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On Virtual Mass of Water contained in a Rectangular Tank whose side-Walls are Vibrating

(Received Feb. 14, 1959)

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Abstract

When the side-walls of a rectangular water tank are vibrating, the water contained therein will also make vibratory motion. In the present paper, the amount of the kinetic energy of water which is in vibratory motion is estimated, and therefrom an approximate formula for the so called virtual mass is deduced. The study is made with respect to four cases namely; (A) The tank is full of water, and the side-walls are vibrating in the same phase each other. (B) The tank is also full of water, and the side-walls are vibrating in opposite phases each other. (C) The tank is almost full of water, but there is a free surface left on top, and the side-walls are vibrating in the same phase each other. (D) The same as the case (C), but the side-walls are vibrating in opposite phases each other. The vibration is assumed to be of infinitesimal amplitude, and the water is regarded as an ideal fluid.

I. Introduction

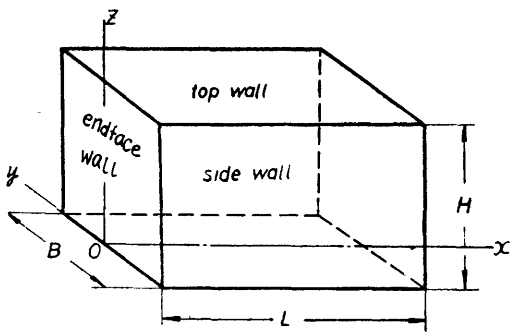

Fig. 1. A sketch of rectangular water tank

Let us consider a rectangular water tank, as shown in Eig. 1. Let us assume that ist length is L, its height H, and its breadth B. In what follows, we shall call, merely for convenience, the two faces of this tank having the dimensions $L\times H$ its side-face walls, while the two faces with dimensions $B\times H$ we shall call its end-faces. The two faces with dimensions $B\times L$ we shall name the top and bottom faces. When the tank is full of water, and the side-walls are making vibratory motion, the particles of water contained in this tank will also vibrate. It is the object of the present paper to estimate the amount of the kinetic energy of water which is in vibratory motion thus set up, and thence to deduce a practical formula for the evaluation of a virtual mass of water contained in the tank. The study is made with respect to four different cases, namely:

(A) The tank is completely full of water, and the side-walls are vibrating in the same phase each other. (B) The tank is also completely full of water, and the side-walls

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are vibrating in oppositephases each other. (C) The tank is almost full of water, but there is a free surface left on top, and the side-walls are vibrating in the same phase each other. (D) The same as the case (C), but the side-walls are vibrating in opposite phase each other.

For each case, the formula giving the amount of the kinetic energy of the whole water was obtained, from which we could deduce the amount of the virtual mass of water corresponding to each state of vibration. The vibration throughout the present paper is assumed to be of small amplitude, and the water is regarded to be an ideal fluid.

II. *Case A*. The tank is full of Water, and the Side Walls are Vibrating in the Same Phase Each Other

Referring to Fig. 2 which shows us the plan of the tank, we first assume that the two side-walls are vibrating in the same phase each other. The tank is assumed to be completely full of water. The transverse displacement of the side wall may be expressed by

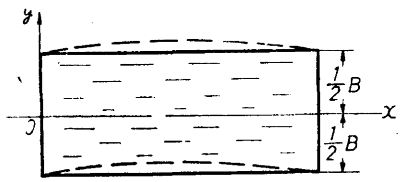

Fig. 2. Vibration mode of the side walls for Case A.

$$w = W_0 \sin mx \sin sz \sin \omega t \tag{1}$$

which the transverse velocity of vibration is given by

$$\frac{\partial w}{\partial t} = A_o \sin mx \sin sz \cos \omega t \tag{2}$$

where W_o is the maximum amplitude of vibration of the side-wall, and $A_o = \omega W_o$. m and s and two constants viz., $m = \pi/L$ and $s = \pi/H$

Assuming the water to be incompressible and non-viscous, the vibratory motion of water is determined by the velocity potential ϕ which satisfies the Laplace's equation $\nabla^2\phi = 0$. Moreover, the velocity potential ϕ must satisfy the suitable boundary conditions. In the present instance, assuming that the two side-walls are vibrating in the same manner expressed by Eq. (2), while the other remaining four walls are kept rigid, we must impose the following conditions upon ϕ:——

(a) at $z = 0$ or H, $\dfrac{\partial \phi}{\partial z} = 0$

(b) at $x = 0$ or L, $\dfrac{\partial \phi}{\partial x} = 0$

(c) at $y = \pm \dfrac{1}{2} B$

$$\frac{\partial \phi}{\partial y} = A_o \sin mx \sin sz \cos \omega t$$

Let us put, as a trial,

$$\phi = B_{oo} f_{oo}(y) + \sum_i{}' \sum_j{}' B_{ij} f_{ij}(y) \cos(mix) \cos(sjz) \tag{3}$$

where $i=0, 1, 2, \cdots\cdots$; $j=0, 1, 2, \cdots\cdots$, and the double summation is to extend to all values of i and j, except i and j equal to zero at the same time. B_{oo} and B_{ij} are the unknown constants not yet determined.

