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On Validity of Infinite-Series Expansion of the Form $\Sigma A_n \cos \lambda_n x$

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Abstract

The validity of infinite-series expansion of the form $\sum A_n \cos \lambda_n x$, of a given function $f(x)$, is examined. Here, λ_n ($n=1, 2, 3, \cdots\cdots$) are successive (positive) roots of the equation $x \sin x + K \cos x = 0$. The infinite series expansion of the form $\sum A_n \sin \mu_n x$, where μ_n ($n=1, 2, \cdots\cdots$) are successive positive roots of the equation $x \cos x + K \sin x = 0$, is well known in connection with the problem of heat-conduction of a spherical body. So that, the author's task was merely to follow the line of thoughts of the older problem. The conclusion quite similar to that of the older problem is arrived at.

I. Introduction

In the author's study about vibration of water contained in a rectangular tank,[1] there arose the need to expand a given function $f(x)$ into an infinite series of the form

$$\sum_{n=1}^{\infty} A_n \cos \lambda_n x \tag{1}$$

where $\lambda_n (n=1, 2, 3, \cdots\cdots)$ are successive (positive) roots of the equation

$$\lambda \sin \lambda + K \cos \lambda = 0 \tag{2}$$

K being a postive constant. But there, the question of validity of the expansion (1) was not examined. And so, the author intends to consider this subject in the present report. The infinite series of the form

$$\sum_{n=1}^{\infty} A_n \sin \mu_n x \tag{3}$$

where $\mu_n (n=1, 2, 3, \cdots\cdots)$ are successive roots of the equation

$$\mu \cos \mu + K \sin \mu = 0 \tag{4}$$

has already been studied in connection with the problem of heat-conduction of a

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1) This PROCEEDINGS, Vol. 13 No. 49. p. 10

spherical body.[2] So that, the author's task in examining the validity of expansion of the form (1), will be to follow, as closely as possible, the line of thoughts in the discussion of the infinite series (3), which is well established.

II. General Theorems

The followng theorems, which is already proved, may be restated here, since they are required in the following treatment.

THEOREM A. Let $\varphi(n, \alpha, t)$ be a function of real variables n, α, t, which, when considered for values of α lying in the interval $a<a'<\alpha<b'<b$, satisfies the following three relations in which n is restricted to positive integers, and in which ε represents a positive quantity which may be taken arbitrarily small:

(I)

$$\lim_{n\to\infty}\int_0^t \varphi(n, \alpha, t)\,dt=\begin{cases}-\frac{1}{2} & \text{when } a-\alpha\leqq t\leqq -\varepsilon\\ \frac{1}{2} & \text{when } \varepsilon\leqq t\leqq b-\alpha\ .\end{cases}$$

Moreover, let these limits be approached uniformly for all of the same values of α and t.

(II)

$$\int_0^t \varphi(n, \alpha, t)\,dt<A\ . \qquad -\varepsilon\leqq t\leqq \varepsilon$$

where A is a constant independent of n, α and t.

(III) $|\varphi(n, \alpha, t)|<B$, $\qquad a-\alpha\leqq t\leqq -\varepsilon$ or $\varepsilon\leqq t\leqq b-\alpha$

where B is a constant independent of n, α and t.

Also let $f(x)$ be any function satisfying the following two conditions:

(i) Throughout the interval $(a\leqq x\leqq b)$, $f(x)$ remains finite with the possible exception of a finite number of points, and is such that the integral

$$\int_a^b |f(x)|\,dx$$

exists.

(ii) In an arbitrarily small neighbourhood about the (special) point $x=\alpha$ $(a'<\alpha<b')$, $f(x)$ has limited total fluctuation.

Then we shall have, for the special value of α (as mentioned in (ii)),

$$\lim_{n\to\infty}\int_a^b f(x)\varphi(n, \alpha, x-\alpha)\,dx=\tfrac{1}{2}\left[f(\alpha-0)+f(\alpha+0)\right]$$

Moreover, if instead of condition (ii), $f(x)$ is continuous throughout the interval (a', b'), [the points $u=a'$, $x=b'$ included], and has limited total fluctuation throughout an interval (a_1, b_1) such that $a<a_1<a'<b'<b_1<b$, then we shall have uniformly for all values of α in (a', b'),

$$\lim_{n\to\infty}\int_a^b f(x)\varphi(n, \alpha, x-\alpha)\,dx = f(\alpha).$$

THEOREM B. Let $\varphi(n, \alpha, t)$ be a function of the real variables n, α, t which, when considered for the special value $\alpha=b$, satisfies the following three relations (in which n is restricted to positive integral values, and ε represents a positive number which may be taken arbitrarily small):

(I)

$$\lim_{n\to\infty}\int_0^t \varphi(n, b, t)\,dt = -G,$$

$$a-b \leqq t \leqq -\varepsilon, \qquad (b>a)$$

G being a constant, independent of t.

(II)

$$\left|\int_0^t \varphi(n, \alpha, t)\,dt\right| < A, \qquad \alpha=b, \ -\varepsilon \leqq t \leqq 0,$$

where A represents a positive constant, independent of n, α, and t.

