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Graphical, Analog, and Digital Solution of Dr. Goodwin's Nonlinear Business Cycle Model *

(Received Sept. 9, 1959)

Kei MORI**

Abstract

The research reported in this paper treats three kinds of solution of Dr. Goodwin's nonlinear business cycle model by the three different approaches.

The first case is a graphical solution by applying a new graphical method which is based upon the Prof. Cunningham's contribution. The second case is an analog solution by the Hitachi's low-speed analog computer comparing with Prof. Strotz and others' analog solution by an electrical analog circuit. And, the final case is a digital solution by the transistor automatic digital computer, Mark IV, which has experimentally produced by Electrotechnical Laboratries in Nov, 1957. Its coding is programmed according to the same numerical method based upon our graphical method.

All methods showed in this papar are directly applicable to Dr. Goodwin's mixed difference-differential equation without expanding the difference term into Taylor's series.

I. Introduction

Goodwin's[1] model (as well as Hick's[2] and Kaldor's[3]) is a representative one of the nonlinear business cycle models that successfully explained the Juglar cycles. The original model of Kaldor's has been clarified by Prof. Yasui[4] of Tōhoku University, and that of Hicks' by Prof. Baumol[5] of the Princeton University and Ass't Prof. Fukuoka of Keio University. However, the study of Goodwin's model

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* Original report in Japanese has been published in the Keio University Centunary Memorial Publication Faculty of Engineering, Nov., 1958.

** 森　　敬 Assistant of Industrial Engineering.

1) R. M. Goodwin; The Nonlinear Accelerator and the Persistence of Business Cycle, *Econometrica*, 19. pp. 1-22, Jan., 1951.

2) J. R. Hicks; *A Contribution to the Theory of the Trade Cycle*, 1950.

3) N. Kaldor; A Model of the Trade Cycle. *Economic Journal*, March, 1940.

4) T. Yasui; Nonlinear Self-excited Oscillations and Business Cycles, Report on the Chicago Meeting, Dec., 1952. *Econometrica*, 21, pp. 470-1, July, 1953.

5) W. J. Baumol; Topology of Second Order Linear Difference Equations with Constant Coefficients, *Econometrica*, 26, pp. 258-285, Apr., 1958.

has made no remarkable progress because of the peculiar form of its equation system —— the differential-difference equation. In view of this fact, the writer has attempted to analyse Goodwin's model from new angles and now wishes to report the result of his research in this paper.

Prior to the writer's work, there have been three studies on Goodwin's model —— the approximative solution by Dr. Goodwin himself[1]; the equivalent linearization solution by Mr. Bothwell[6], a mathematician of the U. S. Navy; and the analog circuit solution[7] by Prof. Strotz of the Northwestern University and two collaborators of his. The methods employed by the writer are to some extent different from these three.

The three different methods were employed in approaching Goodwin's model in such a manner that they could work out complementarily, and the writer obtained three solutions more minute and precise: firstly a graphical solution in a new method[8] as proposed by Prof. Cunningham of the Yale University, secondly an analog solution with the aid of a low-speed analog computer, and lastly a digital solution by an automatic digital computer. The programming for the last is based upon the same method of numerical calculation, on which the said graphical solution stands.

II. Structure of Goodwin's Model

What is referred to as Goodwin's model in this paper means the final form of several models given in Dr. Goodwin's thesis[1]. This final model is made up of two structural equations and one definition.

Consumption Function.

$$C(t)=\alpha Y(t)-\varepsilon\dot{Y}(t)+C_0 \qquad (1)$$

$C(t)$ represents total consumption in period t; $Y(t)$, total income or total output in period t; and C_0, the amount of consumption when income is zero, i. e., the consumption amount required to maintain the minimum standard of living. Parameter α is a marginal propensity to consume, and ε is a time constant related to expenditure lag. $\dot{Y}(t)$ is dY/dt.

Nonlinear Investment Function:

$$I(t)=\phi[\dot{Y}(t-\theta)] \qquad (2)$$

6) F. E. Bothwell; The Method of Equivalent Linearization, *Econometrica*, 20, pp. 269-283, Apr., 1952.

7) R. H. Strotz, J. C. McAnulty and J. B. Naines Jr.; Goodwin's Nonlinear Theory of Business Cycles; An Electro-Analog Solution, *Econometrica*, 21, pp. 390-411, 1953.

8) W. J. Cunningham; Graphical Solution of Certain Nonlinear Differential Difference Equations, *Journal of the Franklin Institute*, 261, pp. 621-9, June, 1956.

$I(t)$ stands for the amount of induced investment; ϕ, the nonlinear acceleration coefficient as shown in Fig. 1. θ is a time lag between rate of change of income and induced investment, i. e., an investment lag. The length of investment lag depends mainly upon the gestation period. The higher the productive stage is, the longer is the investment lag.

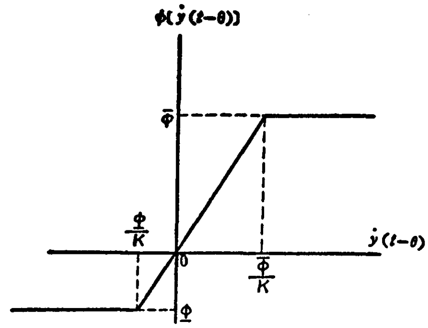


Fig. 1.

Difinition of Income:

$$Y(t)=C(t)+I(t)+A(t) \tag{3}$$

A is spontaneous investment which arises outside the system. Variables are expressed by real value with fluctuation of nominal value having been deflated by the price-index, and they are shown in billions of dollars.

Reduced Equation for Income Fluctuations:

$$\varepsilon\dot{Y}(t)+(1-\alpha)Y(t)=\phi[\dot{Y}(t-\theta)]+C_0+A \tag{4}$$

Let us suppose that (C_0+A) is invariable with respect to time, then Eq. (4) may be transformed into the following relations in terms of $y(t)$ which is an income deviation from unstable equilibrium value $(C_0+A)/(1-\alpha)$:

$$\varepsilon\dot{y}(t)+(1-\alpha)y(t)=\phi[\dot{y}(t-\theta)] \tag{5}$$

$$\varepsilon\dot{y}(t+\theta)+(1-\alpha)y(t+\theta)=\phi[\dot{y}(t)] \tag{6}$$

Let us call either of the above nonlinear differential-difference equations Goodwin's fundamental equation. In accordance with Kuznets' data of the American economy, values of the parameters are given by Goodwin as follows:

$$\varepsilon=0.5\,\mathrm{T}, \qquad \alpha=0.6, \qquad \theta=1.0\,\mathrm{T}, \qquad k=2.0\,\mathrm{T},$$

$$\bar{\phi}=9.0\,\mathrm{M}\cdot\mathrm{T}^{-1}, \quad \underline{\phi}=-3.0\,\mathrm{M}\cdot\mathrm{T}^{-1}.$$

III. Graphical Solution

The graphical method of solution treated here is applicable directly to fundametal equation (5), unlike Goodwin's method in which he expanded the difference term of fundamental equation (6) into a Taylor series and disregarded the third and succeeding terms of the series in an attempt to approximate Eq. (6) to a differential equation.

Since Eq. (5) is a differential-difference equation, an arbitrary function $y(t)$ should be given, as the initial condition, interval from $t=-\theta$ to $t=0$. For convenience, this will be approximated by four points for the $-\theta+\Delta t$, $-\theta+2\Delta t$, $-\theta+3\Delta t$, 0.

$$y(t)=a+b\sin\left(\frac{2\pi}{T}\right)t \tag{7}$$

then we obtain

$$\dot{y}(t)=\left(\frac{2\pi}{T}\right)b\cos\left(\frac{2\pi}{T}\right)t \tag{8}$$

The initial curve is formed by as many initial points as $\frac{\theta}{\Delta t}$. Let Δt be 1/4 year, then the initial curve is given by four initial points because $\theta=1.0$ year.

