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On Virtual Mass of Water Contained in a Rectangular Tank whose Side-Walls are Vibrating — IV.

(Received Dec. 15, 1960)

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Abstract

When side-walls of a rectangular tank, which is filled up with water, is vibrating, the inside water will also make a vibratory motion. Accordingly, the value of natural frequency of vibration of side-wall will be considerably lowered, due to the presence of water. This effect is represented by "virtual mass" of water.

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But, there may arise the questions as to, (a) what degree of approximation it will give? (b) what will take place, if the factor K was not so large in comparison with unity? In the present report, these questions are studied theoretically. As before, the water is assumed to be an incompressble ideal fluid, and the vibration to be of infinitesimally small amplitude. It is shown, by taking up the numerical example for the case of a rectungular water tank having the proportion of $H:B:L=3:3:5$, that, if the value of cofficient K is as large as 1000, the author's approximate formula (given in Report I), is suffiiciently accurate for a practical use, but that if the value of K is as low as 10 it requires considerable modifiication.

I. Introduction

Let us consider a rectungular tank inside which water is filled up. Side-walls of the tank is regarded to be made up of rectangular elastic plates. When these side-walls make a vibratory motion, the inside water will also vibrate. Due to

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this fact, the natural frequency of vibration of side-walls will be considerably lowered. This effect is conveniently expressed as the value of "virtual mass" of water. The author have made theoretical study about this subject, and reported the result of study in this Journal, under the same headings.[1] [2] [3]

In these reports, two states at the top level of water were considered, namely (*a*) the top of water is directly in contact with a rigid plane wall. (*b*) the top of water is left free, forming a state of "free surface".

As to the case (*b*), an approximate formula for the "virtual mass" of water was obtained, by assuming that value of the factor $K=\omega^2H/g$ (wherein ω=angular frequency of vibration, H=depth of the tank, g=acceleration due to gravitation of the earth) is very large in comparison with unity. The object of this formula being mainly for estimation of natural frequency of vibration of rectangular wall in a ship (such as a super-tanker), this assumption is seen to be usually satisfied.

But, there may arise the questions as to, (*a*) what degree of approximation it will give? (*b*) what will take place, if the factor K was not so large in comparison with unity? In what follows, these questions will be studied theoretically. Here, the water is assumed to be an incompressible inviscid fluid, and vibration to be of infiinitesimally small amplitude, as in previous papers. Also, as before, we shall use the following notations:

ϕ=velocity potential for the vibratory motion of the water, w=transverse displacement of rectangular flat-plate, A=amplitude of vibration of ditto, $W_0=\omega A$ (amplitude of transverse velocity of the flat-plate), ω=angular frequency of vibration, L=length of the rectangular water-tank, H=height of ditto, B=breath of ditto, ρ_w=density of water, T_w=kinetic energy of water in vibration (alloted to one panel LH of flat plate, in contact with water on one side), $K=\omega^2H/g$, $m=\pi/L$, $s=\pi/H$, g=acceleration due to gravitation of the earth=9.80 m/sec², $s_j=\xi_j/H$, ξ_j=non-dimensional factor (j=1, 2, 3, ……).

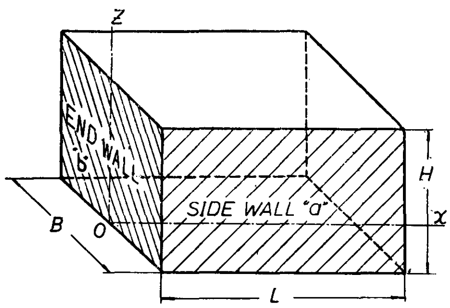

Fig. 1. Sketch of a rectangular water tank

1) F. Kito; This Proceedings, Vol. 11, No. 40 (1958)
2) F. Kito; This Proceedings, Vol. 12, No. 46 (1959)
3) F. Kito; This Proceedings, Vol. 12, No. 47 (1959)

II. Expression for the velocity-potential ϕ, for the case in which top-surface is a free-surface

Let us assume that the water is almost completely filled up in the tank, but there is left a small vacancy, the top-surface being in the state of "free-surface". If we assume that the motion of water is a potential flow, there will exist the velocity potential ϕ, which satisfies the Laplace's equation $\nabla^2\phi=0$.