When we put this assumption (3) into the Laplace's equation $\nabla^2\phi=0$, namely

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

we see that, we must have

$$\frac{d^2 f_{oo}(y)}{dy^2} = 0$$

$$\frac{d^2 f_{ij}(y)}{dy^2} - [(mi)^2 + (sj)^2] f_{ij}(y) = 0$$

The solutions of these equations suited to the present case is seen to be

$$f_{oo}(y) = y$$

$$f_{ij}(y) = \sinh(n_{ij}\, y)$$

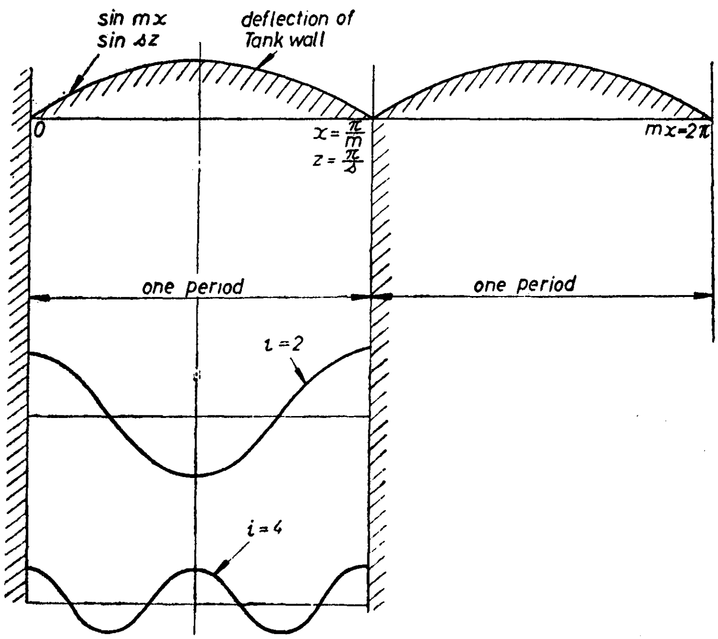

Fig. 3. Wave form for Fourier series expansion (Case A)

with

$$n_{ij} = [(mi)^2 + (sj)^2]^{1/2}$$

By the assumption of (3), the above mentioned conditions (a) and (b) are satisfied by themselves. The boundary condition (c) reduces here to;

$$\begin{aligned} B_{oo} + \sum{}' \sum{}' B_{ij}\, n_{ij} \cosh(\tfrac{1}{2} n_{ij} B) \cos(mi\,x) \cos(sj\,z) \\ = A_o \sin mx \sin sz \cos \omega t \end{aligned} \tag{4}$$

One way of satisfying this relation for $0<x<L$, $0<z<H$ is to regard the left hand side of Eq. (4) as a double Fourier series which represents the right hand side of Eq. (4). This is illustrated in Fig. 3.

In this figure a periodic curve is shown which is obtained by arranging the curve of $\sin mx$ or $\sin sz$ in sucsession in rows (with periode of L or of H).

In order to represent this periodic curve in a Fourier series, the values of i and j must be even integers (zero inclusive). The actual values of B_{oo}, B_{ij} can be obtained from Eq. (4), by multiplying both sides of (4) by 1, $\cos mi\,x$. ($i=0, 2, 4, \cdots\cdots$) and $\cos(sj\,z)$ ($j=0, 2, 4, \cdots\cdots$), and integrating over the range of $0 \leqq x \leqq L$, $0 \leqq z \leqq H$. Thus we have, in turn,

$$B_{oo} \int_0^L dx \int_0^H dz = B_o \int_0^L dx \int_0^H dz \sin mx \text{ s n } sz \cos \omega t$$

whence

$$B_{oo} = \frac{4}{\pi^2} A_o \cos \omega t \tag{5}$$

and

$$B_{ij}\, n_{ij} \cosh\left(\frac{1}{2} n_{ij} B\right) \cdot \frac{\pi^2}{\varepsilon ms} = A_o \int_0^L \sin mx \cos(mix)\, dx$$

$$\times \int_0^H \sin sz \cos(sjz)\, dz \times \cos \omega t$$

where ε is a numerical factor such as for $i \neq 0$, $j \neq 0$; $\varepsilon = 4$, but if $i=0$, $j \neq 0$ or $i \neq 0$, $j=0$, then $\varepsilon = 2$.

Now, we have

$$\int_0^L \sin mx \cos(mix)\, dx = \frac{1}{2} \int_0^L [\sin\{(1+i)\,mx\} + \sin\{(1-i)\,mx\}]dx$$

$$= \frac{1}{2} \Big[\frac{-1}{m(1+i)} \{\cos(1+i)\pi - 1\}$$

$$+\frac{-1}{m(1-i)}\{\cos(i-1)\pi-1\}\Big]$$

$$=\frac{-2}{m(i^2-1)} \qquad (i=0, 2, 4, \cdots\cdots)$$

Also we have

$$\int_0^H \sin sz \ \cos(sjz)\, dz = \frac{-2}{s(j^2-1)} \qquad (j=0, 2, 4, \cdots\cdots)$$

Whence,

$$B_{ij} = \frac{4}{ms(i^2-1)(j^2-1)}\cdot\frac{\mathcal{E}ms}{\pi^2}\cdot\frac{1}{n_{ij}}\cdot\frac{A_o \cos\omega t}{\cosh(\frac{1}{2}n_{ij}B)}$$

$$=\frac{4\mathcal{E}}{\pi^2}\cdot\frac{1}{(i^2-1)(j^2-1)}\cdot\frac{1}{n_{ij}}\cdot\frac{A_o \cos\omega t}{\cosh(\frac{1}{2}n_{ij}B)} \tag{6}$$

Thus all the unknown constants in the Eq. (3) have been determined.

The Kinetic Energy of the Fluid Motion

According to a general theorem in Hydrodynamics, the kinetic energy T_1 of a motion of fluid given by the velocity potential ϕ can be obtained by the formula

$$T_1=\frac{\rho_w}{2}\int\int\phi\frac{\partial\phi}{\partial n}\,dS \tag{7}$$

where ρ_w is the density of the fluid. The double (surface) integral is to extend to the boundary surface of a fluid region now in consideration. dS is the surface element, and $\partial/\partial n$ means the differentiation in the direction of inwardly-drawn normal to the boundary surface.