(III)

$$|\varphi(n, b, t)| < B,$$
$$a-b \leqq t \leqq -\varepsilon,$$

B being a constant independent of n and t.

(IV) Let $f(x)$ be any function of x which satisfies the following condition:

Throughout the interval (a, b), $f(x)$ remains finite (with the possible exceptions of a finite number of points), and the integral

$$\int_a^b |f(x)|\,dx$$

exists. Moreover, the function $f(x)$ has limited total fluctuation in an arbitrarily small neighbourhood at the left of the point $x=b$.

Under these conditions, we shall have,

$$\lim_{n\to\infty}\int_a^b f(x)\varphi(n, b, x-b)\,dx = Gf(b-0).$$

In what follows, we shall be concerned with the case of $a=0$, $b=1$.

III. Expansion of a given function $f(x)$ in the form $\sum A_n \cos \lambda_n x$

Let us put

$$U(\lambda_n, x)=\cos(\lambda_n x) \qquad (n=1, 2, 3, \cdots\cdots) \tag{5}$$

where λ_n are successive (positive) roots of the equation (2). The roots λ_n can be obtained by taking the points of intersection of the plane curve $y=\cot x$ with the straight line $y=-x/K$. And we see that there are an infinite number of them. Moreover, by actual integration, we see that

$$\int_0^1 U(\lambda_m, x) U(\lambda_n, x)\, dx=0, \qquad (m \neq n)$$

$$K_n=\int_0^1 [U_n(\lambda_n, x)]^2 dx=\frac{1}{2}\left(1-\frac{1}{K}\sin^2 \lambda_n\right). \tag{6}$$

Thus, for a given function $f(x)$, of variable x, we may write formally, an infinite series expansion of the form (1), in which we put,

$$A_n=\frac{1}{K_n}\int_0^1 f(u)\, U(\lambda_n, u)\, du \tag{7}$$

K_n being defined by (6). There remains the question of validity of the expansion (1).

We now put

$$f_n(x)=\int_0^1 f(\xi) \sum_{n=1}^{n} \frac{1}{K_n} U(\lambda_n, x)\, U(\lambda_n, \xi)\, d\xi \tag{8}$$

or,

$$f_n(\alpha)=\int_0^1 f(\xi)\, \varphi(n, \alpha, \xi-\alpha)\, d\xi \tag{9}$$

where we put

$$\varphi(n, \alpha, \xi-\alpha)=\sum_{n=1}^{n} \frac{1}{K_n} U(\lambda_n, \alpha)\, U(\lambda_n, \xi) \tag{10}$$

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2) See, for example, W. B. Ford, Studies on Divergent Series and Summability (Michigan Science Series Vol. II), 1916. This literature will, in what follows, be referred to as "Ford".

This may also be rewritten as follows:

$$\varphi(n,\ \alpha,\ \xi)=\sum_{n=1}^{n}\frac{U(\lambda_n,\ \alpha)}{K_n}U(\lambda_n,\ \xi+\alpha) \tag{11}$$

whence we have

$$\int_0^{\xi}\varphi(n,\ \alpha,\ \xi)d\xi=\sum_{n=1}^{n}\frac{U(\lambda_n,\ \alpha)}{K_n}\int_0^{\xi}U(\lambda_n,\ \xi+\alpha)d\xi\ . \tag{12}$$

It is a known fact[3] that, for a given value of α, if we construct functions $\theta(z)$ and $\psi(z)$ of a complex variable z, such that;

(a): $\theta(z)$ is analytic throughout the finite z-plane,

(b): $\psi(z)$ is analytic throughout the finite z-plane,

(c): we have

$$\psi'(\lambda_n)u'(\lambda_n)=\psi(\lambda_n)u''(\lambda_n) \tag{13}$$

wherein λ_n ($n=1,\ 2,\ 3,\cdots\cdots$) are zeroes of the function $u(z)$,

(d): we have also,

$$\theta'(\lambda_n)=\frac{U(\lambda_n,\ \alpha)}{K_n\psi(\lambda_n)}[u'(\lambda_n)]^2\int_0^t U(\lambda_n,\ \alpha+t)dt\ , \tag{14}$$

then we shall have

$$\int_0^t\varphi(n,\ \alpha,\ t)dt=\frac{1}{2\pi i}\int_{C_n}\frac{\theta(z)\psi(z)}{[u(z)]^2}dz\ , \tag{15}$$

where the contour-intgral on the right hand side of eq. (15) is to be made around a contour C_n, which encloses n roots $\lambda_1.\ \lambda_2,\cdots\cdots\ \lambda_n$ of equation $u(z)=0$.

For our purpose, we shall take

$$u(z)=z\sin z+K\cos z\ . \tag{16}$$

IV. Non-existence of complex root

In § 3, it was pointed out that the eq. (2) has an infinite number of positive roots. We see also that the eq. (2) has an infinite number of negative roots. Here we shall examine whether or not the eq. (2) has complex roots.

