

Title	On magnetic field generated by a solenoid having spheroidal core
Sub Title	
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Publisher	慶應義塾大学藤原記念工学部
Publication year	1960
Jtitle	Proceedings of the Fujihara Memorial Faculty of Engineering Keio University Vol.13, No.50 (1960.) ,p.96(12)- 109(25)
JaLC DOI	
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Notes	
Genre	Departmental Bulletin Paper
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=KO50001004-00130050-0012

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On Magnetic Field generated by a Solenoid having Spheroidal Core

(Received May 24, 1961)

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Abstract

An iron core, which is in a form of prolate spheroid, has a solenoid wound up around a part of its surface. When a continuous current is made to flow through the solenoid, a magnetic field will be set up in the space outside the iron core. In the present paper, the author gives formula for the evaluation of this magnetic field. Some results of numerical calculations are also given.

I. Introduction

An iron core, which is in a form of prolate spheroid, has a solenoid wound up around its surface, as sketched in Fig. 1. When a direct current is made to flow through the solenoid, a magnetic field will be set up in the space outside of the iron-core. It is the object of the present report to study the field strengths of this magnetic field.

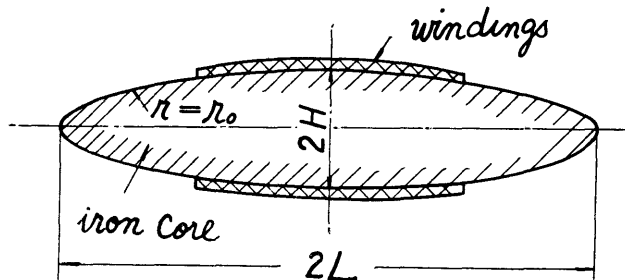


Fig. 1. Windings around an iron-core of prolate-spheroidal form

This problem is a rather simple one. The author is not aware whether it has been already reported elsewhere or not. But the author has presented it here, as he thought that it may interest someone connected with the use of solenoids.

II. General Formula relating to Prolate Spheroid

Referring to rectangular coordinates (x, y, z) , the equation of a family of prolate spheroid is given by

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$$\frac{x^2+y^2}{C_0^2(r^2-1)} + \frac{z^2}{C_0^2r^2} = 1, \quad (1)$$

where we take z -axis to be the axis of the spheroid, its major and minor diameters being $2C_0r$ and $2C_0\sqrt{r^2-1}$, respectively. r is a non-dimensional number having real positive value greater than 1.

In a problem relating to prolate spheroid, we use a system of orthogonal curvilinear coordinates (r, θ, φ) . The relation between the curvilinear coordinates (r, θ, φ) and rectangular coordinates (x, y, z) is given by

$$\begin{aligned} x &= C_0\sqrt{r^2-1} \sin \theta \cos \varphi, \\ y &= C_0\sqrt{r^2-1} \sin \theta \sin \varphi, \\ z &= C_0r \cos \theta \end{aligned} \quad (2)$$

The distance ρ of any point P from the origin, and the angle $P0z = \theta'$ (see Fig. 2), are given respectively by

$$\sin \theta' = \frac{C_0}{\rho} \sqrt{r^2-1} \sin \theta,$$

$$\rho = (x^2 + y^2 + z^2)^{1/2} = C_0(r^2 - \sin^2 \theta)^{1/2}$$

Putting the line-element ds in the form

$$(ds)^2 = h_1^2(dr)^2 + h_2^2(d\theta)^2 + h_3^2(d\varphi)^2,$$

we have

$$h_1^2 = C_0^2 \frac{r^2 - \cos^2 \theta}{r^2 - 1}, \quad h_2^2 = C_0^2 (r^2 - \cos^2 \theta),$$

$$h_3^2 = C_0^2 (r^2 - 1) \sin^2 \theta$$

and, the Laplace's equation can be written;

$$\begin{aligned} \nabla^2 V &\equiv \frac{1}{C_0^2} \left[\frac{\partial}{\partial r} \left\{ (r^2-1) \frac{\partial V}{\partial r} \right\} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left\{ \sin \theta \frac{\partial V}{\partial \theta} \right\} \right. \\ &\quad \left. + \frac{r^2 - \cos^2 \theta}{(r^2-1) \sin^2 \theta} \frac{\partial^2 V}{\partial \varphi^2} \right] = 0 \end{aligned} \quad (3)$$

Special solution of this equation (3), which is symmetrical about the z -axis, may be given by,

$$V_n = [AP_n(r) + BQ_n(r)] \times [CP_n(\cos \theta) + DQ_n(\cos \theta)] \quad (4)$$

Where P_n, Q_n are Legendre functions. A, B, C and D are arbitrary constants. n being taken as positive integers, the functions $P_n(\xi)$ are polynomials of degree n , and, they are fitted for solution inside the spheroid. The functions $Q_n(r)$ becomes $\rightarrow 0$ as $r \rightarrow \infty$, and so, they are fitted for solution outside of the spheroid.