In the present case, taking the water region to be the whole content of the rectangular tank (Fig. 1), the boundary surfaces consist of its sixface walls. On the four rigid faces among them, we have $\partial\phi/\partial n=0$. On the two remaining faces which are vibrating (Fig. 2), we have

$$\frac{\partial\phi}{\partial n}=\frac{\partial\phi}{\partial y}=A_o \sin mx \sin sz \cos\omega t \tag{8}$$

Putting the values of Eqs. (3) and (8) into Eq. (7), we obtain (integrating over both faces)

$$T_1=\rho_w\int_0^L dx\int_0^H dz\Big[\phi\frac{\partial\phi}{\partial y}\Big]=\rho_w\int_0^L dx\int_0^H dz\Big[A_o \sin mx \sin sz \cos\omega t\Big]$$

$$\times\left[\frac{1}{2}B_{oo}B+\sum{}'\sum{}' B_{ij}\cos(mix)\cos(sjz)\sinh\left(\frac{1}{2}n_{ij}B\right)\right]$$

$$=\rho_w\left[\frac{4}{ms}B_{oo}\times\frac{1}{2}BA_o+A_o\sum{}'\sum{}' B_{ij}\frac{4}{ms(i^2-1)(j^2-1)}\cdot\right.$$

$$\left.\sinh\left(\frac{1}{2}n_{ij}B\right)\right]\cos\omega t$$

Again, putting into this expression, the values of B_{oo} and B_{ij} which we found in the above,

$$T_1=\rho_w A_o^2 LH\left[\frac{8}{\pi^4}B+\sum{}'\sum{}'\frac{4\varepsilon}{\pi^4}\right.$$

$$\left.\times\left\{\frac{1}{(i^2-1)(j^2-1)}\right\}^2\frac{4}{n_{ij}}\tanh\left(\frac{1}{2}n_{ij}B\right)\right]\times\cos^2\omega t \tag{9}$$

We consider that this amount of the kinetic energy is participated by the two faces. Then we have for the kinetic energy per each face-wall;

$$T_w=\frac{1}{2}T_1=\frac{1}{2}\rho_w[A_o^2\cos^2\omega t]\times LHB\times M \tag{10}$$

where M is a non-dimensional factor which is given by

$$M=\frac{8}{\pi^4}+\sum{}'\sum{}'\frac{16\varepsilon}{\pi^4}\left\{\frac{1}{(i^2-1)(j^2-1)}\right\}^2\times\frac{1}{Bn_{ij}}\tanh\left(\frac{1}{2}Bn_{ij}\right) \tag{11}$$

where we have

$$Bn_j=B[(mi)^2+(sj)^2]^{\frac{1}{2}}=\pi\left[\left(\frac{iB}{L}\right)^2+\left(\frac{jB}{H}\right)^2\right]^{\frac{1}{2}}$$

the double summation in Eq. (11) being made for all even integers of i and j, viz., $i=0, 2, 4,\cdots\cdots$; $j=0, 2, 4,\cdots\cdots$ except that the case of $i=0$, $j=0$ simultaneously is not taken in the summation.

III. *Case B.* The tank is full of Water, and the Side Walls are Vibrating on Opposite Phase Each Other

When the tank is completely full of an incompressible fluid, there is no possibility of vibration of the form such as shown in Fig. 4, because this would bring about change in the total volume of the fluid. So that, if we are to consider the case of vibration of the side faces on the opposite phase each other, it must be the case of an even number of half-waves such as is shown in Fig. 5. In the case of Fig. 5, where there occur two half-waves, the motion of the side-faces will be represented by

$$\left.\begin{aligned} w &= W_0 \sin 2mx \sin sz \sin \omega t \\ \frac{\partial w}{\partial t} &= A_0 \sin 2mx \sin sz \cos \omega t \end{aligned}\right\} \tag{12}$$

for one face and with negative sign attached to this expression for the other face, where $A_0 = \omega W_0$ and $m = \pi/L$, $s = \pi/H$, as before.

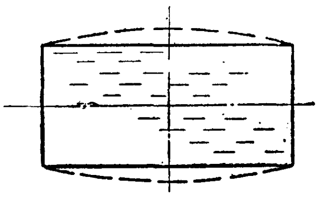

Fig. 4. Vibration of the side-walls on opposite phases each other

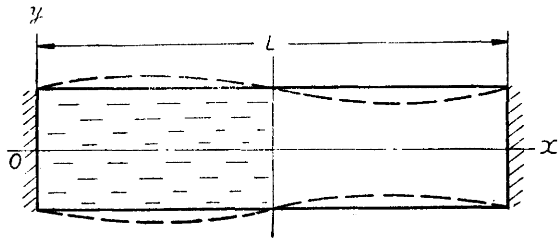

Fig. 5. Vibration mode of the side walls for Case B.

We shall assume the velocity potential ϕ corresponding to the fluid motion thus set up to be of the form:——

$$\phi = \Sigma' \Sigma' B_{ij} f_{ij}(y) \cos(mix) \cos(sjz) \tag{13}$$

wherein

$$f_{ij}(y) = \cosh n_{ij}\, y$$

$$n_{ij} = [(mi)^2 + (sj)^2]^{1/2}$$

and the summation is to extend to all integral values $i = 0, 1, 2, \cdots\cdots$; $j = 0, 1, 2, \cdots\cdots$ except $i = j = 0$ at the same time. Here the term for $i = 0$, $j = 0$ is lacking because of the vibration on opposite phase.

With this assumption of (13), the boundary conditions

$$\frac{\partial \phi}{\partial z} = 0 \qquad \text{for} \quad z = 0 \quad \text{and} \quad H,$$

$$\frac{\partial \phi}{\partial x} = 0 \qquad \text{for} \quad x = 0 \quad \text{and} \quad L,$$

are satisfied. In order to satisfy the condition at the two side-faces $y = \pm \frac{1}{2} B$, we must have

$$\sum' \sum' B_{ij}\, n_{ij} \sinh\left(\frac{1}{2} n_{ij} B\right) \cos(mix) \cos(sjz) \tag{14}$$

$$= A_0 \sin 2mx \sin sz \cos \omega t$$

Thus there arises the same question as before, about representing the right hand side of Eq. (14) in a double Fourier series of the left hand side of Eq. (14). With respect to the variable z, the circumstance is the same as in the previous section. But, with respect to the variable x, the circumstance is quite different, and becomes as shown in Fig. 6.