Now, in the theory of functions of a complex variable z, it is shown that, if a function $f(z)$ is analytic in a region enclosed by a closed contour C drawn on the complex plane z, then we shall have,

$$N=\frac{1}{2\pi}\Delta_C\arg\{f(z)\}$$

3) Ford, § 58.

wherein N denotes the number of zeroes of $f(z)$ which exist inside the contour C, Δ_C denotes the variation of $\arg\{f(z)\}$ round the contour C.

In our case of the function $u(z)$, we have, by putting $z=\xi+i\eta$,

$$\text{r. p. } u(z)=\cosh\eta\left[\left(1-\frac{\eta}{K}\tanh\eta\right)\cos\xi+\frac{\xi}{K}\sin\xi\right]$$

$$\text{i. p. } u(z)=\cosh\eta\left[\frac{\xi}{K}(\tanh\eta)\cos\xi+\left(-1+\frac{\eta}{K}\right)\sin\xi\right].$$

Thus, for a very large positive value of η we have, approximately,

$$\text{r. p. } u(z)\doteqdot\cosh\eta\left[\left(1-\frac{\eta}{K}\right)\cos\xi+\frac{\xi}{K}\sin\xi\right]$$

$$\text{i. p. } u(z)\doteqdot\cosh\eta\left[\left(-1+\frac{\eta}{K}\right)\sin\xi+\frac{\xi}{K}\cos\xi\right]$$

while for a very large negative value of η, we have approximately,

$$\text{r. p. } u(z)=\cosh\eta\left[\left(1-\frac{|\eta|}{K}\right)\cos\xi+\frac{\xi}{K}\sin\xi\right]$$

$$\text{i. p. } u(z)=\cosh\eta\left[\left(1-\frac{|\eta|}{K}\right)\sin\xi-\frac{\xi}{K}\cos\xi\right].$$

So that, the variation of r. p. and i. p. of the function $u(z)$ for $(m-1)\pi\leqq\xi\leqq m\pi$ will be as shown in a rough sketch of Fig. 1, wherein m denotes an even integer.

Moreover, we have, for $z=s\pi+i\eta$, (s being an integer);

$$u(z)=(-1)^s\left[\cosh\eta-\frac{\eta}{K}\sinh\eta+i\frac{s\pi}{K}\sinh\eta\right]$$

In Fig. 2, a rectangle $a\ b\ c\ d$ is made up of four sides, each represented by

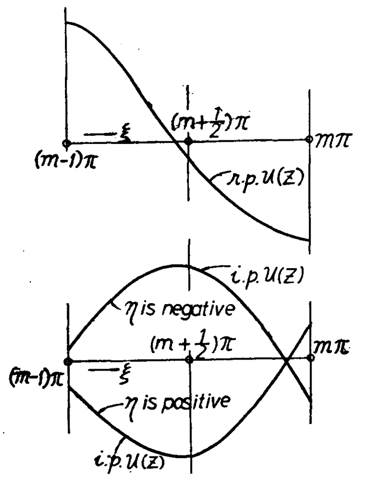

Fig. 1. Rough sketch of variation of r. p. $u(z)$ and i. p. $u(z)$ for very large value of $|\eta|$

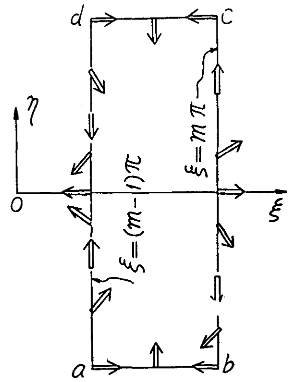

Fig. 2. Illustrating the change of arg $\{u(z)\}$ around the contour $abcd$.

$\xi=(m-1)\pi$, $\xi=m\pi$, $\eta=$a very large positive constant, $\eta=$a very large negative constant, respectively. From the above inference, we readily see that the vector representing complex number $u(z)$ turns round its directions as sketched in Fig. 2, when we travel once around the contour $a\ b\ c\ d$.

Thus, we see that for the contour C of Fig. 2, we have

$$\Delta_C \arg\{u(z)\} = 2\pi$$

which shows us that $N=1$. As there exist single real root inside this contour, we infer that there can exist no complex root inside this contour.

V. Evaluation of the integral expression $\int_0^t \varphi(n, a, t)\,dt$

After these preliminary discussions, we now turn to the evaluation of integral expression (12). In what follows, we shall use real variables α and t. It is assumed that they lie in the range;

$$\left.\begin{array}{l} 0<a'<\alpha<b'<1, \\ -\alpha\leqq t\leqq 1-\alpha \\ 0<a'<\alpha\leqq 2\alpha+t\leqq 1+\alpha<1+b'<2 \end{array}\right\} \quad (R)$$

it is to be noted that we have $|t|<1$. Thus α takes only the positive value, while t may take positive or negative values. We first note that, by the relation

$$\lambda_n \sin\lambda_n + K\cos\lambda_n = 0\ ,$$

we can write, instead of (6),

$$K_n = \frac{K(K-1)+\lambda_n^2}{2(K^2+\lambda_n^2)}\ .$$

On the other hand, we have

$$\begin{aligned} u(z) &= z\sin z + K\cos z \\ u'(z) &= z\cos z - (K-1)\frac{u(z)-K\cos z}{z} \\ u''(z) &= -u(z) + 2\cos z \end{aligned}$$

and hence,

$$\begin{aligned} u'(\lambda_n) &= \frac{\cos\lambda_n}{\lambda_n}[\lambda_n^2 + K(K-1)] \\ u''(\lambda_n) &= 2\cos\lambda_n\ . \end{aligned}$$