For $n=0, 1, 2, \dots$, the actual values of $P_n(\xi)$ and $Q_n(\xi)$ are :—

$$P_0(\xi) = 1, \quad P_1(\xi) = \xi, \quad P_2(\xi) = \frac{1}{2}(3\xi^2 - 1), \quad P_3(\xi) = \frac{1}{2}(5\xi^3 - 3\xi), \quad \dots$$

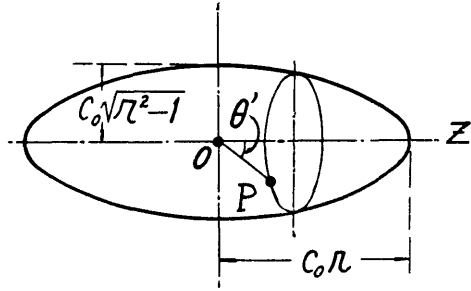


Fig. 2. A surface of prolate spheroid

$$\begin{aligned} Q_0(\xi) &= \frac{1}{2} \log \frac{\xi+1}{\xi-1}, \quad Q_1(\xi) = \frac{1}{2} \xi \log \frac{\xi+1}{\xi-1} - 1, \\ Q_2(\xi) &= \frac{1}{2} \frac{3\xi^2-1}{2} \log \frac{\xi+1}{\xi-1} - \frac{3}{2} \xi, \dots \end{aligned}$$

When we take up the prolate spheroid with major and minor diameters $2L$ and $2H$ respectively, we must have

$$2C_0 r_0 = 2L, \quad 2C_0 \sqrt{r_0^2 - 1} = 2H$$

whence we have,

$$\begin{aligned} r_0 &= 1 / \sqrt{1 - \left(\frac{H}{L}\right)^2}, \\ C_0 &= L \sqrt{1 - \left(\frac{H}{L}\right)^2} \end{aligned} \quad (5)$$

Value of curl \mathbf{A} expressed by curvilinear coordinates (r, θ, φ)

Let \mathbf{A} be a vector. In terms of general orthogonal curvilinear coordinates (λ, μ, ν) , the curl of vector \mathbf{A} , viz. curl \mathbf{A} , is expressed by

$$\begin{aligned} \xi &= \frac{1}{h_2 h_3} \left\{ \frac{\partial}{\partial \mu} (h_3 A_3) - \frac{\partial}{\partial \nu} (h_2 A_2) \right\}, \\ \eta &= \frac{1}{h_3 h_1} \left\{ \frac{\partial}{\partial \nu} (h_1 A_1) - \frac{\partial}{\partial \lambda} (h_3 A_3) \right\}, \\ \zeta &= \frac{1}{h_1 h_2} \left\{ \frac{\partial}{\partial \lambda} (h_2 A_2) - \frac{\partial}{\partial \mu} (h_1 A_1) \right\} \end{aligned}$$

In the present case we have $\lambda = r$, $\mu = \theta$, $\nu = \varphi$; h_1 , h_2 and h_3 being given by the above-mentioned formula. Also, noting that A_r , A_θ and A_φ are independent of φ , we have,

$$\left. \begin{aligned} \xi &= \frac{1}{C_0 \sqrt{r^2 - \cos^2 \theta} \sin \theta} \frac{\partial}{\partial \theta} \left\{ \sin \theta A_\varphi \right\}, \\ \eta &= \frac{-1}{C_0 \sqrt{r^2 - \cos^2 \theta}} \frac{\partial}{\partial r} \left\{ \sqrt{r^2 - 1} A_\varphi \right\}, \\ \zeta &= \frac{\sqrt{r^2 - 1}}{C_0 (r^2 - \cos^2 \theta)} \left\{ \frac{\partial}{\partial r} (\sqrt{r^2 - \cos^2 \theta} A_\theta) - \frac{\partial}{\partial \theta} \left(\frac{\sqrt{r^2 - \cos^2 \theta}}{\sqrt{r^2 - 1}} A_r \right) \right\} \end{aligned} \right\} \quad (6)$$

III. Magnetic Field generated by a Surface Distribution of Electric Current around Iron-Core in form of Prolate Spheroid

Let us denote by \mathbf{B} the magnetic induction, by \mathbf{H} the magnetic field strength, and by \mathbf{I} the electric current distributed over a surface. These are all vectors. Also, we denote by μ the magnetic permeability.

