_1m, n_2m, \ldots\ldots, n_km)$ may be considered also as the characteristic function of the distribution function $f(v_1, v_2, \ldots\ldots, v_k)$ in k-dimensional space G. It is easily proved from the properties of the characteristic function $\varphi(n_1m, n_2m, \ldots\ldots, n_km)$ of the distribution function $f(v_1, v_2, \ldots\ldots, v_k) \in L_1$ in k-dimensional space G that as $m \to \infty$,

$$\sum_{n_1}' \sum_{n_2}' \ldots\ldots \sum_{n_k}' C_{n_1} C_{n_2} \ldots\ldots C_{n_k} e^{i \sum_{l=1}^{k} x_l n_l m} \varphi(n_1m, n_2m, \ldots\ldots, n_km) \to 0. \quad (100)$$

Hence we may say that as $m \to \infty$,

$$F(\xi_1, \xi_2, \ldots\ldots, \xi_k) \to \left\{ \frac{\xi_1 \xi_2 \ldots\ldots \xi_k}{(2\pi)^k} \right\}. \quad (101)$$

By extending the method of proof adopted in case of mixing puff pastry, the above result may be proved easily from the fact that the mixing mechanism of the bedding method applied to the locally randomized material flow satisfies the condition (90) for the statistical regularity of the event

$$A = \{y_1, y_2, \ldots\ldots, y_k;\ 0 \leqq y_1 \leqq \xi_1,\ 0 \leqq y_2 \leqq \xi_2, \ldots\ldots,\ 0 \leqq y_k \leqq \xi_k\}$$

as $m \to \infty$. Thus, we come to the conclusion that the position of a system of finite particles in the material flow may be considered as a realization of the uniformly distributed independent random variables in the interval $[0, 2\pi]$, when the material is thoroughly mixed (i. e. $m \to \infty$) by the bedding method. This situation corresponds to the fact stated in the former sections, i. e. the fact that the Poisson process with the mean value function $\lambda(t)$ tends to the temporally homogeneous Poisson process with the mean value function $\bar{\lambda}'(t-a)$ by the bedding operation (when $m \to \infty$).

By using Dr. Shewhart's words, we may say that this situation in limit represents "a statistical state constituting a limit to which we may hope to go in improving the uniformity of quality". In this case, the mixing operation by the bedding method is an operation of statistical control.

In practice, we cannot take m (the number of noodles in the bedding method) so large. So we must investigate into the asymptotic behavior of the mixing mechanism of the bedding method with a properly desgined mixer. In order to assure a certain degree of uniformity of bulk material, the number of noodles needed in the bedding method depends on the scale and efficiency of the local mixing operation by a mixer. If the scale and efficiency of the local mixing operation is large, we can reduce considerably the number of noodles adopted in the bedding method. Here we can investigate into the asymptotic behavior of the mixing mechanism of bedding operation under tne fairly general conditions, by studying the properties of the function $\varphi(n_1m, n_2m, \ldots\ldots, n_km)$.

It should be noted that the properties of the function $\varphi(n_1m, n_2m, \ldots\ldots, n_km)$ depends upon the distribution function $f(v_1, v_2, \ldots\ldots, v_k)$ which represents the scale and efficiency of the mixing operation by the mixer. When the mixer is well designed, we may put

$$f(v_1, v_2, \ldots\ldots, v_k) = \frac{1}{(2\pi)^{k/2}|V|^{1/2}} e^{-\frac{1}{2}\{V^{-1}(\mathfrak{V}-\mathfrak{X})(\mathfrak{V}-\mathfrak{X})\}} \tag{102}$$

approximately, where V denotes a variance-covariance matrix in normal multivariate distribution, and

$$\mathfrak{V}=(v_1, v_2, \ldots\ldots\ldots\ldots\ldots, v_k),$$
$$\mathfrak{X}=(x_1, x_2, \ldots\ldots\ldots\ldots\ldots, x_k),$$
$$\mathfrak{N}=(n_1, n_2, \ldots\ldots\ldots\ldots\ldots, n_k),$$

Then we can easily calculate

$$\varphi(n_1m, n_2m, \ldots\ldots, n_km) = e^{i(\mathfrak{N}, \mathfrak{X})m - \frac{1}{2}\{V\mathfrak{N}, \mathfrak{N}\}m^2} \tag{103}$$

