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Oscillation Represented by the Third Order Differential Equation (Part II)

(Received Sept. 15, 1959)

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Abstract

In the last paper, Part I, published 1956 by the same auther,[1] the nonlinear oscillation represented by the third order differential equation was discussed in the phase space whose axes were displacement x, velocity $y=dx/dt$ and accelaration $Z=d^2x/dt^2$. Hartley and Colpittz Oscillators were analized in the paper. Here, in Part II, two themes are discussed; the first theme is the relation between the coefficient of the third order constant coefficient differential equation and its solution; the second theme is a simple numerical calculation method which we call *Energy function method* (E function method).

IV. Consideration on the Coefficients of the Constant Coefficient Third Order Differential Equation and its Solution

The second order differential equation which represents an oscillation is as follows:

$$\frac{d^2x}{dt^2}+2h\frac{dx}{dt}+\omega^2x=0 \tag{6-1}$$

It can be solved as follows:

$$x=Ae^{-ht}\sin\{(\omega^2-h^2)t+\varphi\} \tag{6-2}$$

In this equation A and φ are arbitrary constants. $2h$, the coefficient of the first order derivative, is called the damping coefficient. Because the oscillation decreases for a positive value of $2h$, and increases for a negative. And for a large value of $|2h|$, the oscillation increases rapidly. ω, the coefficient of x in (6-1), is nearly equal to the angular frequency of the solution (6-2). Thus, we can understand directly the behavior of the second order constant coefficient differential equation without solving it. However, for the third order differential equation, we may not find such a simple character.

If the solution of the third order constant coefficient differential equation is

$$x=Ke^{-at}+Ae^{-ht}\sin(\omega t+\varphi) \tag{6-3}$$

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1) H. Fujita: This Proceedings 8 30 (1955)

then, the differential equation is

$$\frac{d^3x}{dt^3}+(\alpha+2h)\frac{d^2x}{dt^2}+(h^2+\omega^2+2\alpha h)\frac{dx}{dt}+\alpha(h^2+\omega^2)x=0 \tag{6-4}$$

which is obtain easily from (6-3), by differentiating (6-3) three times and eliminating K, A and φ.

In (6-4), we may not find an aparent character of the solution: what term represents the angular frequency ω or damping constant h, or decaying constant α.

But when we add the third order derivative with a small valued coefficient to the second order differential equation, then one can show that the added coefficient of the third order derivative represents a negative damping, proportional to the square of angular frequency ω. Consider the following differential equation;

$$\varepsilon\frac{d^3x}{dt^3}+\frac{d^2x}{dt^2}+h'\frac{dx}{dt}+\omega'^2x=0 \tag{6-5}$$

where $1\gg\varepsilon>0$. Let $\lambda_0=-h+j\omega$ be the characteristic root for $\varepsilon=0$, and λ for $\varepsilon\neq 0$. λ may be expanded to power series

$$\lambda=\lambda_0+\varepsilon\lambda_1+\varepsilon^2\lambda_2+\cdots\cdots\cdots \tag{6-6}$$

The characteristic equation for (6-5) is

$$\varepsilon\lambda^3+\lambda^2+h'\lambda+\omega'^2=0 \tag{6-7}$$

Substitute (6-6) into (6-7) and put the coefficient of ε being zero and we obtain

$$\lambda_0^3+2\lambda_0\lambda_1+h'\lambda_1=0 \tag{6-8}$$

From (6-8),

$$\lambda_1=-\frac{\lambda_0^3}{2\lambda_0+h'}$$

The first approximate solution is

$$\lambda=-h+j\omega-\varepsilon\frac{(-h+j\omega)^3}{2(-h+j\omega)+2h}$$

If we assume $|h|\ll 1$ and neglect the higher power of h, then

$$\lambda=-h+j\omega+\varepsilon\left(\frac{\omega^2}{2}+j\frac{3h\omega}{2}\right)$$

$$=-h+\frac{\varepsilon}{2}\omega^2+j\left(\omega+\frac{3\varepsilon}{2}h\omega\right) \tag{6-9}$$

In the real part of λ, $\frac{1}{2}\varepsilon\omega^2$ is added to $-h$. It is proportional to ω^2 and has opposite sign to $-h$. Therefore, we could say it negative damping.