(a) Inside a region wherein there exist no electric current, we have,

$$\left. \begin{aligned} \operatorname{div} \underline{B} &= 0, & \underline{B} &= \operatorname{curl} \underline{A}, \\ \operatorname{div} \underline{A} &= 0, & \underline{H} &= \frac{1}{\mu} \operatorname{curl} \underline{A}, \\ \nabla^2 \underline{A} &= 0 \end{aligned} \right\} \quad (A)$$

Here, \underline{A} is a magnetic vector potential.

(b) Value of vector potential \underline{A} is given by,

$$\underline{A} = \frac{\mu}{4\pi} \iiint \frac{dv}{R} \underline{I} \quad (7)$$

if the current is distributed in a volume, whose volume-element being expressed by dv . R is the distance between the two points.

But, if the electric current is distributed over a surface, whose surface-element is dS , we have,

$$\underline{A} = \frac{\mu}{4\pi} \iint \frac{dS}{R} \underline{I} \quad (8)$$

(c) Boundary conditions to be satisfied at a boundary-surface, which separates two regions having different values of magnetic permeability, and along which a surface distribution of electric current is given.

In Fig. 3, region (1) has a magnetic permeability of μ_1 , while in the region (2) the magnetic permeability is μ_2 . The boundary surface between two regions (1) and (2) is denoted by S . Over the surface S (or, a part of it), a surface distribution of electric current $I' = i't$ is given, where t = the thickness of current layer, i' = current density in it.

Let the outwardly drawn normal to the surface S with regard to the region (1) be n_1 , and that with regard to the region (2) be denoted by n_2 .

The boundary conditions to be satisfied at the surface S , by the vector potentials \underline{A}_1 and \underline{A}_2 pertaining to two regions (1) and (2), may be written ;

$$\left. \begin{aligned} \frac{\partial \underline{A}_1}{\partial n_1} + \frac{\partial \underline{A}_2}{\partial n_2} &= \underline{I}', \\ \frac{1}{\mu_1} \underline{A}_1 &= \frac{1}{\mu_2} \underline{A}_2 \end{aligned} \right\} \quad (9)$$

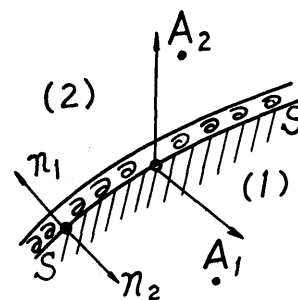


Fig. 3. Boundary surface between two media

Current vector \underline{I}' has its direction tangent to the surface S .¹⁾ In terms of magnetic

1) See, for example, Prof. H. Nukiyama, *Electricity and Magnetism*, Vol. I, 7th Ed. 1946.

inductions B_1 and B_2 pertaining to two regions, the boundary condition may also be written²⁾;

$$\left. \begin{aligned} \left(\frac{B_1}{\mu_1} - \frac{B_2}{\mu_2}\right) (\text{in direction tangent to surface } S) &= 0, \\ (B_1 - B_2) (\text{in direction normal to surface } S) &= 0 \end{aligned} \right\} \quad (9')$$

(d) Fundamental Solution for the Region outside of the Iron-Core.

In the region or space lying in outside of the iron-core, there exist no electric current, and we have by our fundamental equation (A);

$$\text{curl } B = \text{curl}(\text{curl } A) = \text{grad}(\text{div } A) - \nabla^2 A = 0$$

So that we may take that the magnetic induction B has a potential ϕ .

In the problem treated below, the given current distribution on the surface of iron-core is symmetrical with respect to the plane $z=0$. Noting that $z = C_0 r \cos \theta$, and that we need only odd functions of z , a simplest form for ϕ , which satisfies the Laplace's equation (3) is seen to be given by,

$$\phi_a = K_1 \cos \theta \left[\frac{1}{2} r \log \frac{r+1}{r-1} - 1 \right]$$

Corresponding to this value of ϕ , we have,

$$\left. \begin{aligned} B_r &= \frac{\partial \phi_a}{h_1 \partial r} = \frac{\sqrt{r^2-1}}{C_0 \sqrt{r^2-\cos^2 \theta}} K_1 \cos \theta \times \left[\frac{1}{2} \log \frac{r+1}{r-1} - \frac{r}{r^2-1} \right], \\ B_\theta &= \frac{\partial \phi_a}{h_2 \partial \theta} = \frac{-K_1 \sin \theta}{C_0 \sqrt{r^2-\cos^2 \theta}} \left[\frac{1}{2} r \log \frac{r+1}{r-1} - 1 \right] \end{aligned} \right\} \quad (10)$$

(e) Fundamental Solution for the Region inside of the Iron-Core.