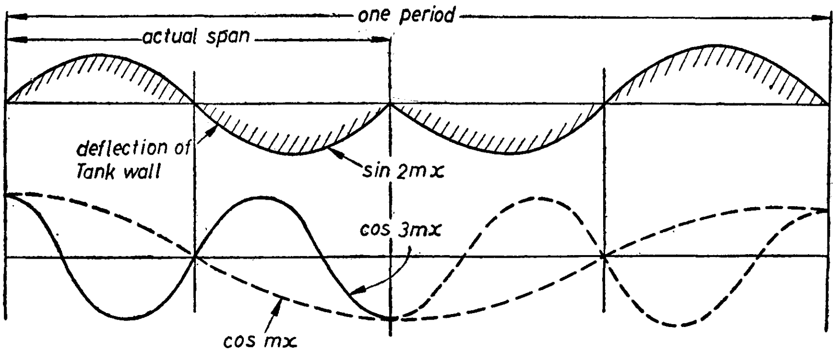

Fig. 6. Wave-form for Fourier series expansion (Case B)

In this figure, a deflection curve of the side-walls is completed into a curve of one period, by reflecting the given curve with respect to the point $x=L$. From the figure, we see that we should take $i=1, 3, 4, \cdots\cdots$ while we should take $j=0, 2, 4, \cdots\cdots$ as before.

Now we have by the actual calculation, i being an odd integer,

$$\int_0^L \sin 2mx \cos mix\, dx = \frac{1}{2}\int_0^L [\sin\{(2+i)mx\} + \sin\{(2-i)mx\}]\, dx$$

$$= \frac{1}{2}\Big[\frac{-1}{(2+i)m}\{\cos(2+i)\pi - 1\}$$

$$+ \frac{-1}{(2-i)m}\{\cos(2-i)\pi - 1\}$$

$$= \frac{1}{2}\Big[\frac{-1}{m}\Big\{\frac{(-1)^i-1}{(2+i)} + \frac{(-1)^i-1}{(2-i)}\Big\}\Big] = \frac{-4}{m(i^2-4)}$$

Also, we have, for $j=0, 2, 4, \cdots\cdots$,

$$\int_0^H \sin sz \cos(sjz)\, dz = \frac{-2}{s(j^2-1)}$$

Thus we have, by multiplying both sides of Eq. (14) by $\cos(mix)\cos(sjz)$ and integrating over the range $0 \leqq x \leqq L$, and $0 \leqq z \leqq H$;

$$B_{ij} = \frac{-4}{m(i^2-4)} \cdot \frac{-2}{s(j^2-1)} \cdot \frac{\mathcal{E}\, ms}{\pi^2} \cdot \frac{1}{n_{ij}} \cdot \frac{A_0 \cos \omega t}{\sinh(\frac{1}{2} n_{ij} B)}$$

or,

$$B_{ij}=\frac{8\mathcal{E}}{\pi^2}\frac{1}{(i^2-4)(j^2-1)}\cdot\frac{A_o\cos\omega t}{u_{ij}\sinh(\frac{1}{2}u_{ij}B)} \tag{15}$$

where $\mathcal{E}=4$ if $i\neq 0$, $j\neq 0$, and $\mathcal{E}=2$ if $i\neq 0$, $j=0$. Thus the unknown constant in the left hand side of Eq. (14) are completely determined.

The Kinetic Energy of the Fluid Motion

As to the kinetic energy of fluid motion, we have, as before

$$T_1=\frac{1}{2}\int\int\phi\frac{\partial\phi}{\partial n}\,dS$$

the integral being to extend to the two faces which are vibrating. As in the previous section, this value of T_1 can be evaluated as follows:——

$$T_1=\rho_w\int_0^L dx\int_0^H dz\left[\phi\frac{\partial\phi}{\partial y}\right]=\rho w\int_0^L dx\int_0^H dz\left[A_o\sin 2mx\sin sz\cos\omega t\right]$$

$$\times\left[\sum{}'\sum{}' B_{ij}\cos(mix)\cos(sjx)\cosh\left(\frac{1}{2}n_{ij}B\right)\right]$$

$$=\rho_w A_o\cos\omega t\left[\sum{}'\sum{}' B_{ij}\frac{8}{ms(i^2-4)(j^2-1)}\cosh\left(\frac{1}{2}n_{ij}B\right)\right]$$

$$=\rho_w A_o^2\cos^2\omega t\left[\frac{8}{ms}\cdot\frac{8\mathcal{E}}{\pi^2}\cdot\sum{}'\sum{}'\left\{\frac{1}{(i^2-4)(j^2-1)}\right\}^2\frac{1}{n_j}\coth\left(\frac{1}{2}n_{ij}B\right)\right]$$

$$=\rho_w A_o^2\cos^2\omega t\ LH\cdot\frac{128}{\pi^4}\sum{}'\sum{}'\left\{\frac{1}{(i^2-4)(j^2-1)}\right\}^2\left(\frac{\mathcal{E}}{2}\right)$$

$$\frac{1}{n_{ij}}\coth\left(\frac{1}{2}n_{ij}B\right) \tag{16}$$

The kinetic energy of the fluid motion which is thought to be participated per one face of the side wall will, therefore, be given by

$$T_w=\frac{1}{2}\rho_w\left[A_o^2\cos^2\omega \mathrm{t}\right]LHB\times M \tag{17}$$

where M is a numerical constant of no-dimension and expressed by

$$M=\frac{128}{\pi^4}\sum{}'\sum{}'\left\{\frac{1}{(i^2-4)(j^2-1)}\right\}^2\cdot\left(\frac{\mathcal{E}}{2}\right)\frac{1}{n_{ij}B}\coth\left(\frac{1}{2}n_{ij}B\right) \tag{18}$$

where the double summation is to be extended for all the values of i and j such that $i=1,3,5,\cdots\cdots;\ j=0,2,4,\cdots\cdots$.

