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Almost Periodic Oscillations in Parametrically Excited Circuits

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Abstruct

In nonlinear circuits which have sinusoidal external sources, sometimes it can be seen that amplitudes and phases of currents and voltages in the circuits are modulated automatically.[1)~4)] We call the phenomena *almost periodic oscillations*, and study them in the case of parametrically excited circuit, and clarify their physical meanings. Their theoretical analysis are developed by phase space method, and we consider that almost periodic oscillaton coresponds to unstable periodic one. It is shown that the method is very useful to investigate such kinds of phenomena.

I. Introduction

It has been known that in nonlinear forced oscillatory circuits automodulating phenomena of oscillatory currents and voltages can be seen,[1)~4)] that is, amplitudes and phases of oscillations do not proceed in steady state. Since their physical meanings has not been clarified yet, we shall investigate them theoretically by method of phase space in this paper, and compare their theoretical results with experimental results.

II. Equations of circuit and their solutions

Before the construction of circuit equations, we set up the following assumptions;

1. Hysteresis phenomena of saturable inducthrs in Fig. 1 can be ignored.

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2. The relation between flux $\tilde{\phi}$ and magnetizing force $\widetilde{H}$ in saturable inductors can be expressed by $\widetilde{H}=\alpha\tilde{\phi}+\beta\tilde{\phi}^3$.
3. Two saturable inductors, T_1 and T_2 in Fig. 1 have the same characteristics.

As shown in Fig. 1 each inductor has winding for DC bias currents, therefore origin of coordinate in $\widetilde{H}-\tilde{\phi}$ curve must be moved.
Then for T_1, $\widetilde{H}_1-H_0=H_1$ and $\tilde{\phi}_1-\phi_0=\phi_1$ and for T_2, $\widetilde{H}_2+H_0=H_2$ and $\tilde{\phi}_2+\phi_0=\phi_2$,
where H_0 and ϕ_0 are magnetizing force and flux respectively by DC bias currents, and H_1, H_2, ϕ_1 and ϕ_2 show new coodinates. Let $\phi_1+\phi_2=u$ and $\phi_1-\phi_2=v$ then $\phi_1=(u+v)/2$, $\phi_2=(u-v)/2$,

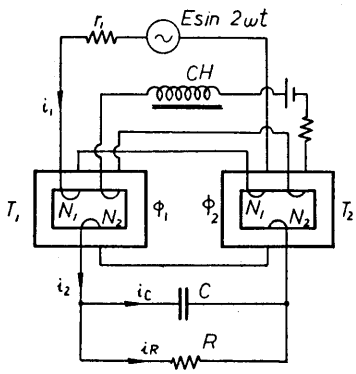

Fig. 1.
Parametrically excited circuits

$$H_1+H_2=\widetilde{H}_1+\widetilde{H}_2=\alpha u+\beta u\left[\frac{1}{4}(u^2+3v^2)+3\phi_0 v+3\phi_0{}^2\right]$$

$$H_1-H_2=\widetilde{H}_1-\widetilde{H}_2=(\alpha+3\beta\phi_0)v+\frac{3}{2}\beta\phi_0(u^2+v^2)+\frac{3}{4}\beta u^2v+\frac{1}{4}\beta v^3$$

In order to construct nondimensional circuit equations, we would derive the following A and B, which have dimensions of magnetizing force and magnetic flux respectively; Thus we decide them, for H A $AT/m=1$ and for ϕ B $weber=1$

From the above relations, circuit equations in Fig. 1 are

$$E\sin 2\omega t=r_1 i_1+BN_1\cdot(\dot{\phi}_1-\dot{\phi}_2) \tag{1}$$

$$BN_2(\dot{\phi}_1+\dot{\phi}_2)+\frac{1}{C}\int i_c\,dt=0 \tag{2}$$

$$\frac{1}{C}\int i_c\,dt=Ri_R \tag{3}$$

$$i_c=i_2-i_R \tag{4}$$

$$\left.\begin{aligned}AH_1l&=N_1i_1+N_2i_2\\ AH_2l&=-N_1i_1+N_2i_2\end{aligned}\right\} \tag{5}$$

where l is the length of magnetic path of inductors shown in metric unit. By differentiation of Eq. (2)

