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A Min Max Solution of an Inventory Problem

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Abstract

H, Scarf's[1] Min Max Solution of inventory problem was solved under the assumption of the two point distribution in demand which leads to the least profit in the case of the investment.

Here some other economical factors which were not considered in Scarf's model were included in constructing the model.

The argument is quite anlogous with Scarf's model, but somewhat complicated. Especially, the keypoint such as the lemma will be given in exactly the same sentences, but with different formulae, in order to show the analogy.

I. The Model

Consider the case in which the commodity is to be sold at the price of $\$r$ per unit, which was purchased at the cost of $\$p$ per unit. It costs $\$h$ per unit to hold the commodity unit time in adequate conditions. There is a demand ξ which is a sample from the demand distribution $\Phi(\xi)$, whose mean is μ and the standard deviation σ. In the case the demand is over the purchase y, the penalty $\$p$ must be paid per unit shortage. The expected profit will be represented by the notation Π.

All the notations mentioned above are listed in Table 1.

As was assumed implicitly in the preceding paragraphs, each economical factor is propotional to the quantity. The expected profit is then

Table 1.

Notations	Conditions	Meanings
c	$c\geq 0$	Unit purchase cost
r	$r\geq 0$ $r\geq c$	Unit selling price
ξ	$\xi\geq 0$	Demand
$\Phi(\xi)$		Demand distribution
y	$y\geq 0$	Amount of purchase
h	$h\geq 0$	Unit holding cost/unit time
p	$p\geq 0$	Shortage penalty
Π		Expected profit

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1) K. J. Arrow, S. Karlin & H. Scarf; "*Studies in the Mathematical Theory of Inventory and Production*" Stanford Univ. Press. (1958)

$$\underbrace{\Pi = r\int_0^y \xi d\Phi(\xi) + r\int_y^\infty y d\Phi(\xi)}_{SELLING} \underbrace{- cy}_{PURCHASE} \underbrace{- h\int_0^y (y-\xi) d\Phi(\xi)}_{HOLDING} \underbrace{- p\int_y^\infty (\xi - y) d\Phi(\xi)}_{PENALTY} \qquad (1)$$

By simple considerations the equation above is rewritten in the form,

$$\pi = (r+h)\int_0^\infty [\min(\xi, y) - \gamma \max(0, \xi - y)]\, d\Phi(\xi) - (c+h)y \qquad (2)$$

where

$$\gamma = \frac{p}{r+h} \geq 0$$

The terms inside the rectangular parentheses after the integral notation is represented graphically in Fig. 1, otherwise in the ordinal algorism

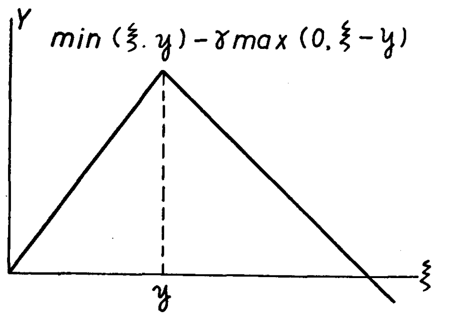

Fig. 1.

$$\left.\begin{array}{ll} Y = \xi & 0 \leq \xi \leq y \\ Y = -\gamma(\xi - y) + y & y \leq \xi \end{array}\right\} \qquad (3)$$

II. The Ordinal Solution and the Min Max Solution

The mathematical model represented in Eq. (2) can be solved easily if the demand distribution is precisely known.

The result is[2)]

$$\Phi(y) = \frac{r - c + p}{r + h + p} \qquad (4)$$

In this article, however as in that of Scarf's, let it be the assumption that the only informations about the demand distribution are the mean μ and the standard deviation σ.

III. The Demand Distribution which gives the Min. Porfit for a Fixed y

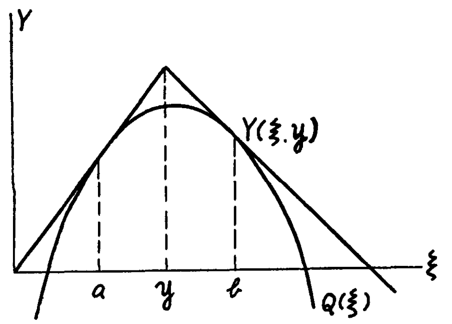

Fig. 2.

The first procedure in solution will then be to find the least profit demand distribution.