IV. *Case C.* The tank is almost full of Water, but there is a Free Surface left, and the side-Walls Vibrating in Same Phase Each Other

Next, suppose that the tank is almost full of water, but there is a thin vacancy left on top, so that the top surface is in a state of a free surface. The two side-walls are assumed to be on the same phase of oscillation each other, as was shown previously in Fig. 2.

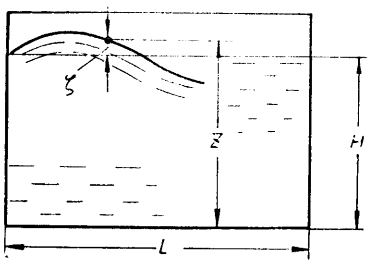

Fig. 7. Surface elevation ζ of the freesurface

Let us first study what condition the velocity potential ϕ must satisfy on the free surface. When in vibration, let the top surface be elevated by an amount ζ. The pressure at any point of altitude z will be given by

$$p = p_0 - \rho_w gz - \rho_w \frac{\partial \phi}{\partial t}$$

for the case of small motions. On the free surface on which $z = H + \zeta$, the pressure must have a constant value. This means that we must have,

$$-\rho_w g\zeta - \rho_w \frac{\partial \phi}{\partial t} = \text{const.}$$

on the free surface. Whence we deduce that

$$-\rho_w g \frac{\partial \zeta}{\partial t} - \rho_w \frac{\partial^2 \phi}{\partial t^2} = 0$$

The velocity of each water particle is given by $V_z = \partial\phi/\partial z$, while on the free surface itself, we have $V_z = \partial\zeta/\partial t$. Combining these relations we have

$$g \frac{\partial \phi}{\partial z} + \frac{\partial^2 \phi}{\partial t^2} = 0$$

on the free surface. This relation may approximately be taken to hold at the original plane surface $z = H$.

Thus we have, for the boundary condition of the free surface, at $z = H$,

$$-\frac{\partial^2 \phi}{\partial t^2} = g \frac{\partial \phi}{\partial z} \tag{19}$$

Now, when the water is vibrating with an angular frequency ω, and with the mode of vibration expressed by

$$\phi = F(y) \cos(mix) \cos(sjz) \cos \omega t \tag{20}$$

we must have, according to the condition (19),

$$\omega^2 \cos(sj\,H) = -sj\ \sin(sj\,H)\cdot g$$

or

$$\cot \xi_j = -\frac{g}{\omega^2 H}\,\xi_j \tag{21}$$

where we put for shortness $\xi_j = sj\,H$.

The roots $\xi_1, \xi_2, \xi_3, \cdots\cdots$ of Eq. (21) can be obtained graphically, by finding the points of intersection of a curve $y = \cot x$ and a straight line $y = (g/\omega^2 H)\,x$, as showh in Fig. 8.

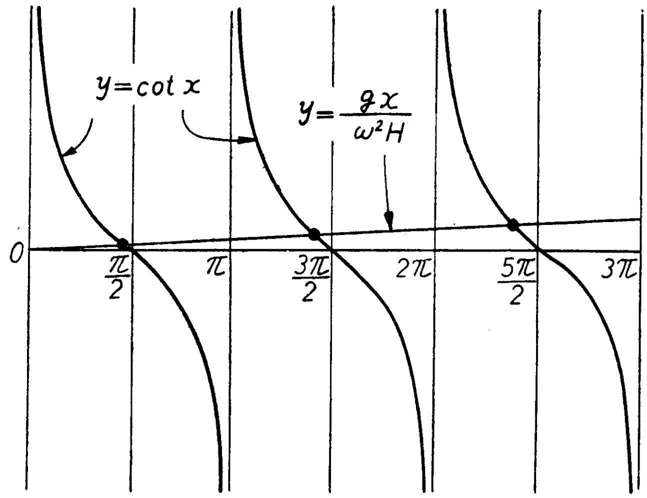

Fig. 8. A graph for finding the roots $\xi_1, \xi_2, \cdots\cdots$ of the equation $\cot x = gx/(\omega^2 H)$

Suppose a tank of height $H = 3$ m, the water contained therein being in vibration with a frequency of $f = 10$ cycles per second. In this case, we have $\omega = 10\times 2\pi$, and so

$$\frac{g}{\omega^2 H} = \frac{9.8}{(20\pi)^2\times 3.1} = \frac{1}{1200}$$

In such a case of comparatively large frequency, the straight line in Fig. 8 becomes very near to x-axis. So that the rook $\xi_1, \xi_2, \xi_3, \cdots\cdots$ (at least the first few of them) is very near to $\pi/2$, $3\pi/2$, $5\pi/2$, $\cdots\cdots\cdots$ and we have approximately

$$sj = \frac{\pi}{H}\left(\frac{1}{2}+\sigma\right) \qquad \sigma = 0, 1, 2, \cdots\cdots$$

This is equivalent to taking, as the condition at the surface $z = H$,

$$p = \text{const.} \qquad \frac{\partial\phi}{\partial t} = 0 \tag{22}$$

instead of Eq. (19), for the case in which the frequency of vibration is comparatively large.