$$BN_2(\ddot{\phi}_1+\ddot{\phi}_2)+\frac{1}{C}i_c=0 \tag{2$'$}$$

From Eq. (5)

$$i_2=\frac{Al}{2N_2}(H_1+H_2)$$

thus Eq. (4) is rewritten in the following form

$$i_c = \frac{Al}{2N_2}(H_1+H_2)+\frac{BN_2}{R}\dot{u}$$

And substitiuting the above equation into Eq. (2)′

$$\ddot{u}+\frac{1}{CR}\dot{u}+\frac{Al}{2CBN_2^2}(H_1+H_2)=\ddot{u}+\frac{1}{CR}\dot{u}$$

$$+\frac{Al}{2CBN_2^2}\Big[\alpha u+\beta u\Big\{\frac{1}{4}(u^2+3v^2)+3\phi_0 v+3\phi_0^2\Big\}\Big]=0 \qquad (6)$$

and from Eqs. (1) and (5).

$$\dot{v}+\frac{A\gamma_1 l}{2BN_1^2}\Big[(\alpha+3\beta\phi_0^2)v+\frac{3}{2}\beta\phi_0(u^2+v^2)+\frac{3}{4}\beta u^2 v+\frac{1}{4}\beta v^3\Big]=\frac{E}{N_1 B}\sin 2\omega t \qquad (7)$$

In practice, there is no parallel resistance R in secondary circnit, but winding resistance γ exists as a series resistance. In resonance circuits, when γ is far smaller than reactance of inductor, γ can be transformed into parallel resistance R, and its values are expressed by $R=1/\omega^2 c^2 \gamma$

Letting $\omega t=\tau$, $\dot{u}=\omega\cdot du/d\tau$ and $\ddot{u}=\omega^2\cdot d^2u/d\tau^2$ and substituting the above relations into Eqs. (6) and (7)

$$\frac{d^2u}{d\tau^2}+a_1\frac{du}{d\tau}+d\Big[4\Big(\frac{\alpha}{\beta}+3\phi_0^2\Big)+(u^2+3v^2)+12\phi_0 v\Big]u=0 \qquad (6)'$$

$$\frac{dv}{d\tau}+a_2\Big[4\Big(\frac{\alpha}{\beta}+3\phi_0^2\Big)v+6\phi_0(u^2+v^2)+3u^2v+u^3\Big]=q\sin 2\tau \qquad (7)'$$

where $a_1=\omega c\gamma_2$, $d=Al\beta/8CBN_2^2\omega^2$, $a_2=A\gamma_1 l\beta/8B\omega N_1^2$, $q=E/N_1B\omega$.

As previously mentioned, in a transformer their winding resistance is generally far smaller than their reactance, then by perturbation method, zero approximate shlution of Eq. (7)′ is expressed in the following form,

$$v=-\frac{1}{2}q\cos 2\tau+z \qquad (8)$$

where z is DC component of flux. When saturation characteristics of inductors are symmetrical, clearly DC component term of the above solution is zero. But in this case their characteristics are non-symmetrical by the superposition of DC bias current, then DC component is not zero and is negative. This is shown simply by considering a type of Eq. (7)′, and if AC component increase, evidently absolute values of DC component increase. Of course this DC component of flux can not be recognized as currents and voltages. Therefore approximate solution of Eq. (7)′ can be put as above mentioned. But if DC component is exact constant, it is contradictory, and this can be shown as follows.

Substituting Eq. (8) into (6)′,

$$\frac{d^2u}{d\tau^2}+a_1\frac{du}{d\tau}+d\Big[\Big(\frac{4\alpha}{\beta}+12\phi_0^2+\frac{3}{8}q^2+12\phi_0 z+3z^2\Big)+u^2-3q(z+2\phi)\cos 2\tau\Big]=0$$

(In strict computation, there is a term of cos 4τ, but this term does not act upon the periodic solution of period 2π, therefore it is ignored.)