Here is the first problem that we face after starting from $[y(0), \dot{y}(0)]$, the last one of the four initial points. Where should the next point $[y(\Delta t), \dot{y}(\Delta t)]$ go? The following two conditions must be considered in determining the next point.

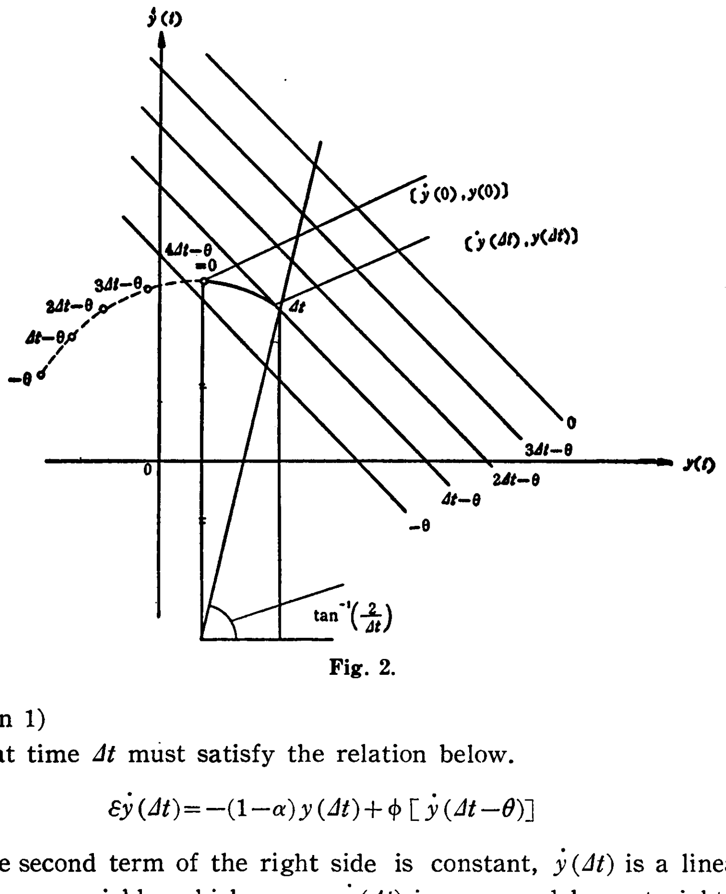


Fig. 2.

(Condition 1)

y and $\dot{y}$ at time Δt must satisfy the relation below.

$$\varepsilon\dot{y}(\Delta t)=-(1-\alpha)y(\Delta t)+\phi[\dot{y}(\Delta t-\theta)] \tag{9}$$

Because the second term of the right side is constant, $\dot{y}(\Delta t)$ is a linear function with $y(\Delta t)$ as a variable, which means $\dot{y}(\Delta t)$ is expressed by a straight line in the $y-\dot{y}$ plane. Thus, a group of straight lines to express y and $\dot{y}$ at time $2\Delta t$, $3\Delta t$ and $4\Delta t$ respectively, may be drawn in the $y-\dot{y}$ plane. Parameter $\phi[\dot{y}]$ is given at θ time prior to the given time, $2\Delta t$, $3\Delta t$, or $4\Delta t$. These straight lines are contours.

In short, Condition 1 means that each one of the wanted points must be on a contour which is to be determined by $\phi[\dot{y}]$ at θ time prior to the given time.

(Condition 2)

Let us give by approximation

$$\frac{\Delta y}{\Delta t}=\dot{y}_{av} \tag{10}$$

then we obtain

$$\Delta y=\dot{y}(0)\Delta t+\left(\frac{\Delta t}{2}\right)\Delta\dot{y} \tag{11}$$

where $\dot{y}_{av}=\dot{y}(0)+\frac{\Delta\dot{y}}{2}$. The first term of the right side is constant, and $\frac{\Delta t}{2}$ is also invariable so long as Δt remains invariable. In the $y-\dot{y}$ plane, therefore, Eq. (11) is expressed by a straight line with a certain incline. For easier graph drawing, let us determine two points for $\Delta y=0$ and $\Delta\dot{y}=0$.
From the former,

$$\dot{y}(\Delta t)=-y(0) \tag{12}$$

is derived, and from the latter,

$$y(\Delta t)=y(0)+\dot{y}(0)\Delta t \tag{13}$$

is derived. If you only note the simple relation given by Eq. (12) and the constancy of the gradient, you would be able to easily draw a straight line satisfying Condition 2.

Here is how to work on the graphs. First of all, produce in the $y-\dot{y}$ plane a contour with $\phi[\dot{y}]$ at time $(\Delta t-\theta)$ as parameter. Next, draw a vertical line to the $y(t)$ axis from the last initial point $[y(0), \dot{y}(0)]$. On the extension of this vertical line, take a point which is equidistant from $y(t)$ axis as $[y(0), \dot{y}(0)]$. Starting from this point, draw a straight line with an incline of $2/\Delta t$ and see where this straight line intersects with the contour that you have drawn. This intersecting point is the wanted position of $[y(\Delta t), \dot{y}(\Delta t)]$. This point $[y(\Delta t), \dot{y}(\Delta t)]$ is the next starting point, and the next contour is determined by $\phi[\dot{y}]$ at time $(2\Delta t-\theta)$. Repeat the same procedure one after another as in the first case,

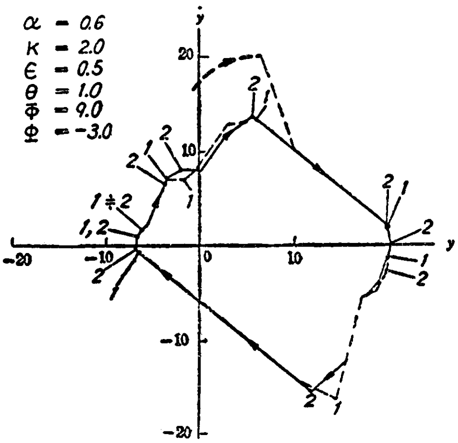


Fig. 3. Graphical Solution; (Mode 1) Major Cycle of Goodwin's Model in the Phase Plane.
$\dot{y}$ axis, unit, a billion dollars/year /year.
y axis, unit, a billion dollars/year.

9) K. Mori, Graphical Solution of Goodwin's Nonlinear Business Cycle Model, *Mita Journal of Economics*, 51, pp. 795-814, Sep., 1958. (in Japanese)

and you will finally obtain the graphical solution of fundamental equation (5) in the $y-\dot{y}$ plane. (See Fig. 3).*

IV. Analog Solution

The analog solution by Prof. Strotz, presented in "The Econometrica" in 1953, differs from the one treated here in that the former was obtained by *the analog circuit of iteration type* whereas the latter by *a low-speed analog computer (ANACOM)*. This difference naturally turns out to be a distinct difference in exactness …… the error of the former is about 10 %; that of the latter is about 1 %.**

In our case, therefore, an electric system equivalent to the economic model can easily be produced by making a block diagram equivalent to the economic model and connecting operators of the computer according to the diagram. If the economic model is thus simulated to a certain degree of exactness, voltage variations to indicate economic fluctuations can be observed in the form of oscillographs or as the movements of beams on oscilloscope.

For convenience's sake, let us transform fundamental equation (5) into

$$\dot{y}(t)=\frac{1}{\varepsilon}\phi[\dot{y}(t-\theta)]-\frac{1-\alpha}{\varepsilon}y(t) \tag{14}$$

The block dirgram equivalent to this relationship requires not only linear operators but nonlinear operators as well. The former consist of an integrator, adder, coefficient multiplier and sign changer; the latter, a function generator of saturative characteristics and a time delay synthesizer.