After this preliminary remark has been made, we shall turn to our problem, stated at the begining of this section. Here also we take, as in *case A*,

$$\phi = B_{oo}\, f_{oo}(y) + \Sigma'\Sigma'\, B_{ij}\, f_{ij}(y)\, \cos(mix)\, \cos(sjz) \tag{23}$$

The boundary conditions will be

(a) at $z=0$ $\quad \dfrac{\partial\phi}{\partial z}=0$

at $z=H$ $\quad \phi=0$

(b) at $x=0$ or L, $\quad \dfrac{\partial\phi}{\partial x}=0$

(c) at $y=\pm\frac{1}{2}B$

$$\frac{\partial\phi}{\partial y} = A_o \sin mx\, \sin sz\, \cos\omega t$$

where we put as before, $m=\pi/L$, $s=\pi/H$. From the condition (b), we must have $i=0, 1, 2, \cdots\cdots$, while by the codition (a) we have $j=\frac{1}{2}+\sigma$ ($\sigma=0, 1, 2, \cdots\cdots$). Also we have, as before,

$$f_{oo}(y)=y, \qquad f_{ij}(y)=\sinh(n_{ij}\,y)$$

$$n_{ij}=[(mi)^2+(sj)^2]^{1/2}$$

In order that the condition (c) may be satisfied, we must have

$$B_{oo}+\sum{}'\sum{}'\, B_{ij}\, n_{ij}\cosh\left(\frac{1}{2}n_{ij}B\right)\cos(mix)\cos(sjz)$$

$$= A_o \sin mx\, \sin sz\, \cos\omega t \tag{24}$$

with $i=0, 1, 2, \cdots\cdots$ and $j=\frac{1}{2}, \frac{3}{2}, \cdots\cdots$.

The Eq. (24) may be considered as an expansion in the double Fourier series, if an arrangement as shown in Fig. 9 is taken. In this figure, two positive half waves followed by the two negative halfwaves are arranged to compose one complete period of a periodic curve. The actual determination of the unknown coefficients can be effected as before, viz.,

$$B_{oo}=0$$

$$B_{ij}\, n_{ij}\cosh\left(\frac{1}{2}n_{ij}B\right)\left[\frac{\pi^2}{\mathcal{E}ms}\right] = A_o\int_0^L \sin mx\, \cos(mix)\, dx \times$$

$$\int_0^H \sin sz \cos(sjz)\, dx\, \cos\omega t \tag{25}$$

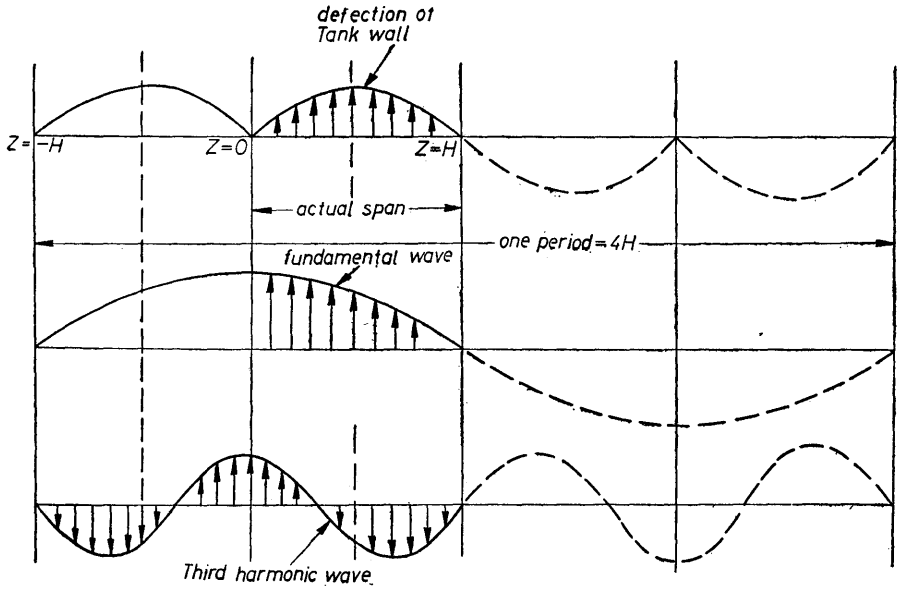

Fig. 9. Wave form for Fourier series expansion (Case C) (Case D)

while we have

$$\int_0^L \sin mx \cos(mix)\, dx = \frac{-2}{m(i^2-1)} \qquad (i=0,1,2,\cdots\cdots)$$

$$\int_0^H \sin sz \cos(sjz)\, dz = \frac{1}{2}\int_0^H [\sin\{(1+j)sz\} + \sin\{(1-j)sz\}]\, dx$$

$$= \frac{1}{2}\left[\frac{1}{(1+j)s} + \frac{1}{(1-j)s}\right] = \frac{-1}{s(j^2-1)}$$

whence, we have

$$B_{ij} = \frac{2}{ms(i^2-1)(j^2-1)} \cdot \frac{\varepsilon ms}{\pi^2} \cdot \frac{1}{n_{ij}} \cdot \frac{A_0 \cos \omega t}{\cosh(\frac{1}{2} n_{ij} B)} \tag{26}$$

The velocity potential ϕ was thus completely determined, and consequently the kinetic energy of the fluid motion can be evaluated. So that we have finally for the *kinetic energy* T_w of the fluid motion refered to each side-face,

$$T_w = \frac{1}{2} \rho_w [A_o^2 \cos^2 \omega t] LHB \times M \tag{27}$$

where

$$M=\frac{8}{\pi^4}\sum{}'\sum{}'\left\{\frac{1}{(i^2-1)(j^2-1)}\right\}^2\left(\frac{\mathcal{E}}{2}\right)\cdot\frac{1}{n_{ij}B}\tanh\left(\frac{1}{2}n_{ij}B\right) \tag{28}$$

$i=0,2,4,\cdots\cdots,\quad j=\frac{1}{2},\frac{3}{2},\frac{5}{2},\cdots\cdots$

$\mathcal{E}=4$ when $i\neq 0$, but $\mathcal{E}=2$ if $i=0$.