If the third order differential equation is following type:

$$\frac{d^3x}{dt^3}+h'\frac{d^2x}{dt^2}+\omega'^2\frac{dx}{dt}+\varepsilon x=0 \tag{6-10}$$

The fourth term of (6–10) represents a negative damping, proportional reversely to ω^2. When $\varepsilon=0$, (6–10) has a following solution:

$$x=Ae^{-ht}\sin(\omega t+\varphi)+K \tag{6–11}$$

K is also arbitrary constant as well as A and φ are arbitrary constants and is regarded like a D. C. bias.

When $0<\varepsilon\ll 1$, we get characteristic root λ in the same way

$$\lambda=-h+\varepsilon\frac{1}{2\omega^2}+j\left(\omega-\frac{\varepsilon h}{2\omega^3}\right)$$

So the added term $\varepsilon\frac{1}{\omega^2}$ represents a negative damping.

Let us consider a case when ε is not so small. In this case the canonical form may be considered as

$$\frac{d^3x}{dt^3}+\frac{d^2x}{dt^2}+\frac{dx}{dt}+x=0 \tag{6–12}$$

This coresponds to the second order canonical differential equation;

$$\frac{d^2x}{dt^2}+x=0 \tag{6–13}$$

Intuitively, the solution of (6–12) is considered as being at a balancing state of lst order positive damping and 3rd order negative damping, or 2rd order positive damping and zeroth order negative damping.

The characteristic equation of (6-13) is

$$(\lambda^2+1)(\lambda+1)=0 \tag{6–14}$$

Therefore, the solution is a linear conbination of e^{-t} aud $\sin(t+\varphi)$.

If the characteristic equation is

$$(\lambda^2+1)(\lambda+a)=0 \tag{6-15}$$

and the differential equation has non-periodic solution e^{-at} and periodic solution $\sin(t+\varphi)$ then differential equation for (6–15) is

$$\frac{d^3x}{dt^3}+a\frac{d^2x}{dt^2}+\frac{dx}{dt}+ax=0$$

Thus, when the coefficients of zeroth and second order are same or the coefficients of the first and the third order are same, the positive damping and negative damping are to be balanced. When a is large, the oscillation is determined mainly from the zeroth and second order terms, and when a is small, the oscillation is determined from the first and the third order terms.

When the oscillation is damped, the characteristic equation is

$$(\lambda^2+h\lambda+1)(\lambda+a)=0 \tag{6–16}$$

and the differential equation is

$$\frac{d^3x}{dt^3}+(a+h)\frac{d^2x}{dt^2}+(ah+1)\frac{dx}{dt}+ax=0 \tag{6–17}$$

It can be considered that the damping of the first order is greater than the third order negative damping, or the damping of the second order is greater than the zeroth order damping.

Now, let us consider the angular frequency of the oscillation, represented by the third order constant coefficient differential equation.

When the characteristic equation is

$$(\lambda^2+\omega^2)(\lambda+a)=0 \tag{6-18}$$

then differential equation is

$$\frac{d^3x}{dt^3}+a\frac{d^2x}{dt^2}+\omega^2\frac{dx}{dt}+a\omega^2x=0 \tag{6-19}$$

The coefficients of the zeroth and the first order terms are proportional to the square of angular frequency ω.