So that, we may also write,

$$K_n = \frac{1}{2}\frac{[u'(\lambda_n)]^2}{\lambda_n^2+K(K-1)}\ . \tag{17}$$

Next, we must determine the function $\psi(z)$ according to the condition that

$$\psi'(\lambda_n)u'(\lambda_n)-\psi(\lambda_n)u''(\lambda_n)=0 .$$

Since we have

$$\frac{u''(\lambda_n)}{u'(\lambda_n)}=\frac{2\lambda_n}{\lambda_n^2+K(K-1)}$$

a possible choice of $\psi(z)$ is

$$\psi(z)=z^2+K(K-1) \tag{18}$$

Taking up this value of $\psi(z)$, the eq. (14) becomes

$$\theta'(\lambda_n)=2\cos(\lambda_n\alpha)\int_0^t\cos[\lambda_n(\alpha+t)]\,dt$$

which suggests us to take,

$$\theta'(z)=2\cos(z\alpha)\int_0^t\cos[z(\alpha+t)]\,dt$$

$$=\int_0^t[\cos\{z(2\alpha+t)\}+\cos zt]\,dt .$$

Thus we take

$$\theta(z)=\int_0^t\left[\frac{\sin zt}{t}+\frac{\sin(2\alpha+t)z}{2\alpha+t}\right]dt. \tag{19}$$

Using this value of $\theta(z)$, the eq. (15) can be written;

$$\int_0^t\varphi(n,\ \alpha,\ t)\,dt$$

$$=\frac{1}{2\pi i}\int_{C_n}\frac{[z^2+K(K-1)]}{[z\sin z+K\cos z]^2}\,dz\int_0^t\left[\frac{\sin zt}{t}+\frac{\sin(2\alpha+t)z}{2\alpha+t}\right]dt$$

$$=\frac{1}{2\pi i}\int_0^t dt\int_{C_n}\frac{z^2+K(K-1)}{[z\sin z+K\cos z]^2}\cdot\left[\frac{\sin zt}{t}+\frac{\sin(2\alpha+t)z}{2\alpha+t}\right]dz . \tag{20}$$

The contour C_n of integration, which encloses n roots $\lambda_1,\ \lambda_2,\cdots\cdots\lambda_n$ of the equation $u(z)=0$, will be chosen to be a rectangle $A\ B\ C\ D$, as shown in Fig. 3. This rectangle has its sides represented respectively by four straight lines

$z=x+iH$ (DC), $z=x-iH$ (AB),
$x=0$ (DA) and $x=k$ (BC).

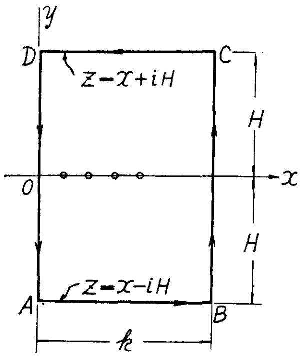

Fig. 3. Contour of Integration C_n

(A) Along the path DA we have $z=x$, and the value of integral for DA is equal to zero, because the integrand in (20) is an odd function of z.

(B) Along the path DC, we have, $z=x+iH$, $dz=dx$. Here we have,

$$J_1=\frac{z^2+K(K-1)}{(z\sin z+K\cos z)^2}\cdot\frac{\sin(2\alpha+t)z}{2\alpha+t}$$
$$\doteqdot\frac{2}{2\alpha+t}\exp\left[(2\alpha+t-2)H\right]\cdot\exp\left[i(2\alpha+t-2)x\right] \tag{21}$$

if the value of H is taken to be as large as we please. Now, if the values of α and t lie in the range of (R), we shall have $2\alpha+t-2<0$.

In that case we have

$$\lim_{H\to\infty}\int_C^D J_1\,dx=0\ .$$

(C) Along the path AB, we have, by the same reasoning,

$$\lim_{H\to\infty}\int_A^B J_1\,dx=0\ .$$

(D) Next, consider the expression

$$J_2=\frac{z^2+K(K-1)}{(z\sin z+K\cos z)^2}\cdot\frac{\sin tz}{t}\ .$$

Putting $z=x+iH$, we have,

$$\frac{\sin tz}{t}\cosh(tH)\left[\frac{\sin(tx)}{t}+i\cos(tx)\frac{\tanh(tH)}{tH}\cdot H\right].$$

Therefore, we have, for a sufficiently large value of H,

$$J_2\to 2H\exp\{(t-2)H\}\exp\{i(t-2)x\}$$

and value of integral of J_2 vanishes as $H\to\infty$, since $|t|<1$. Similarly, for $z=x-iH$.