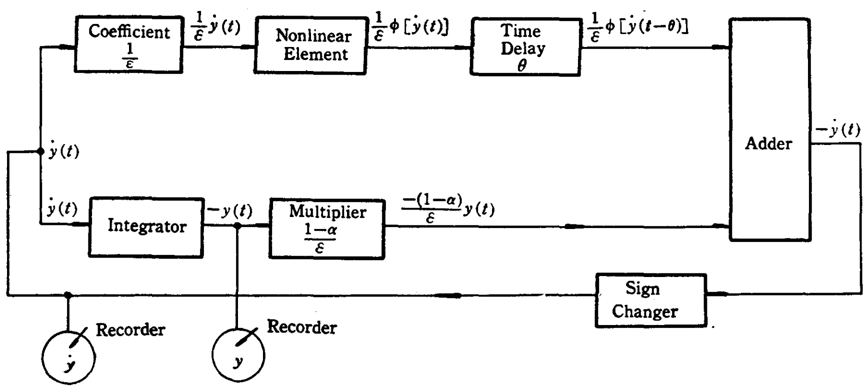


Fig. 4. Block Diagrams of Reduced Equation of Goodwin's Model

$$\dot{y}(t)=\frac{1}{\varepsilon}\phi[\dot{y}(t-\theta)]-\left(\frac{1-\alpha}{\varepsilon}\right)y(t)$$

* See, 4 of Reference (9) for further details of the of the graphical solution.

** Furthermore, ANACOM abounds with operational mobility providing easiness in partial checkups and in the alteration of the system, so that it is suited for setting the structural system (1)——(3) as well as the reduced form (5) or (6).

The time delay used here is of a discrete memory system. Input waves are distributed in sequence to 20 condensers. After a certain period of time, they are taken out of the condensers in order, so that similar waves may be reproduced.

Photos [A] show the solution of fundamental equation (5) with parameters given by Dr. Goodwin. Photo A-a gives a limit cycle in the phase plane $y-\dot{y}$; Photo A-b indicates how $\dot{y}$ changes; and Photo A-c how y changes.*

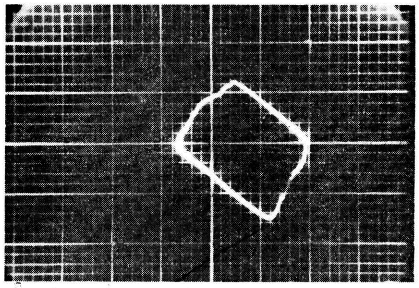
Photos. [A_1]

Analog Solution; (Mode 1) Major Cycle of Goodwin's Model

Phase Plane A-a

Horizontal axis = $y(t)$ axis
(1 div. = a billion dollars/year)

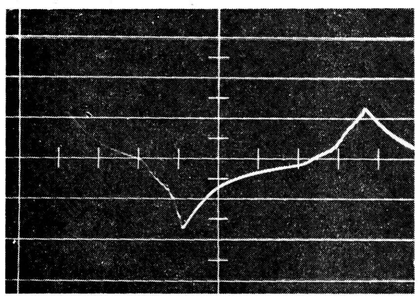
Vertical axis = $\dot{y}(t)$ axis
(1 div. = a billion dollars/year /year)



Time Shape of $\dot{y}(t)$ A-b

Horizonal axis = time axis
(1 div. =1.4 years)

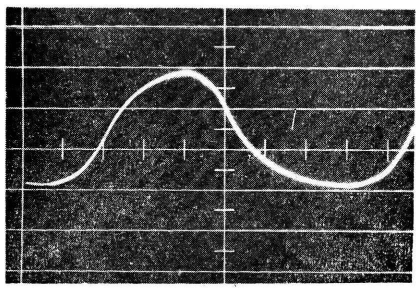
Vertical axis = $\dot{y}(t)$ axis
(1 div. = a billion dollars/year /year)



Time Shape of $y(t)$ A-c

Horizontal axis = time axis
(1 div. = 1.4 years)

Vertical axis = $y(t)$ axis
(1 div. = a billion dollars/year)



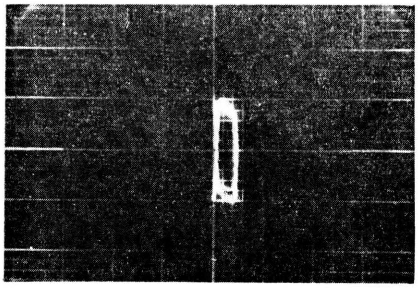
Limit cycles of higher-order mode were observed with only $\bar{\phi}$ changed to 6.0, leaving other parameters unchanged (Photos J'). Cycles of a higher-order mode are inclined to be moved by small noises into those of a lower-order mode.

* The writer used a Hitachi portable low-speed ANACOM. Effects of parameter variations can be revealed more precisely by the digital solution than by the analog solution itself. For this reason, photographs indicating the effects of parameter variations are omitted here.

Photos. [J']

Analog Solution; (Mode 2) Minor Cycle of Goodwin's Model

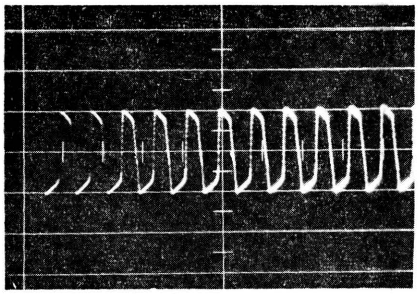
(Case J_2) $\bar{\varphi}=6.0$



Phase Plane J_2-a

Horizontal axis $=y(t)$ axis
(1 div. = a billion dollars/year.)

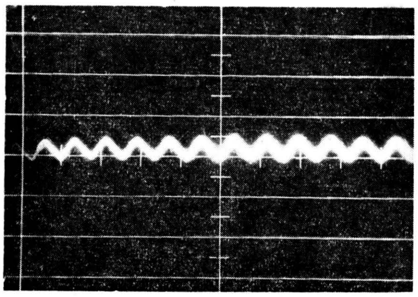
Vertical axis $=\dot{y}(t)$ axis
(1 div. = a billion dollars/year/year.)



Time Shape of $\dot{y}(t)$ J_2-b

Horizontal axis = time axis
(1 div. = 1.4 years)

Vertical axis $=\dot{y}(t)$ axis
(1 div. = a billion dollars/year/year.)



Time Shape of $y(t)$ J_2-c

Horizontal axis = time axis
(1 div. = 1.4 years)

Vertical axis $=y(t)$ axis
(1 div. = a billion dollars/year.)

V. Digital Solution

V. 1 Programming

The graphical solution described in Chap. III of this paper has an advantage that complicated movements of this system can be clarified by a rather simple method. On the other hand, however, it concurrently has a great disadvantage in that such a procedure as is influenced by the graph-drawing skill of an individual concerned must be repeated scores of times, or even hundreds of times. Because of this dependency on the skill, it would be almost impossible to define the limits of error in this method. A way to eliminate the weakness of the graphical solution and to preserve its merit is the use of a digital computer with program-storage. In our case, the programming was performed according to the same method of numerical calculation on which the graphical solution in III stands.*

As described in Chap. III, $y(\Delta t)$ and $\dot{y}(\Delta t)$ are determined by solving the following two equations.

$$\dot{y}(\Delta t)+\frac{(1-\alpha)}{\varepsilon}y(\Delta t)=A \tag{15}$$

where $A=\frac{1}{\varepsilon}\cdot\phi[\dot{y}(\Delta t-\theta)]$.

$$\left(\frac{2}{\Delta t}\right)y(\Delta t)-\dot{y}(\Delta t)=B \tag{16}$$

where $B=\left[\dot{y}(0)+\left(\frac{2}{\Delta t}\right)y(0)\right]$.