Corresponding to the above, the fundamental solution for inside region may be written;

$$\phi_i = G_1 r \cos \theta,$$

from which we obtain,

$$\left. \begin{aligned} B_r &= \frac{\partial \phi_i}{h_1 \partial r} = \frac{\sqrt{r^2-1}}{C_0 \sqrt{r^2-\cos^2 \theta}} G_1 \cos \theta, \\ B_\theta &= \frac{\partial \phi_i}{h_2 \partial \theta} = \frac{-G_1 \sin \theta}{C_0 \sqrt{r^2-\cos^2 \theta}} r \end{aligned} \right\} \quad (11)$$

If we take the boundary surface of the iron core to be given by $r=r_0$, we have, according to the boundary condition (9');

$$\begin{aligned} \frac{\sqrt{r_0^2-1} \cos \theta}{C_0 \sqrt{r_0^2-\cos^2 \theta}} \left[K_1 \left\{ \frac{1}{2} \log \frac{r_0+1}{r_0-1} - \frac{r_0}{r_0^2-1} \right\} - G_1 \right] &= 0, \\ - \frac{\sin \theta}{C_0 \sqrt{r_0^2-\cos^2 \theta}} \left[\frac{G_1}{\mu_i} r_0 - \frac{K_1}{\mu_a} \left(\frac{1}{2} r_0 \log \frac{r_0+1}{r_0-1} - 1 \right) \right] &= I \end{aligned}$$

2) E. Weber, Electromagnetic Fields, 1950, p. 46.

Whence we have

$$G_1 = K_1 \left\{ \frac{1}{2} \log \frac{r_0 + 1}{r_0 - 1} - \frac{r_0}{r_0^2 - 1} \right\},$$

$$I_s = - \frac{\sin \theta}{C_0 \sqrt{r_0^2 - \cos^2 \theta}} M_1 K_1,$$

where we put for brevity,

$$M_1 = \frac{r_0}{\mu_i} \left\{ \frac{1}{2} \log \frac{r_0 + 1}{r_0 - 1} - \frac{r_0}{r_0^2 - 1} \right\}$$

$$- \frac{1}{\mu_a} \left\{ \frac{1}{2} r_0 \log \frac{r_0 + 1}{r_0 - 1} - 1 \right\}$$

μ_a and μ_i are magnetic permeabilities of outside domain and inside core, respectively. Values of I_s thus found will give current distribution over the surface of the iron-core. If I_m is the mean current intensity, we have

$$\pi I_m = E_1 K_1 \int_0^\pi \frac{\sin \theta d\theta}{\sqrt{r_0^2 - \cos^2 \theta}}$$

whence we obtain the relation,

$$K_1 = \frac{\pi}{2M_1 \alpha} I_m$$

where $\alpha = \sin^{-1}(1/r_0)$.

Here, by "mean current intensity" we mean the mean value of I with regard to angle θ (in the range of $\theta = 0$ to $\theta = \pi$) multiplied by C_0 . Of course, this value I_m is used merely as a reference.

The above-mentioned current distribution I_s is shown as a rough sketch in Fig. 4.

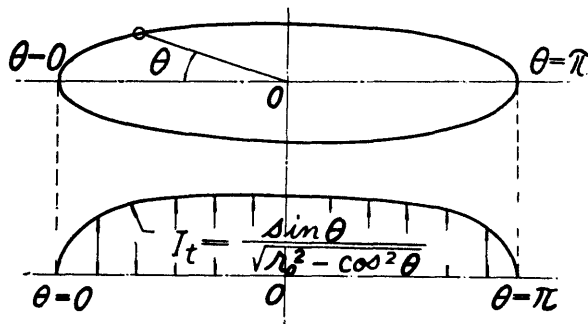


Fig. 4. Simplest case of current distribution over the surface of iron-core

It may be mentioned here that for a fairly slender form of prolate spheroid, the value of r_0 is very nearly equal to unity, and the value of the function

$$\frac{\sin \theta}{\sqrt{r_0^2 - \cos^2 \theta}}$$

is very nearly equal to unity, except at $\theta=0$ or π , where it is equal to zero.

(f) The General Case.

In the above statement, the simplest solution was shown. By taking more general form

$$\phi_a = K_1 Q_1(r) P_1(\cos \theta) + K_3 Q_3(r) P_3(\cos \theta) + \dots,$$

$$\phi_i = G_1 P_1(r) P_1(\cos \theta) + G_3 P_3(r) P_3(\cos \theta) + \dots$$

and, determining constants K_1, K_3, \dots and G_1, G_3, \dots suitably, we can obtain the solution which correspond to any given (symmetrical) current distribution around the surface of iron-core.