V. *Csae D.* The tank is almost full of Water, but there is left a Free Turface, and the Side-Walls Vibrating on the Opposite Phase Each Other

This is the same problem as in the previous section (Case C), the only difference being that here vibration of the two side-walls are on opposite phases (instead of the same phase) each other, as illustrated in Fig. 4. As was already mentioned, when the tank is completely full of water, there is no possibility of vibration of this nature. Aut if there is a vacant space left on top of the tank, even though it be a very thin one, this kind of vibration can take place.

As a velocity potential ϕ corresponding to this case, let us put, as before,

$$\phi=B_{oo}f_{oo}(y)+\Sigma'\Sigma' B_{ij}f_{ij}(y)\cos(mix)\cos(sjz)$$

In the present case of vibration on opposite phases, we must have $B_{oo}=0$. The boundary conditions for the present case are,

(a) at $z=0$ $\quad\frac{\partial\phi}{\partial z}=0$

at $z=H$ $\quad\phi=0$

(b) at $x=0$ or L, $\quad\frac{\partial\phi}{\partial x}=0$

(c) at $y=\frac{1}{2}B$

$$\frac{\partial\phi}{\partial y}=\sum{}'\sum{}'\left[B_{ij}f'_{ij}\left(\frac{1}{2}B\right)\cos(mix)\cos(sjz)\right]=A_o\sin mx\sin sz\cot\omega t \tag{29}$$

From the condition (a), we have $j=\frac{1}{2}+\sigma$, with $\sigma=0,1,2,\cdots\cdots$, while by the condition (b) we have $i=0,1,2,\cdots\cdots$.

Moreover, we have to choose

$$f_{ij}(y)=\cosh(n_{ij}y)$$

$$n_{ij}=[(mi)^2+(sj)^2]^{1/2}$$

In order to find the coefficients B_{ij} in the left hand side of Eq. (29), regarding it as a double Fourier series, the arrangement as shown in Fig. 9 will be taken. Then we have

$$B_{ij} f'_{ij}\left(\frac{1}{2}B\right)\frac{2HL}{\varepsilon} = A_o \int_0^L \sin mx \cos(mix)\, dx \cdot \int_{-H}^{H} |\sin sz| \cos(sjz)\, dz \cdot \cos\omega t$$

but we have

$$\int_0^L \sin mx \cos(mix)\, dx = \frac{1}{2}\left[\frac{-1}{m(1+i)}\{\cos(1+i)\pi - 1\}\right.$$

$$\left. + \frac{-1}{m(1-i)}\{\cos(i-1)\pi - 1\}\right] = \frac{-2}{m(i^2-1)} \quad \text{if } i \text{ is even}$$

$$= 0 \quad \text{if } i \text{ is odd.}$$

$$\int_{-H}^{H} |\sin sz| \cos(sjz)\, dz = 2\int_0^H \sin sz \cos(sjz)\, dz = \frac{2}{2}\left[\frac{-1}{s(1+j)}\{\cos(1+j)\pi - 1\}\right.$$

$$\left. + \frac{-1}{s(1-j)}\{\cos(j-1)\pi - 1\}\right] = \frac{2}{s(1-j^2)}$$

j being equal to $\sigma + \frac{1}{2}$ where $\sigma = 0, 1, 2, \cdots\cdots$. Combining these resulte we have

$$B_{ij} f'_{ij}\left(\frac{1}{2}B\right)\frac{2HL}{\varepsilon} = \left[\frac{-2}{m(i^2-1)}\right]\left[\frac{2}{s(1-j^2)}\right] A_o \cos\omega t$$

where $\varepsilon = 4$ if $i \neq 0$, $\sigma \neq 0$, and $\varepsilon = 2$ if $i = 0$, $\sigma = 0, 1, 2, \cdots\cdots$. Or, rearranging it,

$$B_{ij} = \frac{2}{(i^2-1)(j^2-1)}\left(\frac{\varepsilon}{\pi^2}\right)\frac{1}{f'_{ij}\left(\frac{1}{2}B\right)} \tag{30}$$

Kinetic energy of the Fluid Motion can be obtained quite similarly as in the *cases* A and B:——

$$T_1 = \frac{1}{2}\rho_w \int\int \phi \frac{\partial \phi}{\partial n} dS = \rho_w \int_0^L dx \int_0^H dz \left[\phi \frac{\partial \phi}{\partial y}\right]$$

$$= \rho_w \int_0^L dx \int_0^H dz [A_o \sin mx \sin sz \cos\omega t] \cdot \left[\sum{}' \sum{}' B_{ij} \cos(mix)\cos(sjz) f_{ij}\left(\frac{1}{2}B\right)\right]$$

$$= \rho_w \left[A_o \sum\sum B_{ij} \frac{4}{ms(i^2-1)(j^2-1)} \times \left(\frac{1}{2}\right) \times \cosh\left(\frac{1}{2}n_{ij}B\right)\right]\cos\omega t$$

$$= \rho_w [A_0^2 \cos^2\omega t] \frac{4}{\pi^2 ms} \sum{}' \sum{}' \left\{\frac{1}{(i^2-1)(j^2-1)^2}\right\}^2 \times \frac{\varepsilon}{n_{ij}} \coth\left(\frac{1}{2}n_{ij}B\right)$$

From which, for the kinetic energy to be attributed to each side-wall, we have

$$T_w=\rho_w[A_0^2\cos^2\omega t]\Big[\frac{4}{\pi^4}HL\Big]\times\frac{1}{2}\sum\sum\Big\{\frac{1}{(i^2-1)(j^2-1)}\Big\}^2\frac{\mathcal{E}}{n_{ij}}\coth\Big(\frac{1}{2}n_{ij}B\Big)$$

$$=\frac{1}{2}\rho_w[A_0^2\cos^2\omega t]LBH\times M \tag{31}$$

where

$$M=\frac{8}{\pi^4}\sum{}'\sum{}'\Big\{\frac{1}{(i^2-1)(j^2-1)}\Big\}^2\Big(\frac{\mathcal{E}}{2}\Big)\frac{1}{n_{ij}B}\cdot\coth\Big(\frac{1}{2}n_{ij}B\Big) \tag{32}$$

where

$$n_{ij}B=B[(mi)^2+(sj)^2]^{\frac{1}{2}}=\pi\Big[\Big(\frac{iB}{L}\Big)^2+\Big(\frac{jB}{H}\Big)^2\Big]^{\frac{1}{2}}$$

and $i=0,2,4,\cdots\cdots$; $j=\sigma+\frac{1}{2}$, $\sigma=0,1,2,\cdots\cdots$. $\mathcal{E}=4$ for $i\neq 0$, but $\mathcal{E}=2$ if $i=0$.