Therefore, $y(\Delta t)$ and $\dot{y}(\Delta t)$ are given as follows:

$$y(\Delta t)=\frac{A+B}{\frac{2}{\Delta t}+\frac{1-\alpha}{\varepsilon}} \tag{17}$$

$$\dot{y}(\Delta t)=B-\left(\frac{1-\alpha}{\varepsilon}\right)y(\Delta t) \tag{18}$$

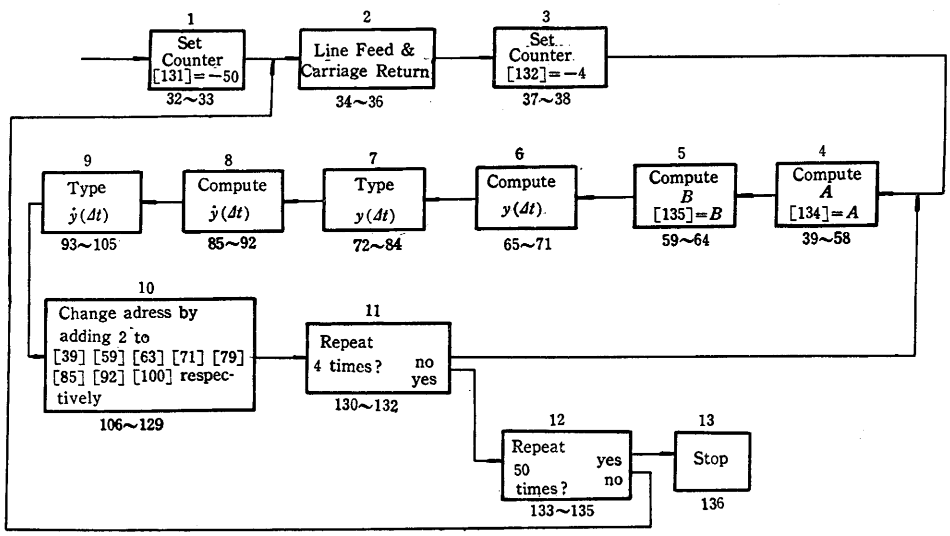


Fig. 5. Programming Flow Chart of Goodwin's Model; (Mode 1)

The flow charting for solving the differential-didifference equation is shown in

* Other methods of numerical calculation would be avairable, of course. However, the use of the same method of numerical calculation as used in the graphical solution would prove more interesting and convenient when you examine the graphical solution or compare results of the graphical solution and the digital solution. The machine used was a transistor digital computer Mark IV, which was experimentally produced by Japan Electrotechnical Laboratries in Nov. 1957.

Fig. 5. Now let us work on coding (See, Table 1) according to this charting (See, Table 2). In doing so, in addition to main program shown in flow chart, we must give orders (or instructions) so that proper addresses can memorize numerical data consisting of parameters, initial values and counters.

This programming is punched in the type, which in turn is set on the computer, then the digital solution of Goodwin's model is typed on paper for every Δt period (See Table 3. in the appendix).

Computational programming for minor cycles is almost the same as that for major cycles. The only differences are that initial values are changed, that a sub-routine to read negative values is added, and that increment Δt is 1/18 as compared with 1/4 in the case of major cycle programming.

Table 1.

Order Code of Mark IV

	Function	*Code*		*Function*	*Code*
A	Add.	02	*Q*	MQR-store	16
*A**	Clear add.	03	*E*	C(Acc) $\geq$ 0 Cond. transfer	20
S	Sub.	04	*G*	C(Acc) $<$ 0 Cond. transfer	24
*S**	Clear sub.	05	*Y*	Round-off	28
V	Multi (+)	06	*L*	Left shift	30
*V**	Clear Multi (+)	07	*R*	Right shift	32
N	Multi (−)	10	*O*	Type out	34
*N**	Clear Multi (−)	11	*W0*	Space	36
D	Division	08	*W1*	Carriage return	36−1
H	Load MDR	12	*W2*	Line feed	36−2
T	Store	14	*W6*	−	36−6
*T**	Clear store	15	*W7*	−	36−7

Table 2.

Programming of Goodwin's Model (Mode 1)

$$A=\frac{1}{\varepsilon}\phi\,[\dot{y}(\Delta t-\theta)] \qquad y(\Delta t)=\frac{A+B}{\dfrac{2}{\Delta t}+\dfrac{1-\alpha}{\varepsilon}} \qquad \dot{y}(t)+(1-\alpha)\,y(t)=\phi\,[\dot{y}(t-\theta)]$$

$$B=\frac{2}{\Delta t}y(0)+\dot{y}(0) \qquad y(\Delta t)=A-\frac{1-\alpha}{\varepsilon}y(\Delta t) \qquad \begin{pmatrix}\varepsilon=0.5 & \theta=1.0 & \bar{\phi}=9\\ \alpha=0.6 & \kappa=2.0 & \underline{\phi}=-3\end{pmatrix}$$

D

		T	14	32	D	
1	32	*S**	05	160	Y	
1	33	*T*	14	137	Y	
			[137] = −50			
2	34	*W*	36	1	Y	
2	35	*W*	36	1	Y	
2	36	*W*	36	2	Y	
3	37	*S**	05	161	Y	
3	38	*T*	14	138	Y	
			[138] = −4			

Group	No.	Op	Code	Address		Remark
	39	*H*	12	171	Y	$\dot{y}(\Delta t-\theta)$
	40	*V**	07	164	Y	
	41	*L*	30	3	Y	
	42	*Y*	28		Y	
	43	*G*	24	52	Y	
	44	*S*	04	165	Y	
	45	*E*	20	49	Y	
	46	*A*	02	165	Y	$0\leqq A<18$
	47	*T*	14	139	Y	[139]
	48	*E*	20	39	Y	
4	49	*A**	03	165	Y	$A=18$
	50	*T*	14	139	Y	[139]
	51	*E*	20	59	Y	
	52	*A*	02	166	Y	
	53	*G*	24	57	Y	
	54	*S*	04	166	Y	
	55	*T*	14	139	Y	
	56	*G*	24	59	Y	
	57	*S**	05	166	Y	
	58	*T*	14	139	Y	$A=-6$ [139]
	59	*H*	12	176	Y	$y(0)$
	60	*V**	07	168	Y	$\frac{2}{\Delta t}y(0)$
	61	*L*	30	3	Y	
5	62	*Y*	28		Y	
	63	*A*	02	177	Y	$\dot{y}(0)$
	64	*T*	14	140	Y	$\frac{2}{\Delta t}y(0)+\dot{y}(0)=B=$ [140]
	65	*A**	03	139	Y	A
	66	*A*	02	140	Y	$A+B$
	67	*T*	14	141	Y	$A+B=$ [141]
6	68	*H*	12	141	Y	
	69	*V**	07	169	Y	
	70	*Y*	28		Y	
	71	*T*	14	178	Y	$y(\Delta t)=$ [178]
	72	*G*	24	79	Y	
	73	*W*	36	5	Y	
	74	*O*	34	3	Y	
	75	*W*	36	7	Y	
	76	*O*	34	2	Y	
	77	*W*	36		Y	
7	78	*E*	20	85	Y	
	79	*S**	05	178	Y	$y(\Delta t)$
	80	*W*	36	6	Y	
	81	*O*	34	3	Y	
	82	*W*	36	7	Y	
	83	*O*	34	2	Y	
	84	*W*	36		Y	
	85	*H*	12	178	Y	$y(\Delta t)$
	86	*V**	07	167	Y	$\times\frac{1-\alpha}{\varepsilon}$
	87	*L*	30	3	Y	
	88	*Y*	28		Y	
8	89	*T*	14	142	Y	$\frac{1-\alpha}{\varepsilon}y(\Delta t)=$ [142]
	90	*S**	05	142	Y	$-\frac{1-\alpha}{\varepsilon}y(\Delta t)$
	91	*A*	02	139	Y	$A-\frac{1-\alpha}{\varepsilon}y(\Delta t)$
	92	*T*	14	179	Y	$\dot{y}(\Delta t)=$ [179]
	93	*G*	24	100	Y	
	94	*W*	36	5	Y	
	95	*O*	34	3	Y	
	96	*W*	36	7	Y	
	97	*O*	34	2	Y	
	98	*W*	36		Y	
9	99	*E*	20	106	Y	
	100	*S**	05	179	Y	
	101	*W*	36	6	Y	
	102	*O*	34	3	Y	
	103	*W*	36	7	Y	
	104	*O*	34	2	Y	
	105	*W*	36		Y	