IV. Some Special Example of Our Problem

As mentioned above, we can obtain the intensity of magnetic field which is generated by any given distribution of current around iron-core in form of prolate spheroid. As an example, let us quote the case which has been brought to attention of the author. In this instance, the coil or windings are arranged as shown in Fig. 5.

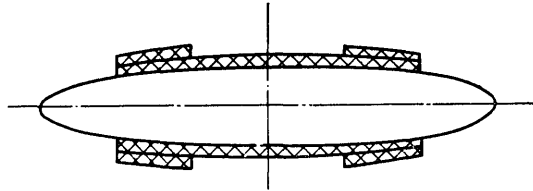


Fig. 5. Special arrangement of windings

As we see from this figure, the middle part has less current (that is ampere turns) per cm than the parts at both ends.

The current distribution, being a function of z , may also be regarded as a function of $\lambda = \cos \theta$, on the surface $r=r_0$. We may construct the function representing the given current distribution of Fig. 5, from three elementary functions as mentioned below;

(a) As the simplest current distribution, we take the term $P_1(\cos \theta) = P_1(\lambda)$. This current distribution is what was already shown in Fig. 4.

(b) Next, let us take up the function $P_3(\lambda) = \frac{1}{2}(5\lambda^3 - 3\lambda)$. Corresponding to this, we have

$$B_\theta = \frac{\partial \phi_1}{h_2 \partial \theta} \div \frac{\sin \theta}{\sqrt{r_0^2 - \cos^2 \theta}} \frac{dP_3(\lambda)}{d\lambda}$$

$$\div \frac{\sin \theta}{\sqrt{r_0^2 - \cos^2 \theta}} \times \frac{3}{2}(5\lambda^2 - 1)$$

(18)

And, the graph of the function $\frac{3}{2}(5\lambda^2-1)$ is roughly as shown in Fig. 6(a).

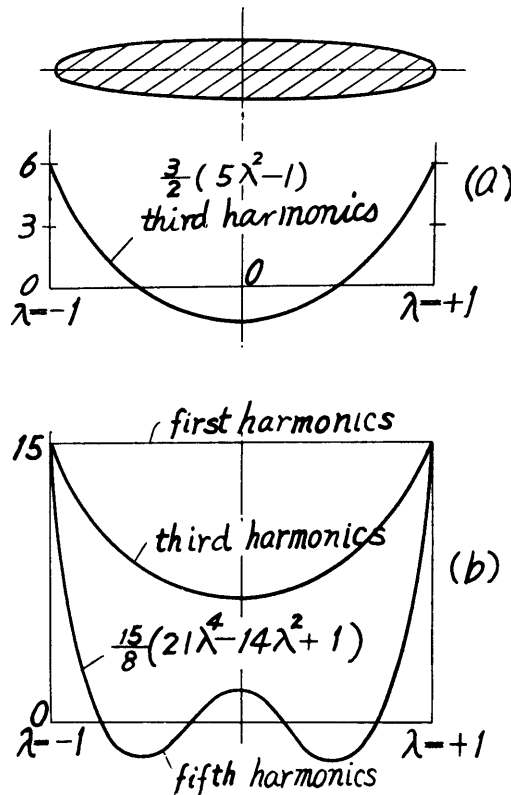


Fig. 6. Values of component distribution of current as functions of λ

(c) Thirdly, let us take the function $P_5(\lambda) = \frac{1}{8}(63\lambda^5 - 90\lambda^3 + 15\lambda)$. Corresponding to this, we have

$$B_\theta \doteq \frac{\sin \theta}{\sqrt{r_0^2 - \cos^2 \theta}} \times \frac{15}{8}(21\lambda^4 - 14\lambda^2 + 1)$$

And, the graph of the function $\frac{15}{8}(21\lambda^4 - 14\lambda^2 + 1)$ is roughly as shown in Fig. 6(b).

From these figures, we see that, by combining these three functions and making a function of the form,

$$15 - \frac{15}{8}(21\lambda^4 - 14\lambda^2 + 1) - \varepsilon \times \frac{3}{2}(5\lambda^2 - 1), \tag{12}$$

where ε is a positive constant, we can obtain the curve C as shown in Fig. 7. It is to be noted that actual current distribution, as a function of θ , is to be obtained by multiplying by

$$\sin \theta / \sqrt{r_0^2 - \cos^2 \theta}$$

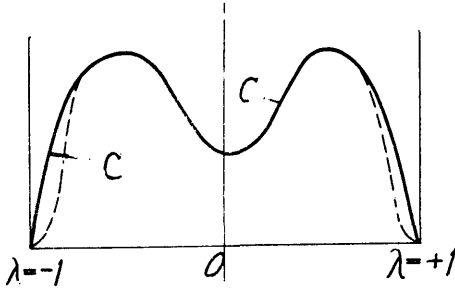


Fig. 7. Special current distribution as function of λ .

