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Equations of Motion for Curved and Twisted Beam with Noncoincident Mass and Elastic Axes*

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Abstract

Tne classical equations of equilibrium for the naturally curved deam of A. E. H. Love were established for the case where the centroid lay on the elastic axis. To the beam whose elastic axis does not coincide with the centroid of cross-section, Love's equations are not applicable.

In this paper, the equations of motion for the curved and twisted beam with noncoincident mass and elastic axes are deduced by the use of the concept of warping function.

I. Nomenclature

The following nomenclatures are used in the paper:

P : generic point on a cross-section, to be chosen close to point G or point S

G : centre of gravity on a cross-section

S : centre of shear on a cross-section

C : locus of point P

x, y, z : a svstem of orthogonal axes; x– and y–axes are directed in parallel with the major and minor centroidal principal axes, through the point on the cross-section of beam. z-axis is in the direction tangential to C.

$\bar{x}, \bar{y}, \bar{z}$: an auxiliary system of axes; $\bar{x}$– and $\bar{y}$–axes are in the directions of the major and minor centroidal principal axes. $\bar{z}$–axis is directed in parallel with z–axis through the point G

u, v, w ; deflections of the point P in x–, y– and z–directions, respectivery

θ : angular deflection about z–axis

s : distance from the root of beam along C

κ_{x0}, κ_{y0}, τ_0 : component curvatures and twist of the curve C in the unstressed state

κ_x, κ_y, τ : component curvatures and twist of the curve C in the stressed state

EI_x, EI_y : flexural rigidity about x– and y–axes, respectively

GJ : torsional rigidity

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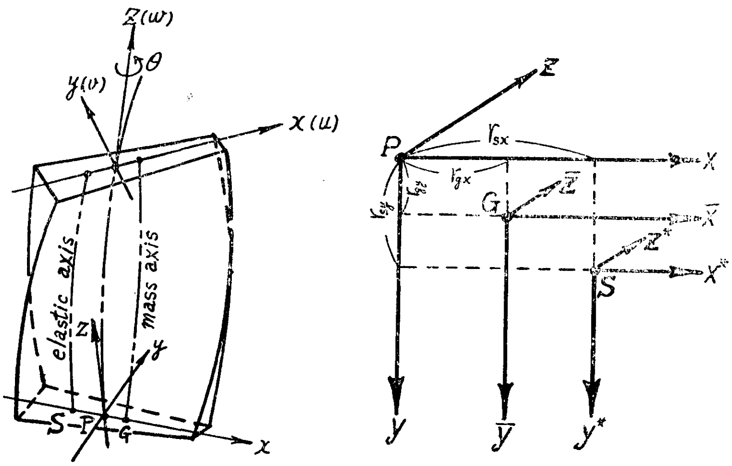
** 佐 藤 武 Lecture at Faculty of Eng., Keio University.



Co-ordinal systems

EC : warping rigidity in reference to x-y-system
A : area of cross-section
ρ : mass density of beam material
l : length of beam
I_G, I_P, $(\equiv I_G + r_g^2 A)$: polar moment of inertia about points G and P, respectively
r_g, r_{gx}, r_{gy} : distance from the point P to the centre of gravity, and its x- and y-components, respectively
r_s, r_{sx}, r_{sy} : distance between the point P and the centre of shear, and its x- and y-components, respectively
φ_P : warping function in reference to x-y-system

$$R_x = \int_A \varphi_P \, y \, dA \,, \qquad R_y = \int_A \varphi_P \, x \, dA \,,$$

Subscripts $\bar{x}$, $\bar{y}$, $\bar{z}$ indicate the value in reference to the $\bar{x}$-, $\bar{y}$-, and $\bar{z}$-axes, respectively.

II. Introduction

The flexural vibrations of a beam, which is twisted like a propeller blade in the unloded state, are being investigated by many authors.[1] These papers are written on the assumptions that the elastic axis and the centroid of a cross-section are initially straight and coincided. In many cases, however, the centure of gravity and the elastic axis do not concide and these axes are naturally curved and twisted, hence Love's equilibrium equations[2] are not applicable. And so far as is known,

1) A. Troesch, M. Anliker & H. Ziegler; Q. Appl. Math., **12** 163. (1954),
R. C. DiPrima & G. H. Handelman; Q. Appl. Math., **12** 241. (1954),
D. D. Rosard, J. Appl. Mech., **20** 241, (1953),
2) A. E. H. Love; A treatise on the Mathematical Theory of Elasticity, Chap. 18, 4th ed. (1952).

general equations of vibration for such a beam have not been found yet.

In this paper, the author deduced the equations of motion for the naturally curved and twisted beam whose elastic axis does not coincide with centroid, under the assumption that the radius of gyration and the distance between the mass and elastic axes are small compared with the beam length.