Lemma: *Let y, μ and σ^2 be fixed, then there exists a quadratic function $Q(\xi) = \alpha + \beta\xi + \delta\xi^2$ such that $Q(\xi) \leq Y(\xi, y)$ for $\xi \geq 0$ with equality holding at only two points a and b. Moreover, there exists a two points distribution situated at a and b, with mean μ and standard deviation σ.*

And here the explanation is given,

2) C. W. Churchman, R. L. Ackoff, & E. L. Arnoff; "*Introduction to Operations Research*" Wiley (1957)

assuming the lemma correct, that the two point distribution described in the lemma *is* the demand distribution that gives the minimum to the profit for a fixed y.

Since

$$\int_0^\infty Y(\xi, y)\, d\Phi(\xi)=\int_0^\infty [Y(\xi, y)-Q(\xi)]\, d\Phi(\xi)+\int_0^\infty Q(\xi)\, d\Phi(\xi) \tag{5}$$

$$\geq \int_0^\infty Q(\xi)\, d\Phi(\xi)=\alpha+\beta\mu+\delta(\mu^2+\sigma^2) \tag{6}$$

According to the lemma,

$$\int_0^\infty Y(\xi, y)\, dF(\xi)=\int Q(\xi)\, dF(\xi)=\alpha+\beta\mu+\delta(\mu^2+\sigma^2) \tag{7}$$

Then

$$\int_0^\infty Y(\xi, y)\, d\Phi(\xi)\geq \int_0^\infty Y(\xi, y)\, dF(\xi) \tag{8}$$

where $F(\xi)$ is the two point distribution described in the lemma.

Hence the two point distribution described in the lemma is the demand distribution which gives the minimum profit for a fixed y.

IV. The Proof of the Lemma

Case 1. $y\leq \frac{\mu^2+\sigma^2}{2\mu}$

In this case we can get the quadratic form explicitly as

$$Q(\xi)=\left\{-\gamma+\frac{2\mu(1+\gamma)}{\mu^2+\sigma^2}y\right\}\xi-\frac{(1+\gamma)y\mu^2}{(\mu^2+\sigma^2)^2}\xi^2 \tag{9}$$

The two point distribution is then,

$$P=\frac{\sigma^2}{\sigma^2+\mu^2} \quad \text{at } \xi=0\ (=a), \qquad P=\frac{\mu^2}{\mu^2+\sigma^2} \quad \text{at } \xi=\frac{\mu^2+\sigma^2}{\mu}(=b)$$

Case 2. $y> \frac{\mu^2+\sigma^2}{2\mu}$

In this case it is difficult to show the quadratic function explicitly, but its existence is proved as follows.

Let $Q_a(b)$ be tangent to $Y(\xi, y)$ at $\xi=a$ $0\leq a<\min(y, \mu)$ and parallel with $Y(\xi, y)$ at $\xi=b=\mu+\frac{\sigma}{\mu-a}$ i.e. the other potion of $Y(\xi, y)$.

And let

$$h(a)=Q_a(b)-Y(b, y) \tag{10}$$

If, in varying a from 0 to $\min(\xi, y)$, $h(a)$ changes its sign, we can deduce, by the continuity of $h(a)$, that there must be a value of a which makes $h(a)$ zero, i.e. there exists the quadratic form described in the lemma. Refering that

(i)	$a=0$;	$h(0)<0$
(ii)	$\mu>y$;	$h(y)>0$
(iii)	$\mu\leq y$;	$h(\mu)>0$

the proof of the lemma is completed.

V. Calculation of the Least Profit for a Fixed y —— $P(y)$

Now we have proved that the two point distribution described in the lemma is the one that gives the least profit for a fixed y. And for one case the explicit form of the quadraic function was given.

Case 1. $$y\leq\frac{\mu^2+\sigma^2}{2\mu}$$

By substituting the coefficients of Eq. (9) into Eq. (7) we get

$$P(y)=-(r+h)\gamma\mu+\left[(r+h)\frac{\mu^2(1+\gamma)}{\mu^2+\sigma^2}-(c+h)\right]y \tag{11}$$

Case 2. $$y>\frac{\mu^2+\sigma^2}{2\mu}$$

As it is known that a two point distribution expressed by a parameter a gives the minimum profit for a fixed y, the distribution that gives the minimum is included in the parametric set of a of the following representation, $F_a(\xi)$.

$$E_F(a,y)=\int_0^\infty Y(\xi, y)\,dF_a(\xi) \tag{12}$$

By differentiating the equation above and equating it to zero, we can get the relation between y and a_0 which gives the *least profit two point distribution* as

$$a_0=y-\sqrt{(y-\mu)^2+\sigma^2} \tag{13}$$

Then by substitution $E_F(a, y)$ becomes a function of y only.
$P(y)$, in this case, is then,

$$P(y)=(r+h)\,E_F(y)-(c+h)\cdot y \tag{14}$$

VI. The Optimal Inventory Level

Now we have come to the stage to decide the optimal level. To simplify, let

$$\rho=+\sqrt{\frac{c+h}{p+r-c}} \tag{15}$$

Case 1. $y\leq\frac{\mu^2+\sigma^2}{2\mu}$

In this case $P(y)$ is a straight line, so that the max. must be at one end of the line, if any. That is, 0 or $\frac{\mu^2+\sigma^2}{2\mu}$ according to the sign of the inclination.