(E) Along the vertical line BC, we have $z=k+iy$, $dz=idy$, and so, we consider the integral;

$$J_3=\frac{1}{2\pi}\int_0^t dt\int^\infty\frac{z^2+K(K-1)}{(z\sin z+K\cos z)^2}\left[\frac{\sin zt}{t}+\frac{\sin(2\alpha+t)z}{2\alpha+t}\right]dy$$

wherein we put $z=k+iy$. Now, if we take $k=(2M+\frac{1}{2})\pi$ (M=positive integer), we shall have $u(z)=z\cosh y\left[1-\frac{ik}{z}\tanh y\right]$ Also, if M be taken sufficiently large,

$$\frac{1}{[z\sin z+K\cos z]^2}=\frac{1}{z^2\cosh^2 y}\left[1-2\frac{iK}{z}\tanh y+3\left(\frac{iK}{z}\tanh y\right)^2+\cdots\cdots\right].$$

And the above expression J_3 may be reduced into as follows;

$$J_3=\frac{1}{2\pi}\int_0^t dt\int_{-\infty}^{\infty}\left[1-\frac{2iK(k-iy)}{k^2+y^2}\tanh y\right]\left[\frac{1}{t}\left\{\sin kt\cosh yt+i\cos kt\sinh yt\right\}\right.$$

$$\left.+\frac{1}{2\alpha+t}\left\{\sin(2\alpha+t)k\cosh(2\alpha+t)y+i\cos(2\alpha+t)k\sinh(2\alpha+t)y\right\}\right]\frac{1}{\cosh^2 y}dy \tag{22}$$

since we have

$$|\tanh y|<1, \qquad \left|\frac{1}{z}\right|<\frac{1}{2M}$$

we may drop terms containing $1/z^2$, $\cdots\cdots$.

There are eight terms in (22), which we shall evaluate, one by one:

(*E 1*)

$$I_{E1}=\frac{1}{2\pi}\int_0^t\frac{\sin kt}{t}dt\int_{-\infty}^{\infty}\frac{\cosh yt}{\cosh^2 y}dy=\frac{f_1(0)}{2\pi}\int_0^t\frac{\sin kt}{\sin t}dt$$

$$+\frac{1}{2\pi}\int_0^t[f_1(t)-f_1(0)]\frac{\sin kt}{\sin t}dt$$

where we put

$$f_1(t)=\frac{\sin t}{t}\int_{-\infty}^{\infty}\frac{\cosh ty}{\cosh^2 y}dy$$

If $t>0$, then we have

$$\lim_{k\to\infty}\int_0^t\frac{\sin kt}{\sin t}dt=\frac{\pi}{2}$$

also,

$$f_1(0)=\int_{-\infty}^{\infty}\frac{1}{\cosh^2 y}dy=2$$

$$\frac{1}{2\pi}\int_0^t [f_1(t)-f_1(0)]\frac{\sin kt}{\sin t}dt = \frac{1}{2\pi}\int_0^\eta \text{''}\, dt + \frac{1}{2\pi}\int_\eta^t \text{''}\, dt$$

η being arbitrarily small. So that, if $f_1(t)$ has limited total fluctuation in the neighbourhood of $t=0$, the last integral tends to 0 as $k\to\infty$.

Thus, we see that

$$\operatorname*{Lim}_{k\to\infty} I_{E1} = \frac{f_1(0)}{2\pi}\cdot\frac{\pi}{2}+0=\frac{1}{2},$$

but, if $t<0$, in like way, we obtain;

$$\operatorname*{Lim}_{k\to\infty} I_{E1} = -\frac{1}{2}.$$

(*E* 2)

$$I_{E2} = \frac{1}{2\pi}(-2iK)\int_0^t \frac{\sin kt}{t}dt\int_{-\infty}^{\infty}\frac{(k-iy)\tanh y\cosh yt}{\cosh^2 y\cdot(k^2+y^2)}dy$$

$$=\frac{K}{k\pi}\int_0^t \frac{\sin kt}{kt}dt\int_{-\infty}^{\infty}\frac{k^2}{(k^2+y^2)}\,\frac{y\tanh y\cosh ty}{\cosh^2 y}dy$$

and, since

$$\frac{k^2}{k^2+y^2}\leqq 1, \qquad \left|\frac{\sin kt}{kt}\right|<1,$$

we see that

$$\operatorname*{Lim}_{k\to\infty} I_{E2} = 0$$

so long as the variable t lies in the range of (R).

(*E* 3) We have

$$I_{E3} = \frac{i}{2\pi}\int_0^t \frac{\cos kt}{t}\int_{-\infty}^{\infty}\frac{\sinh yt}{\cosh^2 y}dy = 0$$

because of antisymmetry of the integrand.