The boundary condition for ϕ will be imposed as follows:

(i) at the bottom surface (which is here assumed to be a rigid wall),

$$\text{at} \quad z=0 \qquad V_z=\partial\phi/\partial z=0 \tag{1}$$

(ii) at two side-walls which are vibrating, for $y=\pm\frac{1}{2}B$,

$$V_y=\frac{\partial\phi}{\partial y}=\pm W_0 \sin mx \sin sz \cos\omega t \tag{2}$$

(iii) at two end-walls which are here assumed to stand still,

$$\text{for} \quad x=0, \quad \text{or} \quad L, \qquad V_x=\frac{\partial\phi}{\partial x}=0 \tag{3}$$

(iv) at the top (free) surface, water pressure $p=$a const. (4)

As a trial, let us take a special solution of Laplace's equation:

$$\phi=f_{ij}(y)\cos(m_i x)\cos(s_j z)\cos\omega t \tag{5}$$

The condition at the end-walls (iii) is satisfied, if we take $i=0, 1, 2, \cdots\cdots$. The condition at the bottom (i) is already satisfied by (5). Lastly, the condition at the free-surface is expressed in the form

$$\text{at} \quad z=H, \qquad \frac{\partial\phi}{\partial z}=-\frac{1}{g}\frac{\partial^2\phi}{\partial t^2} \tag{6}$$

provided that the vibration amplitude is infinitesimally small. Moreover, putting the expression (5) into (6), we see that the equation

$$\cot\xi_j=-\xi_j/K \tag{7}$$

must be satisfied. Here, we put

$$K=\omega^2 H/g, \qquad \xi_j=s_j H.$$

If value of the factor K is very large in comparison to unity, the roots of eq. (7) may approximately be taken to be $\xi_j=(\pi/2)\times j$, ($j=1, 3, 5, \cdots\cdots$). The approximate formula given in Report I was made for this case of very large value of K. But, if the value of K is not so large, roots ξ_j of the equation (7) must be obtained otherwise. For example, they may be obtained graphically, by finding the points of intersection of the plane curve $y=\cot x$ with the straight line $y=-x/K$.

Table 1 shows values ξ_j of the roots of the eq. (7), obtained by this graphical method. It must be noted that when K is not so large in comparison with unity, values of ξ_j ($j=1, 2, \cdots\cdots$) are functions of the angular frequency ω of vibration.

Table 1. Values of roots ξ_j

K	ξ_1/π	ξ_2/π	ξ_3/π	ξ_4/π
1000	0.500	1.500	2.500	3.500
100	0.505	1.517	5.526	3.537
20	0.527	1.577	2.625	3.663
10	0.554	1.653	2.726	3.777

From these preliminary considerations, we are led to the following expression (8) for the velocity potential ϕ of vibratory motion of water:

$$\phi=\sum_i\sum_j B_{ij} f_{ij}(y) \cos(m_i x) \,\mathrm{cis}(s_j z) \tag{8}$$

This expression (8) will give the velocity potential ϕ, for the case in which two end-walls ($B\times H$) and bottom wall ($L\times B$) do not vibrate, while two side-walls ($H \times L$) vibrate in a prescribed manner (2), the top surface being a free-surface. In expression (8), which is a doubly infinite series, B_{ij} are coefficients, and is shown below that they contain the factor $\cos\omega t$. These coefficients B_{ij} must be so chosen as to make the transverse velocity $\partial\phi/\partial y$ (for $y=\pm B/2$) coincide with the value given by (2). Here, we must distinguish the two different modes viz., opposite-phase and in-phase vibration of side walls. (see Fig. 2 and Fig. 3).