If z is constant, the above equation is so-called nonlinearlized Mathieu's differential equation, and it is well known that this has periodic solutions of period 2π under some conditions. In order to investigate the nature of solutions of Eqs. (6)′ and (7)′, first we would study nonlinearlized Mathieu's equation, and which is rewritten in the following form, $\ddot{y}+\mu\dot{y}+(1+\delta+\gamma\cos 2t+\varepsilon y^2)y=0$ In this equation, δ and γ responses for the amplitude of periodic solution become as in Fig. 2. Now, we would return to study Eqs. (6)′ and (7)′ again. When oscillation begins to build up, then it will be seen, from Eq. (7) that absolute value of z increases. As the result of this, coefficients of Eq. (7)′ also vary. Moreover, because γ in Fig. 1 is small enough in our case, variation of z lags behind that of amplitude of u. This result shows

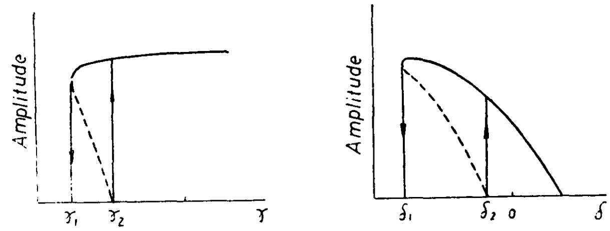

Fig. 2. Natures of periodic solutions for
$\ddot{y}+\mu y+(1+\delta+\gamma\cos 2t)y+\varepsilon y^3=0$

that when oscillation builts up, γ moves into the left hand side in Fig 2-a, that is, γ becomes smaller. And at the moment γ becomes γ_1, oscillation decayes rapidly, and γ moves into the right hand side. When γ bechmes γ_2, oscillation builds up again. This shows that amplitude and phase of u and z are time varying functions, and have limit cycles as relaxation type oscillations.

Such a phenomenon is observed as something of modulated oscillations. For δ, we can guess that the same phenomenon can be observed too. We call it almost periodic or automodulated oscillations. Sometimes such a phenomenon can be seen in circuits with nonliear reactors. Especially, when reactors are biased by *DC* source, it can be easily seen. From the above results, z must be variable.

Let amplitude of u be $\sqrt{\rho}$, and substituting Eq. (8) into (7)′, and picking up only *DC* components which do not contain periodic term, that is, $\cos t$, $\sin t$ and so on, Then (7)′ becomes

$$\frac{dz}{d\tau}+a_1\left[4\left(\frac{\alpha}{\beta}+3\phi_0^2\right)z+6\phi_0\left(\frac{1}{2}\rho+z^2+\frac{1}{8}q^2\right)+\frac{3}{2}\rho z+z^3+\frac{3}{8}q^2z\right]=0$$

and further putting $z+2\phi_0=x$, Eqs. (6)′ and (7)′ become

$$\frac{d^2u}{d\tau^2}+a_1\frac{du}{d\tau}+d\left[\left(3x^2+\frac{4\alpha}{\beta}+\frac{3}{8}q^2\right)+u^2-3qx\cos 2\tau\right]=0 \tag{9}$$

$$\frac{dx}{d\tau}+a^2\left[x^3+\left(\frac{3}{2}\rho+\frac{3}{8}q^2+\frac{4\alpha}{\beta}\right)x-8\phi_0\left(\phi_0^2+\frac{\alpha}{\beta}\right)\right]=0 \tag{10}$$

Equation (9) is rewritten in the following form $d^2u/d\tau^2+u=\nu f(u, du/d\tau, x, \tau)$ where $0<\nu\ll1$ Let $u=\sqrt{\rho}\cos(\tau-\phi)$ and $du/d\tau=-\sqrt{\rho}\sin(\tau-\phi)$ Eq. (9) can be transformed into a polar coodinate system as follows by stroboscopic method,

$$\frac{d\rho}{d\tau}=\nu\frac{\sqrt{\rho}}{\pi}\int_0^{2\pi} f\left(u,\frac{du}{d\tau}, x, \tau\right)\sin(\tau-\phi)\,d\tau,$$

$$\frac{d\phi}{d\tau}=-\nu\frac{1}{2\pi\sqrt{\rho}}\int_0^{2\pi} f\left(u,\frac{du}{d\tau}, x, \tau\right)\cos(\tau-\phi)\,d\tau$$