By differentiation of expressions (2), we can express dx , dy and dz in terms of dr , $d\theta$ and $d\varphi$, as follows:—

$$dx = \frac{r \sin \theta \cos \varphi}{\sqrt{r^2 - \cos^2 \theta}} [h_1 dr] + \frac{\sqrt{r^2 - 1} \cos \theta \cos \varphi}{\sqrt{r^2 - \cos^2 \theta}} [h_2 d\theta] - \sin \varphi [h_3 d\varphi],$$

$$dy = \frac{r \sin \theta \sin \varphi}{\sqrt{r^2 - \cos^2 \theta}} [h_1 dr] + \frac{\sqrt{r^2 - 1} \cos \theta \sin \varphi}{\sqrt{r^2 - \cos^2 \theta}} [h_2 d\theta] + \cos \varphi [h_3 d\varphi],$$

$$dz = \frac{\sqrt{r^2 - 1} \cos \theta}{\sqrt{r^2 - \cos^2 \theta}} [h_1 dr] - \frac{r \sin \theta}{\sqrt{r^2 - \cos^2 \theta}} [h_2 d\theta]$$

But, since

$$dx = l_1 [h_1 dr] + m_1 [h_2 d\theta] + n_1 [h_3 d\varphi],$$

$$dy = l_2 [h_1 dr] + m_2 [h_2 d\theta] + n_2 [h_3 d\varphi],$$

$$dz = l_3 [h_1 dr] + m_3 [h_2 d\theta] + n_3 [h_3 d\varphi],$$

where (l_1, l_2, l_3) , (m_1, m_2, m_3) , (n_1, n_2, n_3) are direction cosines of three line-elements $[h_1 dr]$, $[h_2 d\theta]$, $[h_3 d\varphi]$, the values of these direction cosines are known. Thus we have, remembering that $B_\varphi = 0$,

$$B_x = \frac{\cos \varphi}{\sqrt{r^2 - \cos^2 \theta}} [r \sin \theta B_r + \sqrt{r^2 - 1} \cos \theta B_\theta],$$

$$B_y = \frac{\sin \varphi}{\sqrt{r^2 - \cos^2 \theta}} [r \sin \theta B_r + \sqrt{r^2 - 1} \cos \theta B_\theta],$$

$$B_z = \frac{1}{\sqrt{r^2 - \cos^2 \theta}} [\sqrt{r^2 - 1} \cos \theta B_r - r \sin \theta B_\theta]$$

VI. Magnetic-Field Strength at a Point far away from the Origin

As a point far away from the origin 0, the value of r^2 is large in comparison with 1. For the simplest case mentioned in Section 3-(d), we shall have, for a very large value of r^2 ,

the above expression (12).

V. Change of Coordinates

The components of magnetic induction with respect to curvilinear coordinates (r, θ, φ) is rather difficult to understand. Components with respect to usual rectangular coordinates (x, y, z) can be deduced from them as follows.

$$\left. \begin{aligned} B_r &\doteq \frac{K_1}{C_0} \left(-\frac{2}{3}\right) \frac{\cos \theta}{r^3}, \\ B_\theta &\doteq \frac{K_1}{C_0} \frac{\sin \theta}{r} \left(1 - \frac{1}{r}\right) \end{aligned} \right\} \quad (13)$$

Since at the outside space, the magnetic permeability of the medium is 1, we have, from (13),

$$\begin{aligned} H_x &\doteq \frac{K_1}{C_0^2 r^2} \left(1 - \frac{1}{r}\right) x \cos \theta, \\ H_y &\doteq \frac{K_1}{C_0^2 r^2} \left(1 - \frac{1}{r}\right) y \cos \theta, \\ H_z &\doteq -\frac{K_1}{C_0^3 r^3} \left(1 - \frac{1}{r}\right) (x^2 + y^2), \end{aligned}$$

where r and θ is given approximately by

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} / C_0 \\ \theta &= \tan^{-1} [\sqrt{x^2 + y^2} / z] \end{aligned}$$

When we consider only the case in which x, y, z have positive values, we may take θ to lie in the range of $0 \leq \theta \leq \pi/2$

VII. Relation between the Current Distribution over the Surface of the Iron-Core and Magnetic Field, for the General Case

Let us write, for simplicity,

$$Q_m(r) = P_m(r) Q_0(r) + S_m(r),$$

where we have

$$Q_0(r) = \log \frac{r+1}{r-1}$$

and $S_m(r)$ are polynomials of degree $m-1$, for $m=3, 5, 7, \dots$. Actual values of $P_m(r)$ and $S_m(r)$ are seen to be

$$P_3(r) = \frac{1}{2} (5r^3 - 3r), \quad S_3(r) = -\frac{5}{2} r^2 + \frac{2}{3},$$

$$P_5(r) = \frac{1}{8} (63r^5 - 70r^3 + 15r),$$

$$S_5(r) = -\frac{63}{8} r^4 + \frac{49}{8} r^2 - \frac{8}{15},$$

etc., etc.