10	106	*A**	03	39	Y
	107	*A*	02	162	Y
	108	*T*	14	39	Y
	109	*A**	03	59	Y
	110	*A*	02	162	Y
	111	*T*	14	59	Y
	112	*A**	03	63	Y
	113	*A*	02	162	Y
	114	*T*	14	63	Y
	115	*A**	03	71	Y
	116	*A*	02	162	Y
	117	*T*	14	71	Y
	118	*A**	03	79	Y
	119	*A*	02	162	Y
	120	*T*	14	79	Y
	121	*A**	03	85	Y
	122	*A*	02	162	Y
	123	*T*	14	85	Y
	124	*A**	03	92	Y
	125	*A*	02	162	Y
	126	*T*	14	92	Y
	127	*A**	03	100	Y
	128	*A*	02	162	Y
	129	*T*	14	100	Y
11	130	*A**	03	138	Y
	131	*A*	02	163	Y
	132	*G*	24	38	Y
12	133	*A**	03	137	Y
	134	*A*	02	163	Y
	135	*G*	24	33	Y
13	136	*F*	22		Y

	T	14	160	*D*	
160	pp	50	Y		
161	pp	4	Y		
162	pp	2	Y		
163	pp	1	Y		
164		00400	Y	(κ/ε)	
165		01800	Y	$(\bar{\phi}/\varepsilon)$	
166		00600	Y	$(\underline{\phi}/\varepsilon)$	
167		00080	Y	$\left(\frac{1-\alpha}{\varepsilon}\right)$	
168		00800	Y	$\left(\frac{2}{\Delta t}\right)$	
169		11364	Y	$\left(\frac{1}{\frac{2}{\Delta t}+\frac{1-\alpha}{\varepsilon}}\right)$	
170		00073	Y	$(y(\Delta t-\theta))$	
171		01802	Y	$(\dot{y}(\Delta t-\theta))$	
172		00258	Y	$(y(2\Delta t-\theta))$	
173		01923	Y	$(\dot{y}(2\Delta t-\theta))$	
174		00451	Y	$(y(3\Delta t-\theta))$	
175		01997	Y	$(\dot{y}(3\Delta t-\theta))$	
176		00650	Y	$(y(0))$	
177		02022	Y	$(\dot{y}(0))$	
	E	20	32	*D*	
	pp		Y		

V. 2 Digital Solution

In order to see what effects the parameter variations have on the income fluctuations, the writer studies phases of the major cycle in thirteen different cases…… standard case A through case M. (See Fig. 6 through Fig. 17.)

In the standard case, furthermore, the minor cycle of second-order mode and that of third-order mode were both treated (See Fig. 18 and Fig. 19.).

Table 3.

Digital solution of Goodwin's Model * (Mode 1) Case A.

$y(t)$: *unit; a billion dollars/year*

t	$y(t+\Delta t)$	$\dot{y}(t+\Delta t)$	$y(t+2\Delta t)$	$\dot{y}(t+2\Delta t)$	$y(t+3\Delta t)$	$\dot{y}(t+3\Delta t)$	$y(t+4\Delta t)$	$\dot{y}(t+4\Delta t)$
0	010.25	009.80	012.48	008.02	014.30	006.56	015.79	005.37
1	017.01	004.39	018.01	003.59	018.83	002.94	019.50	002.40
2	020.00	001.56	019.99	−001.63	019.32	−003.70	018.23	−004.98
3	016.72	−007.14	013.71	−016.97	009.85	−013.88	006.70	−011.36
4	004.12	−009.30	002.01	−007.61	000.28	−006.22	−001.13	−005.10
5	−002.29	−004.17	−003.24	−003.41	−004.01	−002.79	−004.64	−002.29
6	−005.16	−001.87	−005.59	−001.53	−005.94	−001.25	−006.22	−001.02
7	−006.45	−000.84	−006.64	−000.69	−006.68	000.34	−006.50	001.12
8	−006.16	001.57	−005.74	001.83	−004.86	005.25	−003.31	007.13
9	−001.49	007.47	000.33	007.06	003.15	015.48	006.67	012.66
10	009.55	010.36	011.90	008.48	013.83	006.94	015.41	005.67
11	016.70	004.64	017.76	003.79	018.62	003.10	019.33	002.54
12	019.91	002.07	020.06	−000.89	019.54	−003.23	018.55	−004.68
13	017.27	−005.54	014.67	−015.30	010.92	−014.74	007.57	−012.06
14	004.83	−009.86	002.59	−008.07	000.76	−006.61	−000.74	−005.41
15	−001.97	−004.42	−002.98	−003.62	−003.80	−002.96	−004.47	−002.42
16	−005.02	−001.98	−005.47	−001.62	−005.84	−001.33	−006.14	−001.09
17	−006.39	−000.89	−006.59	−000.73	−006.68	000.02	−006.57	000.90
18	−006.28	001.46	−005.88	001.78	−005.13	004.18	−003.78	006.62
19	−002.02	007.46	−000.18	007.26	002.56	014.67	006.04	013.17
20	009.03	010.78	011.48	008.82	013.48	007.22	015.12	005.90
21	016.46	004.83	017.56	003.95	018.46	003.23	019.19	002.65
22	019.79	002.17	020.03	−000.22	019.65	−002.80	018.75	−004.40
23	017.53	−005.34	015.23	−013.06	011.68	−015.34	008.19	−012.55
24	005.34	−010.27	003.01	−008.41	001.10	−006.88	000.46	−005.63
25	−001.74	−004.61	−002.79	−003.77	−003.65	−003.08	−004.35	−002.52
26	−004.92	−002.06	−005.39	−001.69	−005.77	−001.38	−006.08	−001.14
27	−006.34	−000.93	−006.55	−000.76	−006.67	−000.18	−006.60	000.72
28	−006.34	001.35	−005.96	001.73	−005.30	003.52	−004.09	006.15
29	−002.41	007.33	−000.57	007.38	001.92	012.54	005.22	013.82
30	008.36	011.31	010.93	009.26	013.03	007.58	014.75	006.20
31	016.16	005.07	017.31	004.15	018.25	003.40	019.02	002.78
32	019.65	002.28	020.01	000.59	019.80	−002.24	019.01	−004.09
33	017.85	−005.16	015.91	−010.37	012.60	−016.08	008.95	−013.16
34	005.96	−010.77	003.51	−008. 1	001.51	−007.21	−000.13	−005.90
35	−001.47	−004.82	−002.57	−003.94	−003.47	−003.22	−004.20	−002.64
36	−004.80	−002.16	−005.29	−001.77	−005.69	−001.45	−006.02	−001.18
37	−006.29	−000.97	−006.51	−000.79	−006.67	−000.46	−006.5	000.60
38	−006.42	001.26	−006.05	001.68	−005.52	002.58	−004.45	005.96
39	−002.80	007.28	−000.95	007.48	001.16	009.39	004.17	014.66
40	007.50	012.00	010.23	009.82	012.46	008.03	014.29	006.57

* As a representative example, only a digital solution of Case A (Mode 1) is shown in Table 3. All of solutions are plotted in phase plane as showed in Fig. 6−19.