III. Equations of Motion

a) Potential energy: V

If we assume that the beam under consideration is long, potential energy V may be written for small vibrations

$$V=\frac{1}{2}\int_0^l\left[\int_A E e_{zz}^2 dA+GJ(\tau-\tau_0)^2\right]ds \tag{1}$$

where, e_{zz} is the axial strain of line C, and it is written as follows

$$e_{zz}=\mathcal{E}+(\kappa_x-\kappa_{x0})y-(\kappa_y-\kappa_{y0})x+\frac{d}{ds}\varphi_p(\tau-\tau_0) \tag{2}$$

The last term $\frac{d}{ds}\varphi_p(\tau-\tau_0)$ expresses the warping strain of the cross-section in the torsinal state, and $\mathcal{E}$, κ_x, κ_y, and τ are expressed as follows[3)]

$$\mathcal{E}=\frac{dw}{ds}-u\kappa_{y0}+v\kappa_{x0} \tag{3}$$

$$\left.\begin{aligned} \kappa_x&=\kappa_{x0}+\theta\kappa_{y0}-\frac{dM_3}{ds}-L_3\tau_0\\ \kappa_y&=\kappa_{y0}-\theta\kappa_{x0}+\frac{dL_3}{ds}-M_3\tau_0\\ \tau&=\tau_0+\frac{d\theta}{ds}+L_3\kappa_{x0}+M_3\kappa_{y0} \end{aligned}\right\} \tag{4}$$

$$\left.\begin{aligned} L_3&=\frac{du}{ds}-v\tau_0+w\kappa_{y0}\\ M_3&=\frac{dv}{ds}+u\tau_0-w\kappa_{x0} \end{aligned}\right\} \tag{5}$$

b) Kinetic energy: T

Kinetic energy T is written as follows

$$T=\frac{1}{2}\int_0^l[\rho A(\dot{u}_G^2+\dot{v}_G^2+\dot{w}_G^2)+\rho I_G\dot{\theta}^2+\rho I_{\bar{x}}\dot{M}_3^2+\rho I_{\bar{y}}\dot{L}_3^2]\,ds \tag{6}$$

where, the dot "·" indicates differentiation with respect to time. $\frac{1}{2}\int_0^l\rho I_{\bar{x}}\dot{M}_3^2\,ds$ and $\frac{1}{2}\int_0^l\rho I_{\bar{y}}\dot{L}_3^2\,ds$ are components of the energy due to the rotatory inertia of the cross-cection. And u_G, v_G and w_G are deflection components of the centre of gravity

3) A. E. H. Love; *ibid.* p. 447.

on a cross-section, and these components are expressed by the deflection components (u, v, w) of the P for small vibrations.

$$\left.\begin{aligned} u_G &= u - r_{gy}\theta \\ v_G &= v + r_{gx}\theta \\ w_G &= w - r_{gx}L_3 - r_{gy}M_3 \end{aligned}\right\} \quad (7)$$

c) Equations of motion

Therefore, the following epurtions of motion can be obtained by the use of Hamiltonian principle.

$$\left[\frac{d^2}{ds^2}-\tau_0^2\right]M_x-\left[2\tau_0\frac{d}{ds}+\frac{d\tau_0}{ds}\right]M_y+\left[\kappa_{y0}\frac{d}{ds}+\frac{d\kappa_{y0}}{ds}+\kappa_{x0}\tau_0\right]M_z$$
$$-\kappa_{x0}T+\frac{d}{ds}K_x+\tau_0K_y-\rho A(\ddot{v}+r_{gx}\ddot{\theta})=0 \quad (8)$$

$$\left[\frac{d^2}{ds^2}-\tau_0^2\right]M_y+\left[2\tau_0\frac{d}{ds}+\frac{d\tau_0}{ds}\right]M_x-\left[\kappa_{x0}\frac{d}{ds}+\frac{d\kappa_{x0}}{ds}-\kappa_{y0}\tau_0\right]M_z$$
$$-\kappa_{y0}T-\frac{d}{ds}K_y+\tau_0K_x+\rho A(\ddot{u}-r_{gy}\ddot{\theta})=0 \quad (9)$$

$$\left[\kappa_{x0}\frac{d}{ds}+\kappa_{y0}\tau_0\right]M_x+\left[\kappa_{y0}\frac{d}{ds}-\kappa_{x0}\tau_0\right]M_y+\frac{d}{ds}T+\kappa_{x0}K_x-\kappa_{y0}K_y$$
$$-\rho A\{\ddot{w}-r_{gx}\ddot{L}_3-r_{gy}\ddot{M}_3\}=0 \quad (10)$$

$$\frac{d}{ds}M_z+\kappa_{x0}M_y-\kappa_{y0}M_x-\{\rho I_P\ddot{\theta}-\rho Ar_{gy}\ddot{u}+\rho Ar_{gx}\ddot{v}\}=0 \quad (11)$$

where

$$M_x=\left[EI_{\bar{x}}(\kappa_x-\kappa_{x0})+\frac{d}{ds}ER_x(\tau-\tau_0)+EAr_{gy}\{\varepsilon+(\kappa_x-\kappa_{x0})r_{gy}-(\kappa_y-\kappa_{y0})r_{gx}\}\right] \quad (12)$$