(*E* 4)

$$I_{E4} = \frac{i}{2\pi}(-2iK)\int_0^t \frac{\cos kt}{t}\int_{-\infty}^{\infty}\frac{(k-iy)\tanh y\sinh yt}{\cosh^2 y\cdot(k^2+y^2)}dy$$

$$=\frac{K}{k\pi}\int_0^t \cos kt\,dt\int_{-\infty}^{\infty}\frac{k^2}{k^2+y^2}\cdot\frac{\tanh y\sinh yt}{t\cosh^2 y}dy$$

since we have

$$y \leqq \left| \frac{\sinh yt}{t} \right| \leqq \sinh y \ ,$$

this integral I_{E4} also tends to zero as $k \to \infty$.

(E 5)

$$I_{E5} = \frac{1}{2\pi} \int_0^t \frac{\sin(2\alpha+t)k}{(2\alpha+t)} dt \int_{-\infty}^{\infty} \frac{\cosh(2\alpha+t)y}{\cosh^2 y} dy$$

putting $2\alpha+t=2-\tau$, we have,

$$I_{E5} = -\frac{1}{2\pi} \int_{2(1-\alpha)}^{\tau} \frac{\sin(2-\tau)k}{2-\tau} d\tau \int_{-\infty}^{\infty} \frac{\cosh(2-\tau)y}{\cosh^2 y} dy$$

or, since $2k=(4M+1)\pi$,

$$I_{E5} = -\frac{1}{2\pi} \int_{2(1-\alpha)}^{\tau} \frac{\sin \tau k}{\sin \tau} f_1(\tau) d\tau$$

where we put,

$$f_1(\tau) = \frac{\sin \tau}{2-\tau} \int_{-\infty}^{\infty} \frac{\cosh(2-\tau)y}{\cosh^2 y} dy$$

as τ lies in the range $(1-b'<\tau<2-a')$, we see that for all the values of α and t in the range (R), the integral I_{E5} tends to zero as $k \to \infty$.

(E 6)

$$I_{E6} = \frac{1}{2\pi}(-2iK) \int_0^t \frac{\sin(2\alpha+t)k}{(2\alpha+t)} dt \int_{-\infty}^{\infty} \frac{(k-iy)\tanh y \cosh(2\alpha+t)y}{\cosh^2 y \cdot (k^2+y^2)} dy$$

Since we have, $|2\alpha+t| < 2$, this integral tends to zero as $k \to \infty$.

(E 7)

$$I_{E7} = \frac{i}{2\pi} \int_0^t \frac{\cos(2\alpha+t)k}{(2\alpha+t)} dt \int_{-\infty}^{\infty} \frac{\sinh(2\alpha+t)y}{\cosh^2 y} dt$$

This integral is null because the integrand is antisymmetrical with respect to y.

(E 8)

$$I_{E8} = \frac{i}{2\pi}(-2iK) \int_0^t \frac{\cos(2\alpha+t)k}{(2\alpha+t)} dt \int_{-\infty}^{\infty} \frac{(k-iy)\tanh y \cdot \sinh(2\alpha+t)y}{\cosh^2 y \cdot (k^2+y^2)} dy$$

Also, this integral tends to zero as $k \to \infty$, since we have $|2\alpha+t|<2$.

In conclusion, we see that the present function $\varphi(n, \alpha, t)$, as given by eq. (15), satisfies the condition (I) of Theorem A, it being understood that here we take $a=0$, $b=1$.

Next, all the integrals of (20), tend to zero, except the first (I_{E1}), and the condition (II) of Theorem A is seen to be satisfied. We can also show that the condition (III) is satisfied. (See Ford, p. 141). Thus, we can apply to our present function, the Theorem A.

VI. Evaluation of the integral expression $\int_0^t \varphi(n, 1, t)\, dt$

We turn now to the evaluation of the integral expression (20), when α is equal to 1. The range of variable t will be taken as $-1 \leqq t \leqq -\varepsilon$.

(A). Along the path DA, we have $z=x$, and the value of the integral for it is equal to zero, as before.

(B). Along the path DC, we have $z=x+iH$, $dz=dx$, Also, since $\alpha=1$, we have $-1<2\alpha+t-2<-\varepsilon$. Hence

$$\lim_{H \to \infty} \int_C^D J_1\, dx = 0 .$$

(C). Along the path AB, we have, by the same reasoning,

$$\lim_{H \to \infty} \int_A^B J_1\, dx = 0 .$$

(D). Next, for the expression

$$J_2 = \frac{z^2+K(K-1)}{(z\sin z+K\cos z)^2} \cdot \frac{\sin tz}{t}$$

we have, as before, for $z=x \pm iH$,

$$J_2 \to 2H \exp\{(t-2)H\} \exp\{i(t-2)x\}$$

so that the integral of J_2 along this line tends to zero as $H \to \infty$.