Table 3. (Cont'd)

t	$y(t+\Delta t)$	$\dot{y}(t+\Delta t)$	$y(t+2\Delta t)$	$\dot{y}(t+2\Delta t)$	$y(t+3\Delta t)$	$\dot{y}(t+3\Delta t)$	$y(t+4\Delta t)$	$\dot{y}(t+4\Delta t)$
41	015.78	005.38	017.00	004.40	018.00	003.60	018.82	002.94
42	019.49	002.41	019.99	001.61	019.99	−001.59	019.33	−003.70
43	018.25	−004.96	016.6	−006.97	013.76	−017.01	009.89	−013.91
44	006.73	−011.38	004.14	−009.31	002.02	−007.62	000.29	−006.23
45	−001.13	−005.10	−002.29	−004.17	−003.24	−003.41	−004.01	−002.79
46	−004.64	−002.29	−005.16	−001.87	−005.59	−001.53	−005.94	−001.25
47	−006.22	−001.02	−006.45	−000.84	−006.64	−000.69	−006.68	000.34
48	−006.50	001.12	−006.16	001.57	−005.74	001.83	−004.86	005.25
49	−000.31	001.21	−001.49	007.47	000.33	007.06	003.15	015.48

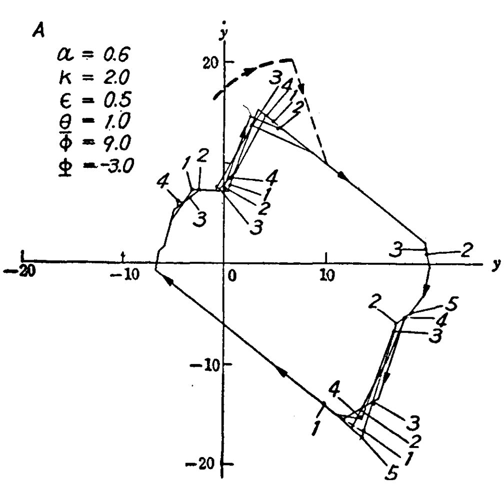


Fig. 6. Digital Solution of Goodwin's Model (Mode 1)

horiz. axis; unit, a billion dollars/year/year
vert. axis; unit, a billion dollars/year

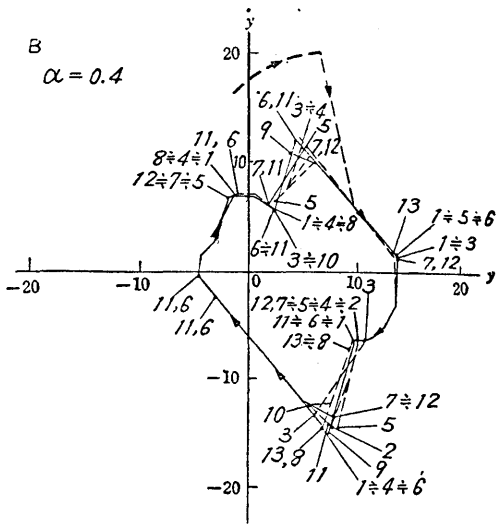


Fig. 7. Digital Solution of Goodwin's Model (Mode 1)

horiz. axis; unit, a billion dollars/year/year
vert. axis; unit, a billion dollars/year

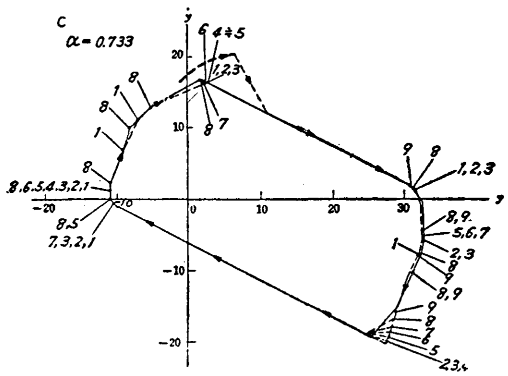


Fig. 8. Digital Solution of Goodwin's Model (Mode 1)
horiz. axis; unit, a billion dollars/year/year
vert. axis; unit, a billion dollars/year

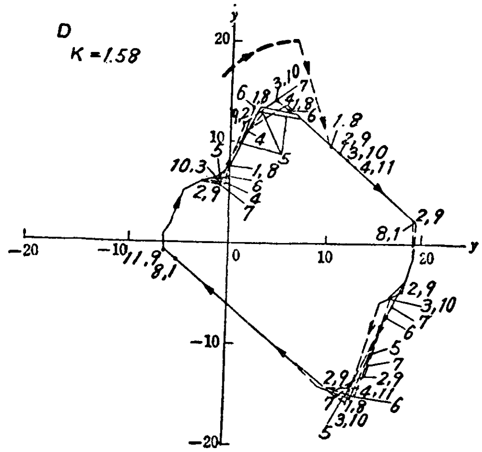


Fig. 9. Digital Solution of Goodwin's Model (Mode 1)
horiz. axis; unit, a billion dollars/year/year
vert. avis; unit, a billion dollars/year

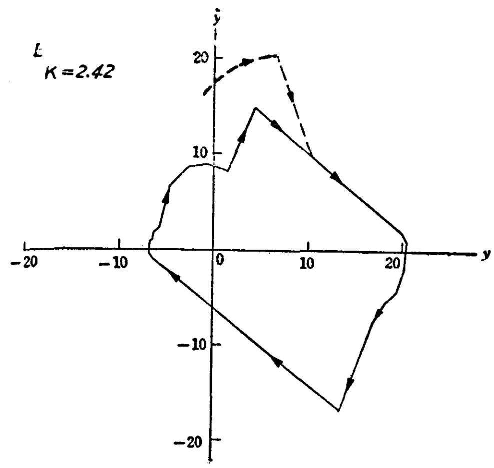


Fig. 10. Digital Solution of Goodwin's Model (Mode 1)
horiz. axis; unit, a billion dollars/year/year
vert. axis; unit, a billion dollars/year

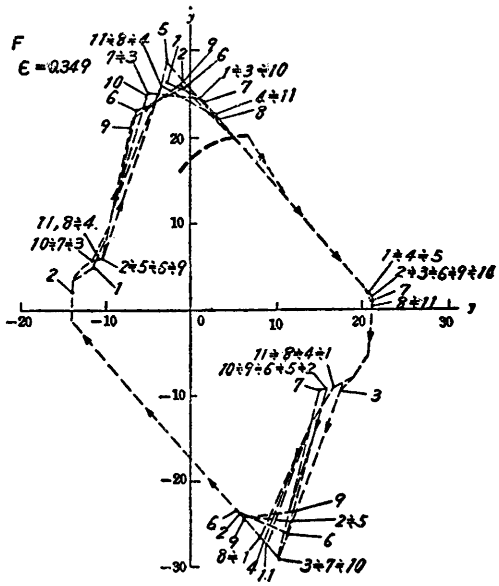


Fig. 11. Digital Solution of Goodwin's Model (Mode 1)
horiz. axis; unit, a billion dollars/year/year
vert. axis; unit, a billion dollars/year

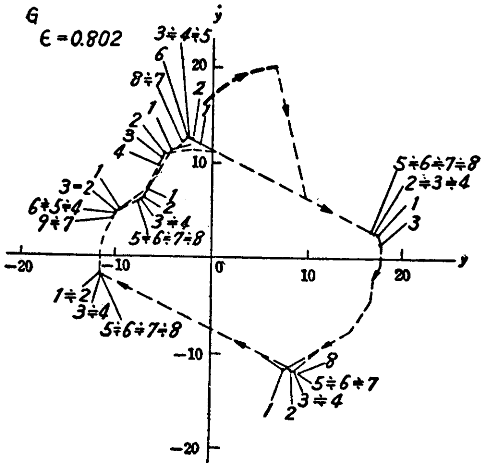


Fig. 12. Digital Solution of Goodwin's Model (Mode 1)
horiz. axis; unit, a billion dollars/year/year
vert. axis; unit, a billion dollars/year