(i) $\rho^2<\frac{\mu^2}{\sigma^2}$

In this case $P(y)$ is increasing, so is the max. at $y=\frac{\mu^2+\sigma^2}{2\mu}$

(ii) $\rho^2>\frac{\mu^2}{\sigma^2}$

In this case the line is decreasing and the max. value is found at 0.
However, when

(iii) $\rho^2=\frac{\mu^2}{\sigma^2}$

the inclination of the line is zero, so that any value between 0 and $\frac{\mu^2+\sigma^2}{2\mu}$ gives the same value to $P(y)$.

Case 2. $y>\frac{\mu^2+\sigma^2}{2\mu}$

In this case we can get the value of y which makes $P(y)$ maximum by differentiating Eq. (14) and equating it to zero, as

$$y=\mu+\sigma\left(\frac{1-\rho^2}{2\rho}\right) \tag{16}$$

so far as it exists in the region of Case 2. And in order to have y of Eq. (16) in this region, we must have

$$0\leq\rho\leq\frac{\mu}{\sigma}$$

Combining the two cases above, the continuity of Eq. (11) and Eq. (14) at the end of their region in mind, we get the resulting optimal level as

1	$y=0$	$\frac{\mu^2}{\sigma^2}<\rho^2$
2	y: any value between 0 and $\mu^2+\rho^2/2\mu$	$\frac{\mu^2}{\sigma^2}=\rho^2$
3	$y=\mu+\sigma\left(\frac{1-\rho^2}{2\rho}\right)$	$0\leq\rho^2\leq\frac{\mu^2}{\sigma^2}$

The corresponding expected profit of the min-max optimal levels described above are then

1	$\Pi = -p\mu$	$\frac{\mu^2}{\sigma^2} < \rho^2$
2	$\Pi = -p\mu$	$\frac{\mu^2}{\sigma^2} = \rho^2$
3	$\Pi = \mu(r-c) - \sigma\sqrt{(c+h)(p+r-c)}$	$\frac{\mu^2}{\sigma^2} \geq \rho^2 \geq 0$

VII. The Graph of $\left(\frac{1-\rho^2}{2\rho}\right)$

In order to get the optimal level described in Eq. (16) simply from a given value of ρ^2, the following graph is provided, which was drawn from the computations by NEAC. 2201, an electronic automatic computer.

For the values of ρ^2 exceeding unity, the values of $\frac{1-\rho^2}{2\rho}$ can be got through the inverse of ρ^2.

Since

$$\frac{1-\rho^2}{2\rho} = -\frac{1-\frac{1}{\rho^2}}{2\frac{1}{\rho}}$$

And

$$\frac{1}{\rho^2} < 1$$

when

$$\rho^2 > 1$$

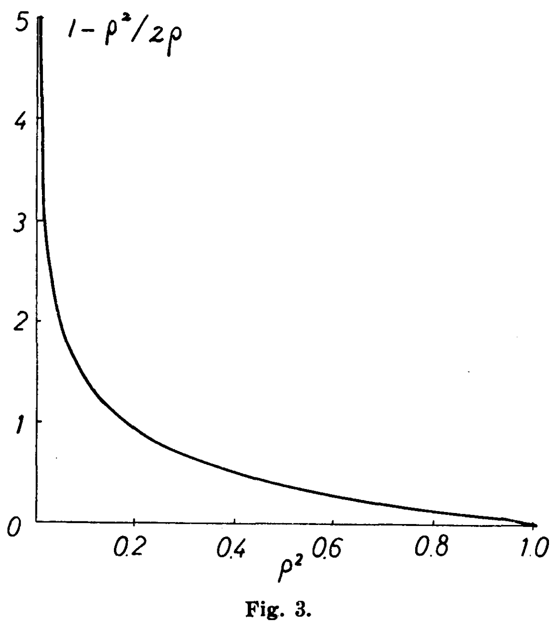

Fig. 3.

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