(E). Along the vertical line BC, we have $z=k+iy$, $dz=idy$, and

$$J_3 = \frac{1}{2\pi} \int_0^t dt \int_{-\infty}^{\infty} \frac{z^2+K(K-1)}{(z\sin z+K\cos z)^2} \left[\frac{\sin zt}{t} + \frac{\sin(2\alpha+t)}{2\alpha+t}\right] dy$$

Taking, as before, $k=(2M+\frac{1}{2})\pi$, we have dropping terms which apparently vanish as $M \to \infty$.

$$J_3=\frac{1}{2\pi}\int_0^t dt\int_{-\infty}^{\infty}\Big[1-\frac{2iK(k-iy)}{k^2+y^2}\tanh y\Big]\Big[\frac{1}{t}\Big\{\sin kt\cosh yt+i\cos kt\sinh yt\Big\}$$

$$+\frac{1}{2\alpha+t}\Big\{\sin(2\alpha+t)k\cosh(2\alpha+t)y+i\cos(2\alpha+t)k\sinh(2\alpha+t)y\Big\}\Big]\frac{1}{\cosh^2 y}dy$$

$$\cdots\cdots (22 \text{ bis})$$

There are eight terms in (22), among them (E 1), (E 2), (E 3) and (E 5) behaving the same as in the preceding section.

(E 5). Putting $2\alpha+t=2-\tau$, we have, since here we take $\alpha=1$;

$$I_{E5}=-\frac{1}{2\pi}\int_0^{\tau}\frac{\sin(2-\tau)k}{2-\tau}d\tau\int_{-\infty}^{\infty}\frac{\cosh(2-\tau)y}{\cosh^2 y}dy$$

Since t lies in the range $-1<t\leqq-\varepsilon$, τ must lie in the range of $\varepsilon\leqq\tau<1$. Now, the above integral is to be made from 0 to τ, while in the former cose of $\alpha<1$, it was to be made from $2(1-\alpha)$ to τ. And so, the former discussion does not apply here. Since $2k=(4M+1)\pi$, we can write

$$I_{E5}=-\frac{1}{2\pi}\int_0^{\tau}\frac{\sin k\tau}{2-\tau}d\tau\int_{-\infty}^{\infty}\frac{\cosh(2-\tau)y}{\cosh^2 y}dy=-\frac{1}{2\pi}\int_0^{\tau}f_1(\tau)\frac{\sin k\tau}{\sin\tau}d\tau$$

where we put

$$f_1(\tau)=\frac{\sin\tau}{2-\tau}\int_{-\infty}^{\infty}\frac{\cosh(2-\tau)}{\cosh^2 y}dy\,.$$

Now, by applying twice the formula of integration by parts, we see that

$$\int_{-\infty}^{\infty}\frac{\cosh\theta y}{\cosh^2 y}dy=\frac{6}{4-\theta^2}\int_{-\infty}^{\infty}\frac{\cosh\theta y}{\cosh^4 y}dy$$

where θ is a constant with regard to y, and such that $|\theta|<2$. Hence, we have

$$f_1(0)=\lim_{\tau\to 0}\frac{\sin\tau}{2-\tau}\int_{-\infty}^{\infty}\frac{\cosh(2-\tau)y}{\cosh^2 y}dy$$

$$=\lim_{\tau\to 0}\frac{6\sin\tau}{\tau(2-\tau)(4-\tau)}\int_{-\infty}^{\infty}\frac{\cosh(2-\tau)y}{\cosh^4 y}dy$$

$$=\frac{3}{4}\int_{-\infty}^{\infty}\frac{\cosh 2y}{\cosh^4 y}dy=2.$$

And, finally we have

$$\lim_{k\to\infty} I_{E5} = -\frac{1}{2\pi} f_1(0) \frac{\pi}{2} = -\frac{1}{2} .$$

$(E\ 6)$. Here we have,

$$I_{E6} = \left(-\frac{K}{\pi}\right) \int_0^{\tau} \frac{\sin(2-\tau)k}{2-\tau} \int_{-\infty}^{\infty} \frac{y \tanh y \cosh(2-\tau)y}{\cosh^2 y \cdot (k^2+y^2)} dy .$$

As the case of $\tau=0$ is included in the integrand, the former argument cannot be applied here. Breaking up the integration with respect to τ into two parts, viz. (1) from $\tau=0$ to $\tau=\delta$, (2) from $\tau=\delta$ to $\tau=\tau$, where δ is a positive constant, arbitrarily small, we see that the part (2) tends to zero as we make $k\to\infty$. As to the part (1) we have;

$$I_{E6(2)} = -\left(\frac{K}{\pi}\right) \int_0^{\delta} \frac{\sin(2-\tau)k}{2-\tau} d\tau \int_{-\infty}^{\infty} \frac{y \tanh y \cosh(2-\tau)y}{\cosh^2 y \cdot (k^2+y^2)} dy .$$