From the above two equations and Eq. (10), we would construct simultaneous equations

$$\frac{d\rho}{d\tau}=-\rho\left(a_1-\frac{3}{2}dq\,x\sin 2\phi\right) \tag{11}$$

$$\frac{d\phi}{d\tau}=-\left[\frac{1}{2}d\left(3x^2+\frac{4\alpha}{\beta}+\frac{3}{8}q^2\right)-\frac{\alpha}{2\beta}+\frac{3}{8}\rho d-\frac{3}{4}dq\,x\cos 2\phi\right] \tag{12}$$

$$\frac{dx}{d\tau}=-a_2\left[x^2+\left(\frac{3}{2}\rho+\frac{3}{8}q^2+\frac{4\alpha}{\beta}\right)x-8\phi_0\left(\phi_0{}^2+\frac{\alpha}{\beta}\right)\right] \tag{13}$$

Periodic solutions of Eqs. (6)' and (7)' are given by ρ, φ and x which satisfy $d\rho/d\tau=0$, $d\varphi/d\tau=0$ and $dx/d\tau=0$ in Eqs. (11), (12) and (13). But these solutions are nonsense as loug as their stabilities are obscure, therefore we must study whether each solutions is stable or unstable. If we form variational equations from Eqs. (11), (12) and (13), stabilities of solutions can be clarifed by Hurwitz's method. Since this method is very complicated in our case, we would study them by graphical method as fhllows.

In Eqs. (11) and (12), let $d\rho/d\tau=0$ and $d\varphi/d\tau=0$ and eliminating φ from these two equations.

Then
$$\rho=\frac{4\alpha}{2d\beta}-4x^2-\frac{16\alpha}{3\beta}-\frac{1}{2}q^2\pm\sqrt{4q^2x^2-\frac{16_1{}^2}{9d^2}}$$

This result shows $\rho\sim x$ curves when $d\rho/d\tau=0$ and $d\varphi/d\tau=0$ (curve I).
Next let $dx/d\tau=0$ in Eq. (13), this also shows $\rho\sim x$ curves when $dx/d\tau=0$ (curve II). From the above two results, we can obtain equilibrium points graphically. In Fig. 3 equilibrium points are shown as intersecting points of these two curves. For their stabilities, we can obtain them by signs of $d\rho/d\tau$ and $dx/d\tau$. On the right hand side of curve II, clearly $dx/d\tau<0$; on the left hand side, $dx/d\tau>0$. On the other hand in inside and outside of curve I, $d\rho/d\tau>0$ and $d\rho/d\tau<0$ respectively, which can be understood by considering stabilities of periodic solutions of non-linearlized Mathieu's equation. Thus we can decide stabilities of solutions graphically as in Fig. 3. From the above discussions we see that there exist seven cases for numbers of equilibrium points and their stabilities. These results are shown in Fig. 4 and Table 1.

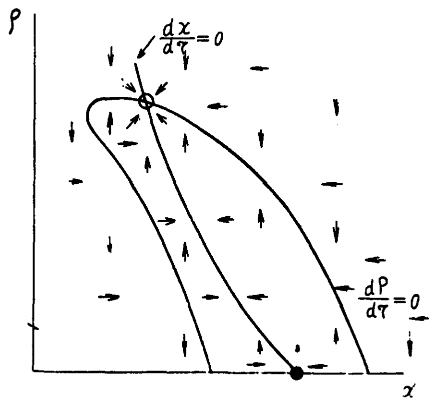

Fig. 3. Stability of periodic solution

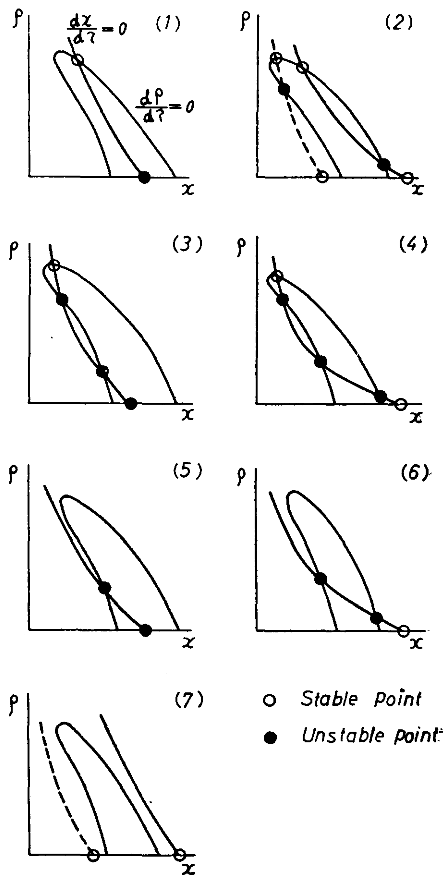

Fig. 4. Classification of equilibrium points.