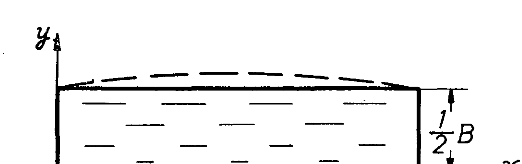

Fig. 2. In-phase Vibration of Side-Walls

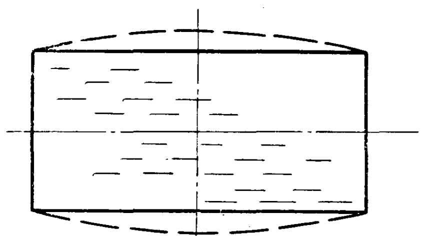

Fig. 3. Opposite-phase Vibration of Side-Walls

Now, in order to make the value of $\partial\phi/\partial y$ (for $y=\pm B/2$) as given by (8), equal to the prescribed value of (2), we must make use of an idea of Fourier series expansion, with regard to the variable x. This is what was done in the previous Report I. As to the variable z, we must make an expansion similar to Fourier series by a sequence of functions,

$$\varphi_j(z) = \cos(\xi_j z/H) \qquad (9)$$
$$(0 \leqq z \leqq H) \quad (j=1,\ 2,\ 3, \cdots\cdots)$$

By actual integration, we can see that the sequence of functions (9) forms an orthogonal system.

Moreover, we have

$$\int_0^H [\varphi_j(z)]^2 dz = \tfrac{1}{2} H k_j \qquad (10)$$

where we put

$$k_j = 1 - (\sin^2 \xi_j)/K \qquad (11)$$

Thus, we can expand the function (2) as an infinite series of functions (9), similar to Fourier Series. Strictly, it is necessary to assure that the sequence of functions (9) form a complete set. This will be discussed in another report.

By this manner, we could obtain values of coefficients B_{ij} in expansion (8), as follows:

(a) case of in-phase vibration $(i=0,\ 2,\ 4, \cdots\cdots;\ \ j=1,\ 2,\ 3, \cdots\cdots)$

$$f_{ij}(y) = \sinh(n_{ij}\, y)$$

$$B_{ij} = \frac{2\varepsilon}{\pi^2} \cdot \frac{(1+\cos\xi_j)/k_j}{(i^2-1)[(\xi_j/\pi)^2-1]} \cdot \frac{W_0 \cos\omega t}{n_{ij}\cosh(n_{ij}B/2)}$$

(b) case of opposite-phase vibration, $(i=0,\ 2,\ 4, \cdots\cdots;\ \ j=1,\ 2,\ 3, \cdots\cdots)$

$$f_{ij}(y) = \cosh(n_{ij}\, y)$$

$$B_{ij} = \frac{2\varepsilon}{\pi^2} \cdot \frac{(1+\cos\xi_j)/k_j}{(i^2-1)[(\xi_j/\pi)^2-1]} \cdot \frac{W_0 \cos\omega t}{n_{ij}\sinh(n_{ij}B/2)}$$

Here we are to take $\varepsilon=2$ if $i=0$, but $\varepsilon=4$ if $i\neq 0$ n_{ij} is given by

$$n_{ij}\, B = \pi\left[\left(i\frac{B}{L}\right)^2 + \left(\frac{\xi_j}{\pi}\frac{B}{L}\right)^2\right]^{1/2} \qquad (12)$$
$$(i=0,\ 2,\ 4, \cdots\cdots;\ \ j=1,\ 2,\ 3, \cdots\cdots)$$

If the value of factor K is very large in comparison with unity, we shall have $\xi_1=(1/2)\pi$, $\xi_2=(3/2)\pi$, $\xi_3=(5/2)\pi$, thus reducing to the case of author's previous Report I.

III. Kinetic energy T_w of vibration of water. Value of coefficients M, M' for virtual mass.

According to hydrodynamical theory, the kinetic energy T_w of motion of water in a region is given by a surface integral

$$T_w = \frac{1}{2}\rho_w \int\int \phi \frac{\partial\phi}{\partial n} dS \qquad (13)$$

which extend to whole boundary surface of the water region. Here, $\partial\phi/\partial n$ represents the normal derivative of ϕ, taken in outward direction as seen from the region. dS is the surface-element of boundary surface.