(a) Solution for the Region outside of the Iron-Core

Putting $\lambda = \cos \theta$, and

$$\phi_a = K_m P_m(\lambda) Q_m(r)$$

we have

$$B_r = \frac{\partial \phi_a}{h_1 \partial r} = \frac{\sqrt{r^2 - 1}}{C_0 \sqrt{r^2 - \cos^2 \theta}} K_m P_m(\lambda) Q'_m(r),$$

$$B_\theta = \frac{\partial \phi_a}{h_2 \partial \theta} = -\frac{K_m \sin \theta}{C_0 \sqrt{r^2 - \cos^2 \theta}} P'_m(\lambda) Q_m(r)$$

(b) Solution for the Region Inside the Iron-Core

Putting

$$\phi_i = G_m P_m(\lambda) P_m(r)$$

and, corresponding to this value of ϕ_i , we have,

$$B_r = \frac{\partial \phi_i}{h_1 \partial r} = \frac{\sqrt{r^2 - 1} G_m}{C_0 \sqrt{r^2 - \cos^2 \theta}} P_m(\lambda) P'_m(r),$$

$$B_\theta = \frac{\partial \phi_i}{h_2 \partial \theta} = -\frac{G_m \sin \theta}{C_0 \sqrt{r^2 - \cos^2 \theta}} P'_m(\lambda) P_m(r)$$

(c) The Current Distribution

Thus, the condition at the surface of the iron core $r=r_0$, is

$$\frac{\sqrt{r_0^2 - 1}}{C_0 \sqrt{r_0^2 - \cos^2 \theta}} P_m(\lambda) [K_m Q'_m(r_0) - G_m P'_m(r_0)] = 0$$

$$-\frac{\sin \theta}{C_0 \sqrt{r_0^2 - \cos^2 \theta}} \left[\frac{K_m}{\mu_a} P'_m(\lambda) Q_m(r_0) - \frac{G_m}{\mu_i} P'_m(\lambda) P_m(r_0) \right] = I$$

From the first equation we obtain

$$G_m = \frac{Q'_m(r_0)}{P'_m(r_0)} K_m$$

and, putting this value of G_m into the second equation we obtain,

$$I = -\frac{\sin \theta}{C_0 \sqrt{r_0^2 - \cos^2 \theta}} K_m P'_m(\lambda) \times \left[\frac{1}{\mu_a} Q_m(r_0) - \frac{Q'_m(r_0)}{P'_m(r_0)} \frac{1}{\mu_i} P_m(r_0) \right]$$

Especially, we observe that $\mu_a = 1$, if the outer space is of air or water, while the value of μ_i is usually very large in comparison with unity (for example, we may have $\mu_i = 500$). In that case, we may write approximately,

$$I = -\frac{\sin \theta}{C_0 \sqrt{r_0^2 - \cos^2 \theta}} K_m M_m P'_m(\lambda), \quad (14)$$

where $M_m = Q_m(r_0)$.

The total amount of current over the surface of iron-core (which may be regarded to represent the number of ampere-turns) is

$$I_t = \int I ds = \frac{2K_m M_m}{\sqrt{r_0^2 - 1}}$$

The value of the constant M_m may also be given by,

$$M_m = P_m(r_0)Q_0(r_0) + S_m(r_0)$$

And, if the given iron-core is fairly slender, we have approximately $r_0^2 - 1 = (H/L)$ (see Fig. 1) and so, approximately, $P_m(r_0) \doteq 1$, $Q_0(r_0) \doteq \frac{1}{2} \log (2L/H)$, and hence,

$$M_m \doteq \frac{1}{2} \log \left(\frac{2L}{H} \right) + S_m(1)$$

(d) The Values at a Point far away from the Origin

For a very large value of r , we have, ⁽³⁾

$$Q_m(r) = \frac{m!}{3 \cdot 5 \cdot \dots \cdot (2m+1)} \left[\frac{1}{r^{m+1}} + \frac{(m+1)(m+2)}{2 \cdot (2m+3)} \frac{1}{r^{m+3}} + \dots \right]$$

and we have approximately,

$$Q_1(r) \doteq \frac{1}{3r^2}, \quad Q_3(r) \doteq \frac{2}{35r^4},$$

$$Q_5(r) \doteq \frac{8}{63r^6}, \quad \text{etc., etc.}$$

for a very large value of r .