Let us show some examples here. The differential equation representing Colpittz oscillator is

$$\dddot{x}+\delta\ddot{x}+\dot{x}+f(x)=0$$

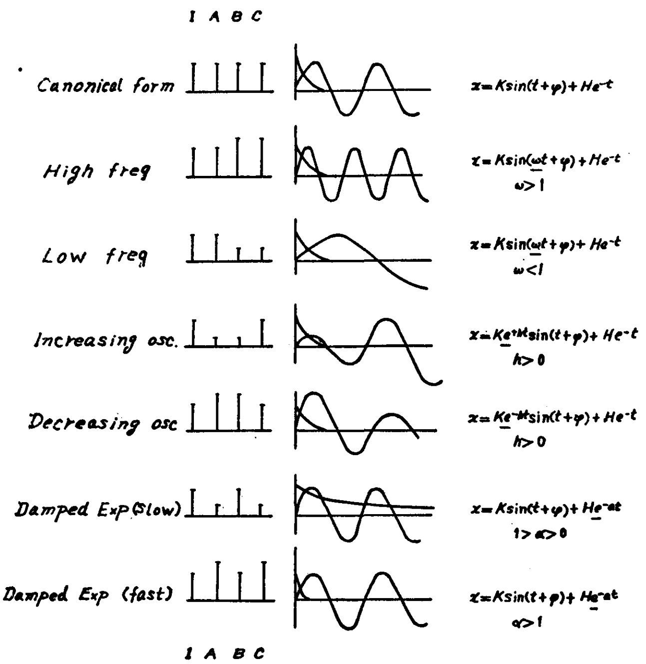
where δ is $1/Q$ and $f(x)$ corresponds to the characteristics of a vacuum tube. When x is not so large, as $f(x)$ is considered linear chracteristics gx, the differential equation is to be considered as

$$\dddot{x}+\delta\ddot{x}+\dot{x}+gx=0$$

When $\delta>g$, the osillation is damped, and when $\delta<g$, it grows up. (See Table. 1.)

Table. 1. Relation between coefficients and solution

$$\frac{d^3x}{dt^3}+A\frac{d^2x}{dt^2}+B\frac{dx}{dt}+Cx=0$$



When x is large and $f(x)$ is saturated to constant value x_0, the differential equation is to be considered as

$$\dddot{x}+\delta\ddot{x}+\dot{x}+x_0=0$$

This has a solution

$$x=Ae^{\alpha t}\sin(\omega t+\varphi)-x_0t+H$$

and oscillation is represented by the 3rd and lst order. Then, 2nd order represents damping.

The summary of the above is in Table. 1..

We do not consider here about the case when the three characteristic roots are all real.

VII. *E* Function Method for the third order differential equation

The customary numerical calculation of differential equations may be so called the isocline method, but recently analogue electronic computers being developed so much, handwork numerical calculations are rarely used. Let us describe on *E* function method which is a simple numerical calculation for differential equations. This method is convenient to observe the solution of differential equations in its broader aspects.

This *E* function method for the second order differential equations is published 1951 by Dr. Tsumura. In this paper *E* function method is applied extensively to the third order differential equations.

Let us rewrite the isocline method in phase plane (or space).

In a rectangular coordinate system whose axes are x, $dx/dt=\dot{x}$ and $d^2x/dt^2=\ddot{x}$,

$$\frac{\dddot{x}}{\ddot{x}}=\frac{d\ddot{x}/dt}{d\dot{x}/dt}=\frac{d\ddot{x}}{d\dot{x}}$$

$$\frac{\ddot{x}}{\dot{x}}=\frac{d\dot{x}/dt}{dx/dt}=\frac{d\dot{x}}{dx}$$

When the third order the differential equation

$$\dddot{x}+F(\ddot{x},\ \dot{x},\ x)=0 \qquad (7\text{-}1)$$

is given, we transform it as follows

$$\left.\begin{aligned}\dot{x}&=y\\ \dot{y}&=z\\ \dot{z}&=-F(x,y,z)\end{aligned}\right\} \qquad (7\text{-}2)$$

From (7-2)

$$\frac{dx}{dy}=\frac{y}{z} \qquad (7\text{-}3)$$

$$\frac{dz}{dy} = -\frac{F(x, y, z)}{z} \tag{7-4}$$

The inclination in $x-y$ plane is determined from (7-3), and the inclination in $z-y$ plane is determined from (7-4). But (7-3) and (7-4) contain divisions by z, so it is troublesome to calculate numerically or to consider intuively.