In the present case of a rectangular water tank, we have $\partial\phi/\partial n=0$ at three boundary faces which are not vibrating, and contribute nothing to the integral (13). As to the two side-faces $(H\times L)$, their contribution to the integral (13) is as follows:

$$T=\rho_w\int_0^L dx\int_0^H dz\Big[W_0\sin mx\sin sz\cos\omega t\Big]$$
$$\times\Big[\sum_i\sum_j B_{ij}\cos(mix)\cos(s_j z)f_{ij}(B/2)\Big] \tag{14}$$

Evaluating this integral, and alloting to each one panel of side-wall, we have

$$T_w=\frac{1}{2}\rho_w\,[W_0\cos\omega t]^2\,[L\,B\,H]\cdot M \tag{15}$$

where M is coefficient which represents the amount of virtual mass of water. Actual values of M are as follows:

(a) For the case of in-phase vibration,

$$M=\frac{4}{\pi^4}\sum_i\sum_j\Big[\frac{1+\cos\xi_j}{(i^2-1)\{(\xi_j/\pi)^2-1\}}\Big]^2\times\frac{\varepsilon}{(n_{ij}B)k_j}\tanh(n_{ij}B/2)$$

(b) For the case of opposite-phase vibration,

$$M=\frac{4}{\pi^4}\sum_i\sum_j\Big[\frac{1+\cos\xi_j}{(i^2-1)\{(\xi_j/\pi)^2-1\}}\Big]^2\times\frac{\varepsilon}{(n_{ij}B)k_j}\coth(n_{ij}B/2)$$

We are to take $i=0,\,2,\,4,\cdots\cdots$, and $j=1,\,2,\,3,\cdots\cdots$. Also, we take $\varepsilon=2$ if $i=0$, but $\varepsilon=4$ if $i\neq0$.

Lastly, we must evaluate the part of integral (13) which correspond to top free-surface. As far as we are discussing the case of small vibration, the integration may approximately be taken over the plane surface $Z=H$. On the free surface $Z=H$ we have $\partial\phi/\partial n=(\omega^2/g)\phi$, by virtue of the conditional equation (6) and the fact that ϕ contains $\cos\omega t$ as a factor, So that we have

$$T_w'=\frac{1}{2}\rho_w\frac{\omega^2}{g}\int_0^L dx\int_{-B/2}^{B/2}\phi^2\,dy \tag{16}$$

On the top plane $z=H$ we have

$$\phi=\sum_i\sum_j B_{ij}f_{ij}(y)\cos(mix)\cos(s_j H)\cos\omega t \tag{17}$$

and, putting this value (17) into the above integral (16), and making integration, the following expression for T_w' is obtained:

$$T_{w}'=\frac{1}{2}\,\rho_w\,[L\,B\,H]\left(\frac{B}{H}\right)^2 M'\,[W_0\cos\omega t]^2 \tag{18}$$

Here T_w' is taken to be the value alloted to each one panel of side-wall.
M' is a coefficient which represents the virtual mass of water. Its actual value is seen to be;

(a) For the case of in-phase vibration,

$$M'=\sum_i\sum_j\sum_k\left(\frac{1}{\varepsilon}\right)[B_{ij}\,B_{ik}M_{ijk}\cos\xi_j\cos\xi_k]$$

where we have, if $j\neq k$,

$$M_{ijk}=\frac{\sinh\{(n_{ij}+n_{ik})B/2\}}{(n_{ij}+n_{ik})B}-\frac{\sinh\{(n_{ij}-n_{ik})B/2}{(n_{ij}-n_{ik})B}$$

while we have, if $j=k$,

$$M_{ijj}=\frac{\sinh(n_{ij}B)}{2(n_{ij}B)}-\frac{1}{2}$$

(b) For the case of opposite-phase vibration,

$$M'=\sum_i\sum_j\sum_k\left(\frac{1}{\varepsilon}\right)[B_{ij}\,B_{ik}K_{ijk}\cos\xi_j\cos\xi_k]$$

where we have, if $j\neq k$,

$$K_{ijk}=\frac{\sinh\{(n_{ij}+n_{ik})B/2\}}{(n_{ij}+n_{ik})B}+\frac{\sinh\{(n_{ij}-n_{ik})B/2\}}{(n_{ij}-n_{ik})B}$$

while we have, if $j=k$,

$$K_{ijj}=\frac{\sinh(n_{ij}B)}{2(n_{ij}B)}+\frac{1}{2}$$

The total value of coefficient of virtual mass (with regard to vibration) of water is the sum of above two, viz., $(M+M')$.