Thus we see that, at a point far away from the origin (that is, from the center of iron-core), the term with the smallest value of the index m matters most.

VIII. Numerical example of Magnetic Field generated by simplest Form of Current Distribution over the Iron-Core

As a numerical example, let us take the case of an iron-core in form of prolate spheroid, the length of which is $2L = 17m$, its diameter at the mid-section being $2H = 1.2m$. For this case, the value of r_0 , which represents the surface of this prolate spheroid, is seen to be equal to

$$r_0 = \frac{1}{\sqrt{1 - \left(\frac{1.2}{17}\right)^2}} = 1.00259.$$

Supposing that the mean current intensity I_m is equal to 1000 amp for the cross section of current sheet, and taking $\mu_i = 500$, $\mu_a = 1$, we find the following values;

$$K_1 = -385.8, \quad M_1 = -2.718, \quad C_0 = 8.478.$$

The current distribution over the surface will be assumed to be given by,

$$I = - \frac{\sin \theta}{c_0 \sqrt{r_0^2 - \cos^2 \theta}} M_1 K_1 \quad (\text{amp/m})$$

The total amount of current is, (since on the surface of core $r = r_0$ cut by the

(3) See for example, Hobson, Spherical and Ellipsoidal Harmonics, 1931.

meridian plane $\varphi = \text{const.}$, we have $d\varphi = 0$, $dr = 0$, and so, $ds = h_2 d\theta = C_0 \sqrt{r_0^2 - \cos^2 \theta} d\theta$,

$$I_t = \int Ids = - \int \frac{M_1 K_1 \sin \theta}{C_0 \sqrt{r_0^2 - \cos^2 \theta}} C_0 \sqrt{r_0^2 - \cos^2 \theta} d\theta = -2 M_1 K_1$$

When M_1 and K_1 have the above mentioned numerical value, we have $I_t = -2100$. This value may be regarded to correspond to total number of amperé-turns of windings.

Based on this numerical data, the author has made numerical estimation of magnetic-field strengths at various points exterior to the iron-core. The final result is shown as contour-curves of B_θ and B_r , as in Figs 8 and 9.

The numbers 100, 50, etc., attached to these distribution curves show values $B_r (= H_r)$ and $B_\theta (= H_\theta)$, corresponding to the above mentioned case of $I_m = 1000$ amps. and $I_t = 2100$ amps.

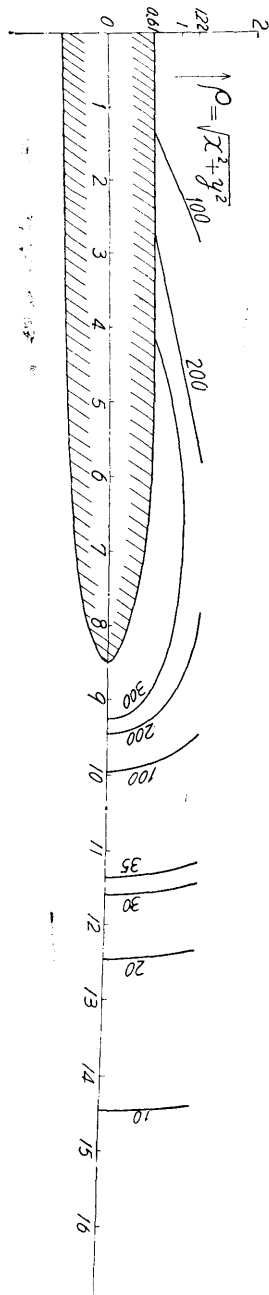


Fig. 8. Distribution of $B_r (=H_r)$ for outside space

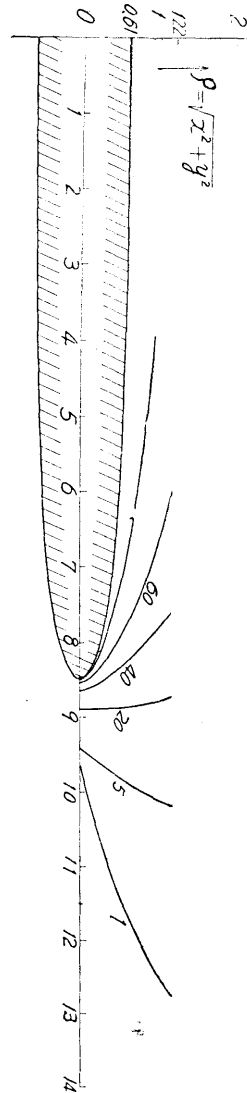


Fig. 9. Distribution of $B_\theta (=H_\theta)$ for outside space