Now, define E function E_1 and E_2 as follows;

$$E_1 \equiv \frac{1}{2}(x^2+y^2) \tag{7-5}$$

$$E_2 \equiv \frac{1}{2}(y^2+z^2) \tag{7-6}$$

(7-5) is sum of potential energy and kinetic energy.

The derivative for (7-5) in x is

$$\frac{dE_1}{dx} = x+y\frac{dy}{dx} = x+y\left(\frac{z}{y}\right) = x+z \tag{7-7}$$

The derivative for (7-6) in y is

$$\frac{dE_2}{dy} = y+z\left(\frac{-F(x, y, z)}{z}\right) = y-F(x, y, z) \tag{7-8}$$

(7-7) decides the inclination of the solution in E_1-x plane and (7-8) decides the inclination in E_2-y plane.

Both (7-7) and (7-8) are very simpe forms and do not contain division though the isocline method shown in (7-3) and (7-4) contains.

There are following properties in the solution in E_1-x plane and E_2-y plane.

(1) When $y=0$, a curve in E_1-x plane is a parabola $E_1=\frac{x^2}{2}$. Any representative point lies always above or on this parabola. Let us call this parabola P curve. This means potential energy curve. When $z=0$, a curve in E_2-y plane is also a parabola $E_2=\frac{y^2}{2}$.

(2) Any representative point in E_1-x plane is corresponding to two values of y which have the same absolute values and opposite signs. In the same way, any representative point in E_2-y plane is corresponding to two values of z.

(3) A curve representing a simple harmonic vibration is strait lines paralled to x axis in E_1-x plane, and paralled to z axis in E_2-y plane.

For, if we put $\quad x=A\sin t$

then $\quad y=A\cos t$

$\quad z=-A\sin t$

and

$$E_1 = \frac{A^2(\sin^2 t+\cos^2 t)}{2} = \frac{A^2}{2} \quad : \textit{constant}$$

In the same way

$$E_2 = \frac{A^2}{2} \quad : \textit{constant}$$

(4) Exponential curve is parabola in E_1-x plane or in E_2-y plane.

For if we put $\quad x=ke^{\alpha t}$

then $\quad y=\alpha ke^{\alpha t}$

$$z=\alpha^2ke^{\alpha t}$$

So
$$E_1=(1+\alpha^2)\frac{x^2}{2}$$

$$E_2=(1+\alpha^2)\frac{y^2}{2}$$

these are parabolas.

(5) Let a vertical line from a representative point $P_1\,(x_0,\,E_{10})$ in E_1-x plane across to P curve at g_1. Let the corresponding points in E_2-y plane be $P_2(y_0,\,E_{20})$ and g_2 respectively. Then

$$\overline{P_1g_1}=\frac{1}{2}y_0^2$$

$$\overline{P_2g_2}=\frac{1}{2}z_0^2$$

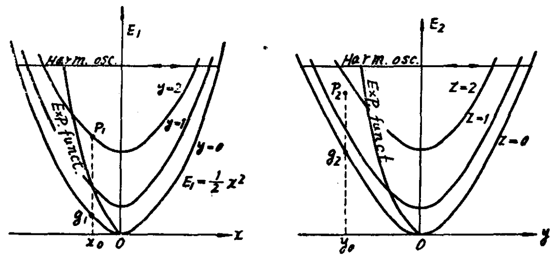


Fig. 1.

These relations are useful for the representative points to transform between E_1-x plane and E_2-y plane.