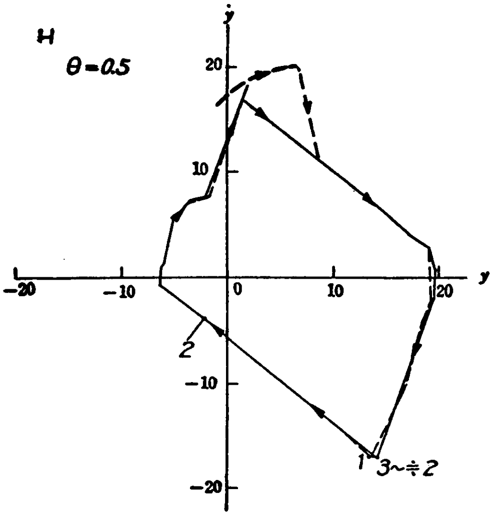


Fig. 13. Digital Solution of Goodwin's Model (Model 1)
horiz. axis; unit, a billion dollars/year/year
vert. axis; unit, a billion dollars/year

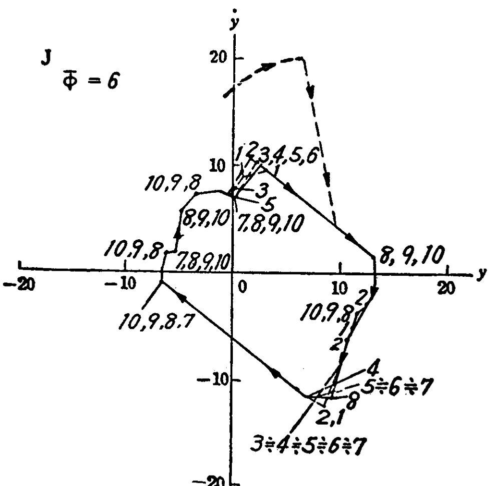


Fig. 14. Digital Solution of Goodwin's Model (Mode 1)
horiz. axis; unit, a billion dollars/year/year
vert. axis; unit, a billion dollars/year

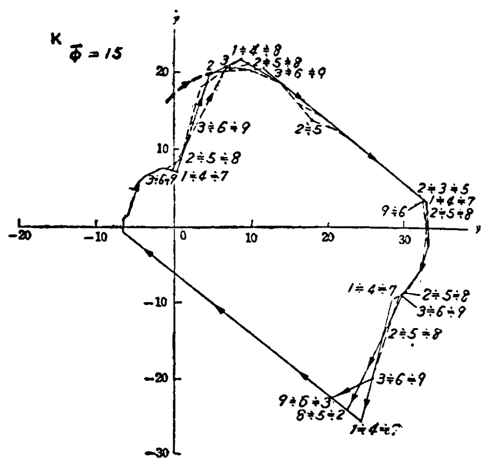


Fig. 15. Digital Solution of Goodwin's Model (Mode 1)
horiz. axis; unit, a billion dollars/year/year
vert. axis; unit, a billion dollars/year

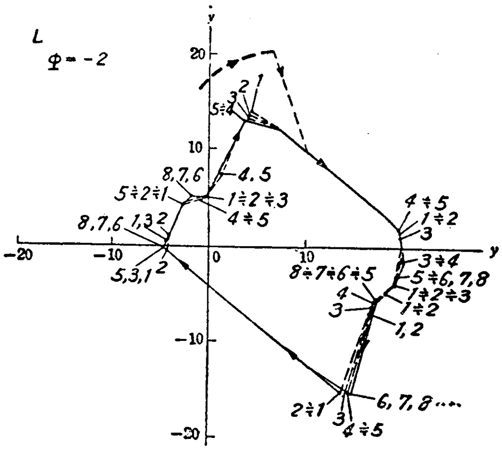


Fig. 16. Digital Solution of Goodwin's Model (Mode 1)
horiz. axis; unit, a billion dolloars/year/year
vert. axis; unit, a billion dollars/year

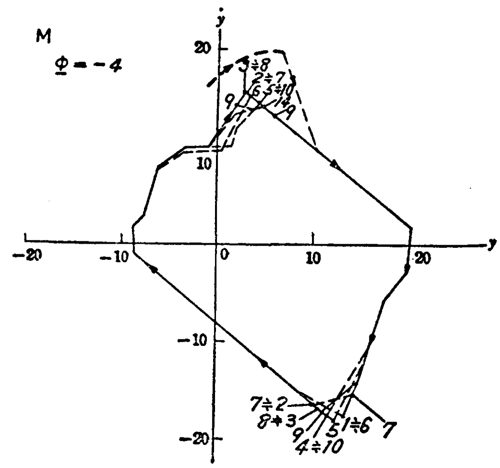


Fig. 17. Digital Solution of Goodwin's Model (Mode 1)
horiz. axis; unit, a billion dollars/year/year
vert. axis. unit, a billion dollars/year

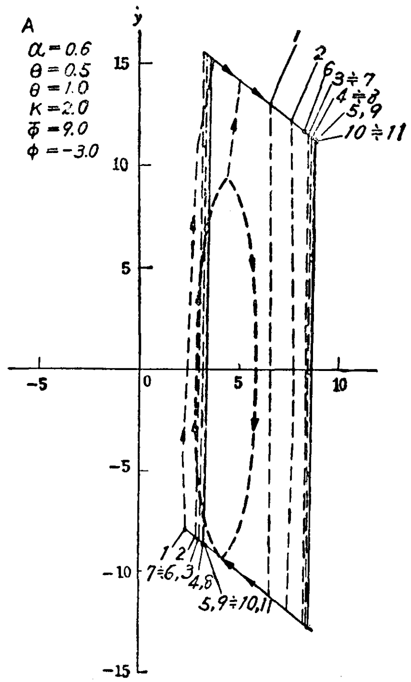


Fig. 18. Digital Solution of Minor Cycle of Goodwin's Model (Mode 2)
horiz. axis; unit, a billion dollars/year/year
vert. axis: unit, a billion dollars/year

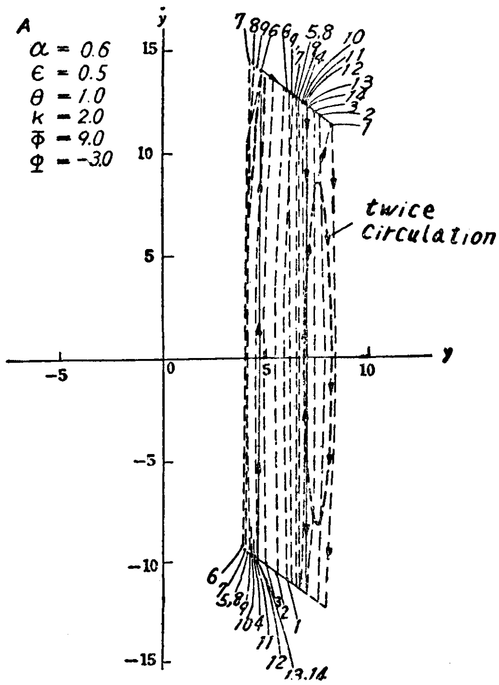


Fig. 19. Digital Solution of Minor Cycle of Goodwin's Model (Mode 3)
horiz. axis; unit, a billion dollars/year/year
vert. axis; unit, a billion dollars/year

Contrary to his conjecture, results were not all definite limit cycles. The writer wishes to mention these peculiar phenomena in the hope that it may be of some help to the study of numerical calculation methods and the theory of functional equations.

i) Definite limit cycles were observed only in cases E, H, J and L, and in minor cycles of second- and third-order modes.

ii) In cases A, B and D, the same cycle occurred every two or three cycles.

iii) In cases C and M, something like a loop, apparently a limit cycle, appeared but it did not make a defiinite loop after all.

iv) In cases F, G and K, the same cycle showed up every two or three cycles, but it was not a decisive one.