Hence

$$F(\xi_1, \xi_2, \ldots\ldots, \xi_k)$$
$$= \left\{\frac{\xi_1 \xi_2 \cdots\cdots \xi_k}{(2\pi)^k}\right\} + \sum_{n_1}{}' \ldots\ldots \sum_{n_k}{}' Cn_1 Cn_2 \ldots\ldots Cn_k\, e^{i\sum_{l=1}^{k} x_l n_l m - \frac{1}{2}\{\sum_{l=1}^{k}\sum_{s=1}^{k} \rho_{ls}\sigma_l\sigma_s n_l n_s\} m^2} \tag{104}$$

From the above relation we can investigate into the asymptotic behavior of the mixing mechanism of the bedding method with a well-designed mixer. When we mix locally a material flow in transit, it will be very effective to apply a small scale bedding operation together with a small mixer. In such a case, we can study the mixing mechanism more completely, by using the result obtained from the above multivariate normal distribution. It may well be concluded that it is very difficult to mix particles having high correlations into a homogeneous state, while it is comparatively easy to mix particles having low correlation into a homogeneous state.

It may easily be assumed that particles stuck fast together such as the powdered coal, have a high positive correlation coefficient ρ_{ij}, and that a congregation of big particles, such as lump coal, have a comparatively low correlation coefficient ρ_{ij}. At the same time, we must pay attention to the fact that the small particles are often much alike in their chemical composition, because they are available from one and the same lump at the time of grinding operation in most cases. Besides, such particles of small size have not a powerful effect on the total variance. Thus, the time series repesenting the chemical contents in the continuous flow composed of such small particles as powdered coal may well be considered as a kind of stationary stochastic process. Hence, we may say that any inital chaotic state of material will finally be brought to a state of material flow which is described by a Gaussian process by the bedding operation with a mixer, going through a state of material flow which is described by a kind of stationary stochastic process. In any case, it may be concluded that small scale mixing operation by bedding method will bring satisfactory results, if this is used together with the large scale bedding operation.

X. Concluding remarks

The author has made a study of the systematic sampling and bedding methods of bulk material, in co-operation with the technical staffs of quality control, Hirohata Iron Works, Fuji Iron and Steel Co.. There were many problems to be solved from the statistical point of view at the iron works, and the author was particularly interested in the subject of this paper, in relation to the quality control of iron ore and coking coal handled on an extentive scale.

A part of the results was published in the book, "The progress of the theory of stochastic inference" (in Japanese), edited by Dr. T. Kitagawa. Another part was published in the author's report delivered at the 32nd Session of the International Statistical Institute.

In this paper, the author has tried to discuss this subject in its broader sense. The author would like to express his thanks to Professor Tosio Kitagawa and Professor Ziro Yamauti for their valuable suggestions, and the technical staffs of quality control, Hirohata Iron Works, Fuji Iron and Steel Co., who have given help and advice in the development of this study in the past several years.

References

1) E. S. Pearson: "Sampling Problems in Industry". Supplement to the Journal of the Royal Statistical Society, Vol.1, No. 2, (1934).
2) A. T. Mckay: "Sampling from Batches". Supplement to the Journal of the Royal Statistical Society, Vol. 1, No. 2, (1934).
3) E. G. Baily: "Accuracy in Sampling Coal". Ind. Eng. Chem., 1 (1909).
4) E. S. Grumell, and others: "Report on the Sampling of Coal, With Special Reference to the Size/Weight Ratio Theory". British Standard Specification 763, (1937).
5) W. Edwards Deming: "On the Sampling of Physical Materials". Presented at the Meeting of the International Statistical Institute held in Berne.
6) Louis Tanner and W. Edwards Deming: "Some Problems in the Sampling of Bulk Materials". Proceedings, Am. Soc. Testing Mats., Vol. 49, (1949).
7) W. Edwards Deming: "Some Theory of Sampling". John Wily, (1950).
8) Josepf Manuele: "Materials Handling for Bulk Sampling". Symposium on Bulk Sampling (A. S. T. M), (1950).
9) Louis Tanner and Melvin Lerner: "Economic Accumulation of Variance Data in Connection with Bulk Sampling". Symposium on Bulk Sampling (A. S. T. M).