$$M_y=\left[EI_{\bar{y}}(\kappa_y-\kappa_{y0})-\frac{d}{ds}ER_y(\tau-\tau_0)-EAr_{gx}\{\varepsilon+(\kappa_x-\kappa_{x0})r_{gy}-(\kappa_y-\kappa_{y0})r_{gx}\}\right] \quad (13)$$

$$M_z=\left[\begin{aligned} &GJ(\tau-\tau_0)-ER_x\frac{d}{ds}(\kappa_x-\kappa_0)+ER_y\frac{d}{ds}(\kappa_y-\kappa_{y0}) \\ &-\frac{d}{ds}\left\{EC_{pbd}\frac{d}{ds}(\tau-\tau_0)\right\}-\frac{1}{2}\left\{\frac{d^2EC_{pbd}}{ds^2}\right\}(\tau-\tau_0) \end{aligned}\right] \quad (14)$$

$$T=EA[\varepsilon+(\kappa_x-\kappa_{x0})r_{gy}-(\kappa_y-\kappa_{y0})r_{gx}] \quad (15)$$

$$K_x=\rho I_{\bar{x}}\ddot{M}_3-\rho Ar_{gy}\{\ddot{w}-r_{gx}\ddot{L}_3-r_{gy}\ddot{M}_3\} \quad (16)$$

$$K_y=\rho I_{\bar{y}}\ddot{L}_3-\rho Ar_{gx}\{\ddot{w}-r_{gx}\ddot{L}_3-r_{gy}\ddot{M}_3\} \quad (17)$$

Eqs. (8)~(11) have the similar form as Love's equations except that moments M_x, M_y, M_z also K_x, K_y and tention T defined by Eqs. (12)~(17) have not the same contents as the Love's. When r_g and r_s tend to zero, however, these Eqs. (8)~(17) are reduced to the Love's expressons.

Otherwise, boundary donditions at $z=0$ and $z=l$ are obtained as follows

$$\left.\begin{aligned}&\delta\left(\frac{du}{ds}\right)\cdot[M_y+\kappa_{x0}M_z^*]=0\\&\delta u\cdot\left[\frac{dM_y}{ds}+2\tau_0 M_x-\kappa_{x0}M_z-\kappa_{y0}\tau_0 M_z^*-K_y\right]=0\end{aligned}\right\}\quad(18)$$

$$\left.\begin{aligned}&\delta\left(\frac{dv}{ds}\right)\cdot[M_x-\kappa_{y0}M_z^*]=0\\&\delta v\cdot\left[\frac{dM_x}{ds}-2\tau_0 M_y+\kappa_{y0}M_z-\kappa_{x0}\tau_0 M_z^*+K_x\right]=0\end{aligned}\right\}\quad(19)$$

$$\delta w[T+\kappa_{x0}M_x+\kappa_{y0}M_y]=0 \quad (20)$$

$$\delta\left(\frac{d\theta}{ds}\right)\cdot M_z^*=0,\qquad \delta\theta\cdot M_z=0 \quad (21)$$

where

$$M_z^*=\left[ER_x(\kappa_x-\kappa_{x0})-ER_y(\kappa_y-\kappa_{y0})+EC_{pbd}\frac{d}{ds}(\tau-\tau_0)+\frac{1}{2}\left\{\frac{dEC_{pbd}}{ds}\right\}(\tau-\tau_0)\right] \quad (22)$$

d) R_x, R_y **and** EC_{pbd}

Expressions for R_x, R_y and EC_{pbd}, which appear in Eqs. (12)~(14), can be found by bhe same procedure that R. Kappus[4] employed.

$$\left.\begin{aligned}&R_x=\int_A\varphi_p y\,dA=-r_{sx}I_{\bar{x}}\\&R_y=\int_A\varphi_p x\,dA=r_{sy}I_{\bar{y}}\end{aligned}\right\}\quad(23)$$

$$EC_{pbd}=EC_{sbd}+EI_{\bar{x}}r_{sx}^2+EI_{\bar{y}}r_{sy}^2 \quad (24)$$

Where EC_{sbd} is warping rigidity in reference to the coordinate system through the point S.

VI. Concluding Remarks

Equations of motion for the curved and twisted beam, of which mass and elastic axes do not coincide, are obtained. These equations include Love's equations as a particular case where $r_s=r_g=0$.

And, general equations obtained in this paper are deduced in connection with the co-ordinate system (x, y, z) through a generic point P on the cross-section. Therefore, this general equations will simplify the solving of many problems which requied very complicated calculations by the earler methods.

But, the demonstrations of the validity and the applications of the deduced general equations must be the subject of future research.

The author wished to express his thanks to Dr. T. Suhara for his invaluable advice in the course of the work.

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4) R. Kappus; Luftahrtforschung, 14 444, (1937)