Let us describe the actual procedure in E function method step by step.

1) Draw parabolas $E_1=\frac{1}{2}x^2$ in E_1-x plane and $E_2=\frac{1}{2}y^2$ in E_2-y plane.

2) Give initial points $P_1\,(x_0,\,y_0)$ in E_1-x plane and $P_2\,(y_0,\,z_0)$ in E_2-y plane.

3) Draw a line from P_1 providing for its inclination is x_0+z_0, and a line from P_2 providing for its inclination is $y_0-F(x_0,\,y_0,\,z_0)$.

4) Take a point P_1' on the line from P_1. The shorter the length $\overline{p_1\;p_1'}$ is, the more exact solution is obtained, but the more troublesome of the procedure increase.

5) Measure the length between P_1' and P curve. It should be measured paralled to E_1 axis.

6) On P curve, try to find a point from which the length to x axis should be equal to the length measured in the previous step. This precedure may be done by the diveder for drawing.

7) Draw a line from the found point parall to E_1 axis. On this line, take a point P_2' which is a crossed point to the line from P_2 in step (3).

Then, P_1' and P_2' are representative points one step advanced from P_1 and P_2.

In these procedures, we took P_1' before P_2', but they might be reversed.: at first P_2', next P_1'.

Remark that the both length in E_1-x plane and E_2-y plane in one step should be restricted in some range in order to keep our accuracy in a specified range.

In this point of view, we must chose the order $P_1'\rightarrow P_2'$ or $P_2'\rightarrow P_1'$.

Now, let us calculate the differential equation representing Colpittz Oscillator with our E function method.

$$\dddot{x}+\frac{1}{Q}\ddot{x}+\dot{x}+g(x)=0$$

Let the nonlinear characteristics of vacuum tube $g(x)$ be

$$g(x)=x-\varepsilon x^3$$

At first we assumed Q of its resonance circuit was about 100.

The result of the numerical calculation by E function is that, for any initia. condition, every trajectry goes to infinity.

For, when $|x|$ increases over the maximum value of $g(x)$, the coefficient of the zeroth order derivative decreases gradually, and will become negative value. This corresponds to case (6) in Table. 1..

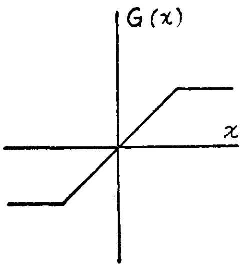


Fig. 2.

Then we represent the nonlinear characteristics $g(x)$ in the broken lines, is shown in Fig. 2.. The solution has a limit cycle but its amplitude is more than 10000.

The condition growing the oscillation up is Q> 1. Assuming that Q=2, the solutions in E_1-x and E_2-y plane are shown in Fig. 3. and 4.. The amplitude of oscillation does not seem to be determined mainly from nonlinearity or loss, but to be determined from other condition—may be condition of grid current.

The steady state solution in E_2-y plane (Fig. 4.) is more distorted than that in E_1-x plane (Fig. 3.), because, in E_2-y plane, the wave is derivated one time more than the one in E_1-x plane.

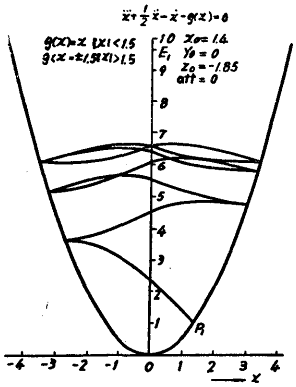


Fig. 3.

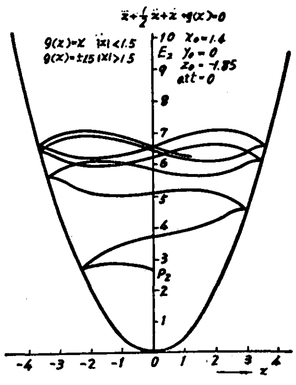


Fig. 4.