IV. The case of partially filled water tank

In the above treatment, the water was assumed to be almost entirely filled up to the level of $z=H$. For the case in which water is only partially filled up to the level $z=h$ ($h<H$), neary the same inference as above can be made. As far as the calculation of water side is concerned, we have only to write h instead of H. But, the expression (2), being transverse velocity of displacement of side-wall, the value of s is still to be taken as $s=\pi/H$. We have also to use the factor $k=\omega^2h/g$

instead of the factor $K=\omega^2 H/g$. Making the calculation similar to author's Report II[2] we obtain,

$$B_{ij}=\frac{W_0\cos\omega t}{N_{ij}\cosh(N_{ij}B/2)}\cdot\left(\frac{\varepsilon}{\pi^2}\right)\cdot\frac{1}{(i^2-1)}\cdot$$
$$\left[\frac{1-\cos(\eta_j-\alpha)\pi}{\eta_j-\alpha}-\frac{1-\cos(\eta_i+\alpha)\pi}{\eta_i+\alpha}\right] \qquad (19)$$

($i=0, 2, 4, \cdots\cdots$; $j=1, 2, 3, \cdots\cdots$) In the formula (19), follwing notation is used:

$S_j=\xi_j/h,\qquad \eta_j=\xi_j/\pi,$
$\alpha=h/H,\qquad k_j=1-(\sin^2\xi_j)/k,$
$k=\omega^2 h/g,$
$N_{ij}=[(mi)^2+(S_j)^2]^{1/2},$
ξ_j=roots of the equation
$\cot\xi_j=-\xi_j/k.$

The subsequent calculation can be made in a similar way as shown in the previous section.

V. Numerical example

Let us take the case of rectangular tank having the proportion of $H:B:L=3:3:5$. The author have made numerical evaluation of coefficients M and M', for the case in which water is almost filled up to the level $z=H$, but yet the top-surface is in a state of free-surface. The numerical evaluation was made for two values of the factor K, namely $K=100$ and $K=10$. Since M is given as a doubly infinite series, and M' as a triply infinite series, we obtained the approximate values of them, by taking $i=0, 2, 4, 6$, and j (and k)$=1, 2, 3, 4$, and by making summation. It was inferred that terms with index $i=6$ or $i>6$ have no practical effect upon the numerical values of M and M'. The result of evaluation is smmarized and shown as Table 2.

Table 2. Values M, M' of coefficients of virtual mass

		$K=1000$	$K=100$	$K=10$
Case of in-phase vibration	M	0.080	0.0805	0.0785
	M'	0.000	0.0026	0.0079
	$M+M'$	0.080	0.0831	0.0864
Case of opposite-phase vibration	M	0.170	0.1610	0.1310
	M'	0.000	0.0040	0.0320
	$M+M'$	0.170	0.1650	0.1630

From the Table 2, we see that when the value of K is as large as 100, there is a small difference as compared with the case of $K\to\infty$. But if the value of K is as low as $K=10$, there appear a considerable difference.

Consider, as an example, the case of a merchant ship, whose screw propeller is 4-bladed and runs at 120 rpm. In this case a vibration of angular frequency $\omega = (120\div 60)\times 5\times 2\pi = 62.8$ rad/sec. would be expected to take place.

If $H=3$ m, we shall have $K=(62.8)^2\times 3\div 9.8=1200$. Thus, we see that, at least when we are concerned with vibration problems of ship's panel boards, it would be a very rare case that the value of K becomes lower than 100. And so, we may infer that the approximate formula given in author's Report I is accurate enough for practical uses.