Now, we can prove the following formula, by integration by parts;

$$\int_{-\infty}^{\infty} \varphi(y) \frac{\cosh \theta y}{\cosh^2 y} dy = \frac{\theta}{4-\theta^2} \int_{-\infty}^{\infty} \varphi'(y) \frac{\sinh \theta y}{\cosh^2 y} dy$$

$$+ \frac{2}{4-\theta^2} \int_{-\infty}^{\infty} \varphi'(y) \frac{\cosh \theta y \sinh y}{\cosh^3 y} dy$$

$$+ \frac{6}{4-\theta^2} \int_{-\infty}^{\infty} \varphi(y) \frac{\cosh \theta y}{\cosh^4 y} dy$$

where θ is a constant such that $|\theta|<2$. Putting $\varphi(y)=\tanh y$ into this formula, and observing that

$$\left| \frac{y}{k^2+y^2} \right| \leqq \frac{\sqrt{2}}{3k}$$

we see that, if we put

$$I_{E6(2)} = -\left(\frac{K}{\pi}\right) \int_0^{\delta} \frac{\sin(2-\tau)k}{2-\tau} f_2(\tau) d\tau$$

$f_2(\tau)$ will be of order of magnitude of $1/k$, for a fixed value of δ, and we shall have

$$I_{E6\,(2)} = \int_0^{\delta} \frac{\sin \tau k}{\tau(2-\tau)} \times \frac{U}{k} d\tau$$

where U has some bounded value. Therefore we have

$$\operatorname*{Lim}_{k\to\infty} I_{E6\,(2)} = 0\,.$$

(E 7). This integral vanishes as before.

(E 8).

$$I_{E8} = \left(\frac{K}{2\pi}\right) \int_0^{t} \frac{\cos(2\alpha+t)}{(2\alpha+t)} dt \int_{-\infty}^{\infty} \frac{k \tanh y \sinh(2\alpha+t)y}{(k^2+y^2)\cosh^2 y} dy$$

$$= \left(-\frac{K}{2\pi}\right) \int_0^{T} \frac{\cos(2-\tau)k}{(2-\tau)} d\tau \int_{-\infty}^{\infty} \frac{k \tanh y \sinh(2-\tau)y}{(k^2+y^2)\cosh^2 y} dy\,.$$

Since

$$\frac{k}{k^2+y^2} \leqq \frac{1}{k}$$

the same argument as for the integral $I_{E6\,(2)}$ can be made. And the limit of I_{E8}, as $k\to\infty$ is seen to be null. Summing up, we see that, to our integral, the part (E 1) (for $t<0$) and (E 5) contribute each $-1/2$, and so we have

$$\operatorname*{Lim}_{n\to\infty} \int_0^{t} \varphi(n, 1, t)\,dt = -1$$

and the condition (I) of Theorem B is seen to be satisfied, if we take $G=-1$. Also, the condition (II), (III) of Theorem B are readily seen to be satisfied.

VII. Conclusion

Summing up the above discussions, we conclude the following theorem.

THEOREM. If $f(x)$ remains finite throughout the interval (0, 1) with the possible exception of finite number of points and is such that the integral

$$\int_0^1 |f(x)|\,dx \tag{α}$$

exists, then the series

$$\sum_{n=1}^{\infty} A_n \cos \lambda_n x \tag{β}$$

in which

$$A_n = \frac{1}{K_n} \int_0^1 f(u) \cos \lambda_n u \, du$$

$$K_n = \frac{1}{2}\left(1 - \frac{1}{K} \sin^2 \lambda_n\right),$$

λ_n being the n th positive root of the equation

$$\lambda \sin \lambda + K \cos \lambda = 0$$

will converge at any point x $(0<x<1)$ in the arbitralily small neighbourhood of which $f(x)$ has limited total fluctuation, and the sum will be

$$\frac{1}{2}[f(x-0)+f(x+0)] .$$

Moreover, the convergence will be uniform to the limit throughout any interval (a', b') which is enclosed within a second interval (a_1, b_1) such that $0<a_1<a'<b'<b_1<1$, provided that $f(x)$ is continuous throughout (a', b') inclusive of the end points $x=a'$, $x=b'$, and has limited total fluctuation throughout (a_1, b_1).

Also, if $f(x)$ remains finite throughout the interval $(0, 1)$, with the possible exception of a finite number of points, and is such that the integral (α) exists, then the series (β) will be summable $(r=1)$ at any point $x\,(0<x<1)$ at which the limits $f(x-0)$, $f(x+0)$ exist, and the sum will be

$$\frac{1}{2}[f(x-0)+f(x+0)] .$$

Moreover, the summability will be uniform to the limit $f(x)$ throughout any interval (a', b') such that $0<a'<b'<1$, provided that at all points within (a', b'), inclusive of the end points $x=a'$, $x=b'$, the function $f(x)$ is continuous.

Under the same conditions for $f(x)$ when considered throughout the whole interval $(0, 1)$, the series (β), when considered for the value $x=1$, will converge to the limit $f(1-0)$, provided $f(x)$ is of limited total fluctuation in the neighbourhood at the left of the point $x=1$, and will be summable $(r=1)$ to the limit $f(1-0)$ whenever this limit exists.

NOTE. The statement and argument of the present report has been made, by following as closely as possible, those made by Ford, as mentioned in the above. The new edition (1950) of the book by Ford has (reportedly) been issued.