Table 1.

case	Stabilities ($\rho=0$)	No. of solutions ($\rho\neq0$)	No. of stable and unstable solutions ($\rho\neq0$) stable	unstable
(1)	unstable	one pair	one pair	zero
(2)	stable	two pairs	one pair	one pair
(3)	unstable	three pairs	one pair	two pairs
(4)	stable	four pairs	one pair	three pairs
(5)	unstable	one pair	zero	one pair
(6)	stable	two pairs	zero	two pairs
(7)	stable	zero	zero	zero

III. Almost Periodic Oscillations

As shown in the previous chapter, if ρ, φ and x remain unmoved at equilibrium points, these points correspond to stable periodic solutions of Eqs. (6)′ and (7)′. But in some conditions such points do not exist, then ρ, φ and x become slowly time varying functions. We have previously showed that limit cycles in $\rho\sim\varphi\sim x$ space can exist, and from this result, we can guess that the time which the representative point takes to revolve once around along the limit cycle is independent of the period of external force. It is very difficult to discuss the existence and stabilities of limit cycles, therefore we would define them as follows. When $\rho=0$ is unstable, and unstable equilibrium points which are not zero exist, in addition stable points do not exist, we consider that limit cycles which circulate around the unstable equilibrium points exist, and that they correspond to almost periodic solutions. Case (5) in classification of equilibrium points is an example of these. But such types of solutions also exist under other conditions which are not shown above. We can guess these conditions from the positions and the natures of equilibrium. These are cases (3), (4) and (5) of previous classifications. For example, in case (3), middle unstable equilibrium point corresponds to almost periodic solutions and it is considered that such solutions can exist only when initial points are near by these equilibrium, namely, it is the case when conditions change from case (5) to case (3) continuously. The same results are expected when they change from (5) to (6), from (3) to (4) and from (6) to (4). These statesments are verified by experimental evidences.

IV. Numerical examples

We would calculate periodic and almost periodic solutions by method of previous chapter, and compare them with experimental values. At first, when $\phi_0=1.0$, and $\omega=314$. we calculate ρ in response to external AC source voltage E in each value of capacity C. These relations are shown in Fig 5. From these results, we can find regions of periodic and almost periodic oscillations, which are shown in E–C plane as Fig. 5. Circuit constants are as follows $A=300\ AT/m$, $B=10^{-3}$ weber, $N_1=N_2=620T$. $\gamma_1=\gamma_2=50\Omega$ $\alpha=0.5$ and $\beta=0.5$.

Next we would calculate ρ in response to frequency of external AC source when $C=10\mu F$ and $E=100$ *volts*. These results are shown in Fig. 7 and ϕ_0 is chosen as parameter. And $\phi_0=0.7$, 1.0 and 1.3 correspond to $30mA$, $50mA$ and $75mA$ of DC bias currents respectively, and V_f is expressed by $V_f=10^{-3}\omega\sqrt{\rho}\,N_2\sqrt{2}$

In order to investigate these phenomena, we analized them by analogue computer. Bilinear characteristics are used in place of saturation of inductors, and it is shown in Fig. 8. Fig. 9 shows result of anaiysis. This does not coincide with numerical result, but is useful for qualitative research, and shows that almost periodic oscillations exist in case (3) in Table 1.