10) A. A. Orning: "Coal Samling Problems". Symposium on Bulk Sampling (A. S. T. M), (1950).
11) W. M. Bertholf: "The Analysis of Variance in a Sampling Experiment". Symposium on Bulk Sampling. (A. S. T. M), (1950).
12) G. H. Jowett: "The Accuracy of Systematic Sampling from Conveyor Belts". Appl. Statist. 1, (1952).
13) J. Visman: "Test on the Binomial Sampling Theory for Heterogeneous Coals". Symposium on Coal Sampling (A. S. T. M), (1954).
14) O. L. Davies: "Statistical Methods in Research and Production". Oliver and Boyd, (1949).
15) W. G. Cochran: "Sampling Techniqes". John Wiley, (1953).
16) H. Sakamoto: "Sampling Problems in the Quality Control". (in Japanese) in the book, "The Progress of the Theory of Stochastic Inference", edited by Dr. T. Kitagawa. Iwanami Shoten, (1953)
17) H. Sakamoto: "Contribution to the Theory of Systematic Sampling and Bedding Methods of Bulk Materials". Bulltin de L'institute International de Statistique. Tome XXXVIII~4e Livraison. Tokyo, (1961).
18) W. A. Shewhart: "Statistical Method from the Viewpoint of Quality Control". The Graduate School, The Department of Agriculture Washington. (1939).
19) W. G. Madow and L. H. Madow: "On the Theory of Systematic Sampling I". Ann. Math. Statist. 15, p. 1-24, (1944).
20) L. H. Madow: "Systematic Sampling and its Relation to other Sampling Designs". Journal Amer. Stat. Assoc., 41, (1946).
21) W. G. Madow: "On the Theory of Systematic Sampling II". Ann. Math. Statist. 20, p. 333-354, (1949).
22) W. G. Madow: "On the Theory of Systematic Sampling III". Ann. Math. Statist. 24, p. 101-106, (1953)
23) F. Yates: "Systematic Sampling". Phil. Trans. A., 241, (1948).
24) W. M. Williams: "The Variance of the Mean of Systematic Samples". Biometrika, Vol. 43, (1956).
25) T. Kitagawa: "Sampling from processes depending upon a continuous parameter". Mem. Fac. Sc., Kyushu University, series A, 5, (1950).
26) T. Kitagawa: "Successive process of statistical inferences (1)-(6)". Mem. Fac. Sci., Kyushu University, series A; 5 (1950), 6 (1951), 7 (1952), 7 (1953), 8 (1953)
27) T. Kitagawa: "Successive process of statistical controls. (1)-(2)". Fac. Sci. Kyushu Uuiversity, series A, 7 (1951), 13 (1959).
28) T. Kitagawa: "Random Integration". Bull. Math. Statistics, 4, (1950).
29) T. Kitagawa: "Empirical Functions and Interpenetrating Sampling Procedures". Mem Fac. Sci., Kyushu University, Series A, 8, (1954).
30) T. Kitagawa: "The t-Distribution concerning Random Integrations". Mem. Fac. Sci., Kyushu Uinversity, Series A, 8, (1953).
31) A. J. Duncun: "Some Measurement Error Consideration in Bulk Sampling with Special Reference to the Sampling of Fertilizer". (A. S. T. M), (1958).
32) B. A. Landry: "A Physical Interpretation of Coal Sampling Variances with Application to Sample Reduction". (A. S. T. M), (1958).

33) W. M. Bertholf: "The Effect of Increment Weight on Sampling Accuracy". (A. S. T. M), (1958).
34) R. S. Bingham, Jr., J. L. Gioele, and V. B. Shelburne: "Studies in Ore Car and Abrasive Grain Sampling Variation". (A. S. T. M), (1958).
35) E. Hopf: "On Causality, Statistics and Probability". J. Math. Physics, Bd. 13, (1934).
36) E. Hopf: "Ergodentheorie". Ergebnisse der Mathematik und ihrer Grenzgebiete. Julius Springer, (1937).
37) A. J. Khintchin: "Die Method der Willkürlichen Function und der Kampf gegen den Idealismus in der Wahrscheilchkeitsrechnung". Soviet-wissenschaft, Naturwissenschaftliche Abteilung, (1954). (German Translation from Russian Paper).
38) A. J. Khintchin: " Mathematical Methods in the Theory of Queing". Griffin's Statistical Monographs & Courses Edited by M. G. Kendall, (1960) (English Tranlation from Russian Paper).
39) A. N. Kolmogoroff: "Grundbegriffe der Wahrscheinlichkeitsrechnung". Ergebnisse der Mathematik und ihrer Grenzgebiete. Julius Springer, (1933).
40) B. W. Gnedenko: "Lehrbuch der Wahrscheinlichkeitsrechnung". Akademie-Verlag Berlin. (1957) (German Translation from Russian Text)
41) S. Bochner and K. Chandrasekharan: "Fourier Transforms". Princeton. (1949).
42) E. C. Titchmarsh: "The Theory of Functions". Oxford. (1932).