There will be much to be examined before we can say whether these peculiar phenomena could be eliminated by the elevation of exactness which may be achieved by making Δt smaller, or they are inevitable phenomena arising from combinations of initial conditions and parameters. Especially, cases F and K are found interesting from the latter point of view.

Table 4.
Comparison of Results
(Income; Unit: a billion dollars/year)

Case	Sort of Solution	Parameter	Period	Maximum level	Minimum level	Amplitude
A_1	Graphical Solution	$\alpha=0.6$	10.00	20.10	−6.68	13.39
	Analog Solution		10.75	19.70	−6.70	12.70
	(Prof. Strotz)	$\kappa=2.0$	(8.12)	(20.00)	(−4.00)	(12.00)
	Digital Solution	$\varepsilon=0.5$	10.00～10.07	19.99～20.06	−6.67～−6.68	13.33～13.37
A_2	Graphical Solution	$\theta=1.0$	1.00	7.95	2.80	2.58
	(Prof. Strotz)		(1.006)	(5.75)	(2.75)	(1.50)
	Digital Solution	$\bar{\phi}=+9.0$	0.96	8.49	3.18	2.66
A_3	(Prof. Strotz)	$\underline{\phi}=-3.0$	(0.535)	(8.25)	(6.75)	(0.75)
	Digital Solution		0.50	6.97	4.61	1.18
B	Analog S.	$\alpha=0.4$	8.50	13.80	−4.40	9.11
	Digital S.		8.04～8.14	13.76～13.87	−4.58～−4.63	9.18～9.26
C	Analog S.	$\alpha=0.74$	14.00	28.40	−10.00	19.20
	Digital S.		11.75～11.79	32.70～32.72	−10.91～−10.92	21.80～21.82

D	Analog *S.*	$\kappa=1.58$	10.50	18.50	−7.20	12.85
	Digital *S.*		9.46∼9.57	19.35∼ 19.50	−6.47∼ −6.50	12.91∼ 13.00
E	Anolog *S.*	$\kappa=2.42$	12.75	19.50	−7.00	13.25
	Digital *S.*		10.50	20.43	−6.81	13.62
F	Analog *S.*	$\varepsilon=0.349$	9.50	22.50	−5.25	13.88
	Digital *S.*		8.14∼8.24	21.02∼ 21.17	−14.02∼ −14.10	17.56∼ 17.62
G	Analog *S.*	$\varepsilon=0.802$	13.00	18.00	−4.60	11.30
	Digital *S.*		11.54∼ 11.57	17.81∼ 17.82	−11.92∼ −11.94	14.87∼ 14.88
H	Analog *S.*	$\theta=0.5$	9.00	16.90	−7.00	11.95
	Digital *S.*	$\theta=0.75$	8.96	19.49	−6.53	13.01
I	Analog *S.*	$\theta=1.5$	11.80	19.00	−7.00	13.00
J_1	Analog *S.*	$\bar{\phi}=6.0$	10.30	12.50	−5.50	9.00
	Digital *S.*		9.00	13.38	−6.68	10.03
J_2	Analog *S.*	$\bar{\phi}=6.0$	1.10	4.50	0.50	2.50
K	Analog *S.*	$\bar{\phi}=15.0$	10.60	30.00	−7.50	18.80
	Digital *S.*		10.64∼ 10.71	33.34∼ 33.44	−6.66∼ −6.68	20.00∼ 20.06
L	Digital *S.*	$\underline{\phi}=-2.0$	11.50	20.05	−4.45	12.25
M	Digital *S.*	$\underline{\phi}=-4.0$	9.71∼ 9.86	20.01	−8.90	14.46

Now let us collect the fruits of the present research from an economic viewpoint. Firstly, the amplitude of income fluctuations is sensitive to the marginal propensity to consume α and the maximum point of investment $\bar{\phi}$, and somewhat responsive to the expenditure lag $\mathcal{E}$ and the minimum point of investment $\underline{\phi}$, but less reactive to the accelerative coefficient κ and the investment lag θ.

However, the period is sensitive to α, and somewhat responsive to κ, $\mathcal{E}$ and $\underline{\phi}$, but least reactive to $\bar{\phi}$. Generally speaking, the amplitude is more sensitive to parameter variations than the period is. These facts would hint something important for the economic stabilization, i. e., the counter business cycle policy which is aimed at the stabilization of the oscillation.

Secondly, we learned the fact that the period is stable to parameter variations —— in other words, the fact advocating or justfying the idea for explaining the Juglar Cycles, which are observed over the period of scores of years, by a model of this kind with constant parameters.

Table 5.

Comparative Dynamics (Income) of Goodwin's Model

Parameter	Sorts of Soluion	Period		Amplitude	
		Sign of Derivative	Sensibility	Sign of Derivative	Sensibility
	Strotz S.	+	Strong	+	Extremely Strong
α	Analog S.	+	Strong	+	Extremely Strong
	Digital S.	+	Strong	+	Extremely Strong
	Strotz S.	+	Weak	+	Extremely Strong
κ	Analog S.	+	Mediate	+	Weak
	Digital S.	+	Mediate	+	Weak
	Strotz S.	+	Strong	↘ ↗ − +	Strong
ε	Analog S.	+	Strong	−	Mediate
	Digital S.	+	Mediate	−	Extremely Strong
	Strotz S.	−	Strong	+	Mediate
θ	Analog S.	+	Mediate	+	Weak
	Digital S.	+	Mediate	+	Weak
	Strotz S.	+	Weak	+	Extremely Strong
$\bar{\varphi}$	Analog S.	↗ ↘ + −	Weak	+	Extremely Strong
	Digital S.	+	Weak	+	Extremely Strong
$\underline{\varphi}$	Digital S.	+	Mediate	+	Mediate

Thirdly, it was confirmed that minor cycles of higher-order mode exist in addition to major cycles. These cycles can never be superimposed on each other because of the non-linearity. The minor cycles are more like triangular waves, and their differential waves are rather like rectangular ones.* This fact means that income level can sometimes go back and forth from one cycle to another if Goodwin's model reflects economic phenomena properly. If this assumption is true, the next problem would be to discover the critical region of the initial condition which confine the system to generate a certain cycle in the income fluctuations. A door to the solution of this problem might be opened, when the anacom is used, by placing initial conditions on the function-generator after defining them closely, and leading the output of the function-generator into the time delay synthesizer in an attempt to test the region in question. When the digital computer is used, the iteration of similar tests would probably work just the same.

Acknowledgement

I am highly indepted to Prof. Y. Chigusa, Ass't Prof. M. Fukuoka, Prof. Z. Nagumo, an expert on nonlinear oscillation, and Prof. Z. Yamauchi of Keio University for their kind guidance and stimulating discussions. In respect to the analog solution, I thank Prof. T. Kose of the Otaru College of Ccmmerce for his valuable suggestion and Mr. T. Numakura of the Central Institute of Hitachi for his joint effort. And I am indebted to Ass't Prof. S. Ura and Lecturer T. Sekine of Keio University and Dr. S. Takahashi of Electrotechnical Laboratries for his helpful advice on the digital solution.

The auther, however, takes personal responsibility for the researches and the conclusions.

* It should be noted that the higher mode of oscillations of digital solution are likely to occur and they are more stable than those of analog solution. In the case of analog solution, the minor cycles seem apt to move the major cycle because of low degree of accuracy of anacom (not because of the proper nature of that system itself).