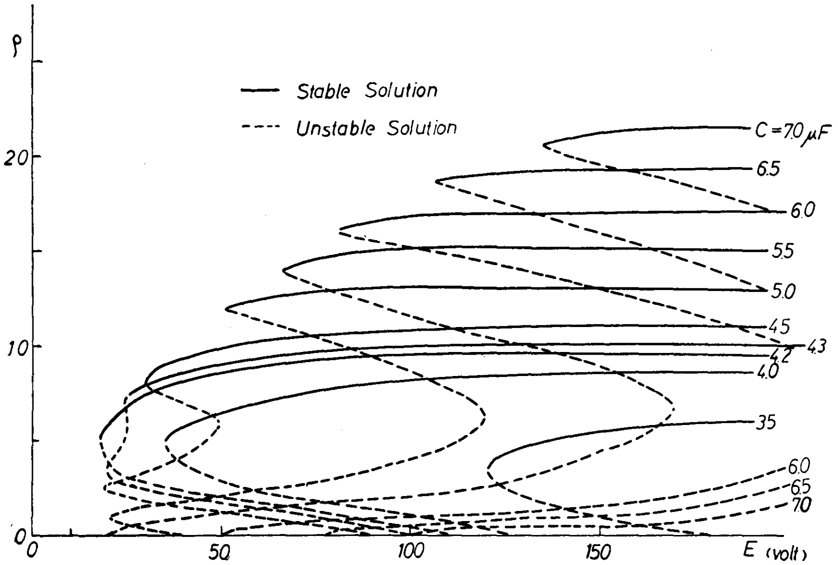

Fig. 5. $E-\rho$ curves for eachvalues of C

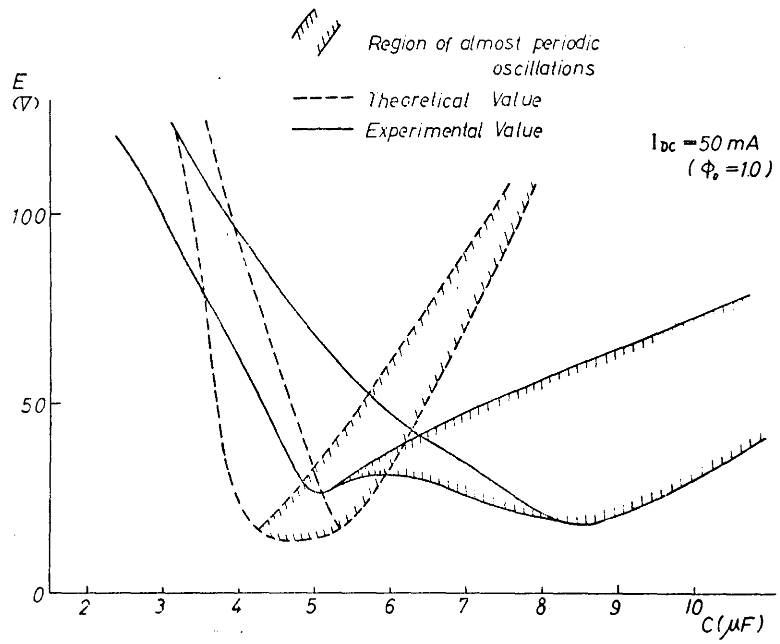

Fig. 6. Regions of almost poriodic oscillations.

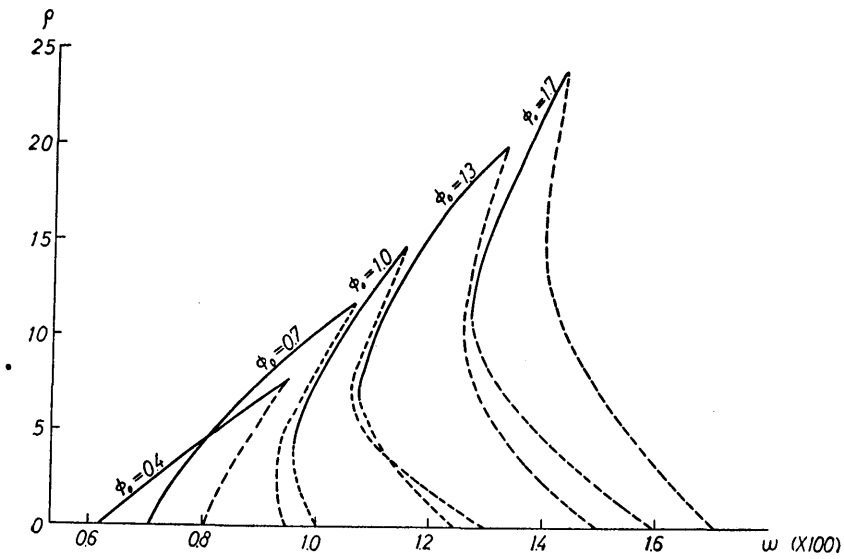

Fig. 7. Frequency response of ρ

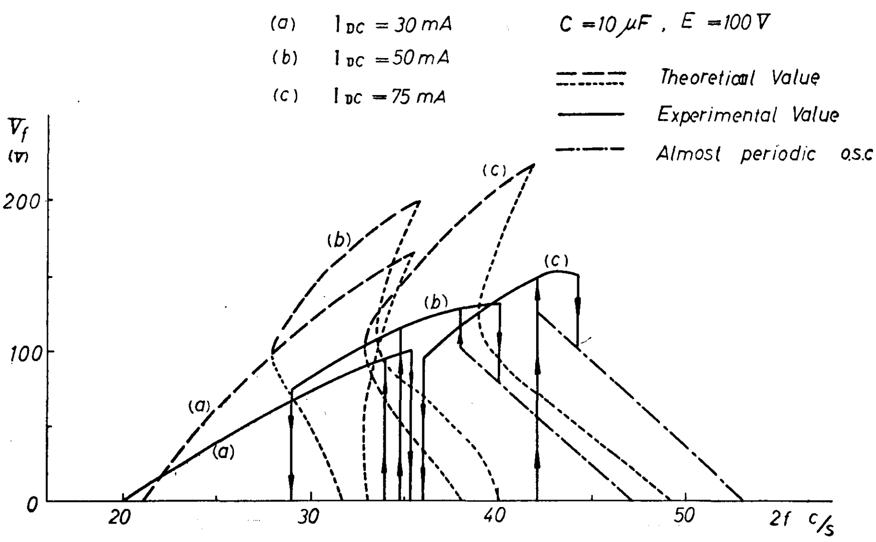

Fig. 8. Frequency response of V_f

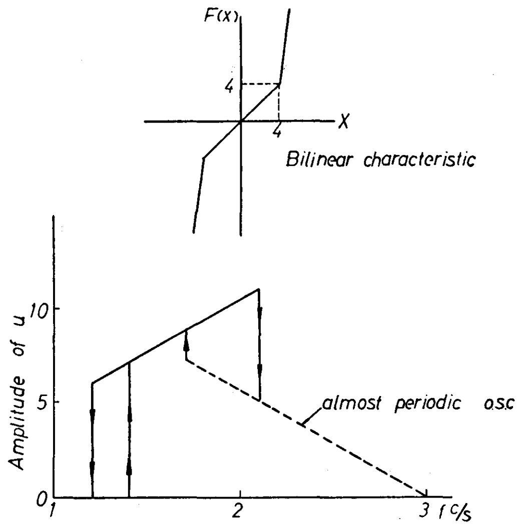

Fig. 9. Analogue computer analysis, for

$$\ddot{u}+a_1\dot{u}+\left[F\left(\frac{u+v}{2}\right)+F\left(\frac{u-v}{2}\right)\right]=0$$

$$\dot{u}+a_2\left[F\left(\frac{u+v}{2}\right)-F\left(\frac{u-v}{2}\right)\right]=q\sin 2\omega t+e_0$$

$a_1=0.1,\ a_2=0.2,\ q=10,\ e_0=3,\ \omega=2\pi f.$

V. Conclusion

We express the parametrically excited circuit as the system of simultaneous differential equatihns of the second and first order. Since the primary circuit of this has nonsymmetrical saturation characteristic because of *DC* bias current, *DC* component of magnetic flux generates in inductors. Therefore coefficients of differential equation which express the secondary circuit vary slowly, and jump phenomena occure, and these phenomena show almost periodic oscillations. Periodic solutions are expressed as equilibrium points in phase space, and we considered that almost periodic oscillations correspond to unstable equilibrium points. It can be seen that results of numerical calculations coincide pretty closely with those of experiments, and is shown that the phase space method is useful for qualitative analysis of automodulating (or almost periodic) oscillations.