VI. The Virtual Mass of Water

When a rectangular elastic plate is vibrating transversally as shown in Fig. 10, its transverse displacement may be expressed by

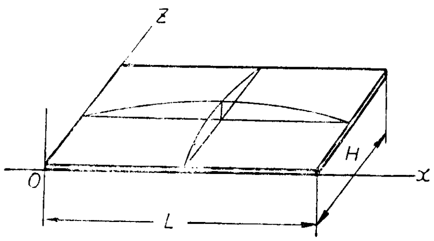

Fig. 10. A mode of vibration of the rectangular plate

$$w=W_o\sin\Big(\frac{\pi}{L}x\Big)\sin\Big(\frac{\pi}{H}z\Big)\sin\omega t$$

and its transverse velocity by

$$\frac{dw}{dt}=A_o\sin\Big(\frac{\pi}{L}x\Big)\sin\Big(\frac{\pi}{H}z\Big)\cos\omega t$$

where $A_o=\omega W_o$. The kinetic energy T_m of the vibration will be given by

$$T_m=\frac{1}{2}\rho_m[A_o\cos\omega t]^2\cdot\frac{1}{4}LHh_m$$

h_m being the thickness of the plate. This expression for T_m may also be written

$$T_m=\frac{1}{2}[W_o\cos\omega t]^2\cdot M_{mv} \tag{33}$$

where $M_{mv}=\frac{1}{4}\rho_m LHh_m=\frac{1}{4}M_m.$

Here M_m is the actual mass of the plate. M_{mv} is a quantity which may be termed (for convenience) the vibrational mass of the plate. In the case of a simple mode of the vibration as sketched in Fig. 10, the value of M_{mv} is just one quarter of the actual mass M_m.

Now, returning to the case of vibration of water, the kinetic energy T_w of the vibrating water contained in the rectangular tank was evaluated, in the previous sections, with regard to the four different cases A, B, C and D, of the vibration of the sidew alls. The result of evaluation was expressed in the form

$$T_w = \frac{1}{2}\rho_w [A_o \cos \omega t]^2 \cdot LHB \cdot M \tag{34}$$

where T_w is the kinetic energy of vibrating water which is to be regarded as participated per each single face-wall of the tank. M is a non-dimentional factor having the values as given by Eqs. (11), (18), (28) and (32) respectively. The values of this numerical factor M were calculated for various combinations of the values of B/H and B/L, and the result is shown graphically in Figs. 11 and 12.

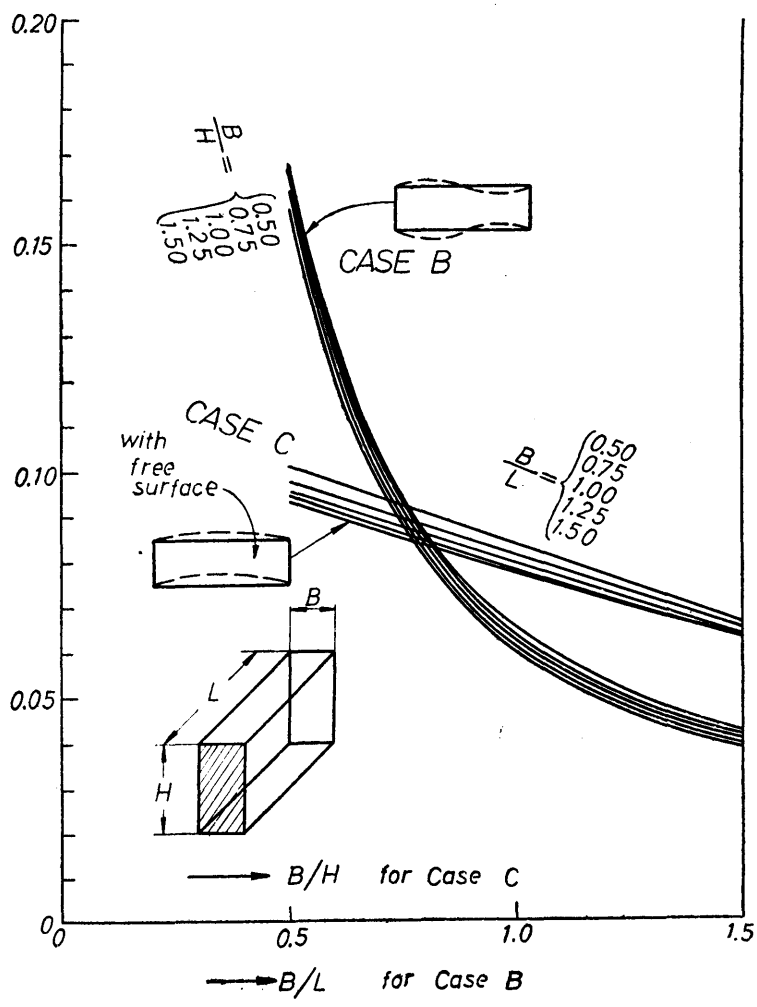

Fig. 11. Value of the numerical factor M for Cases B and C.

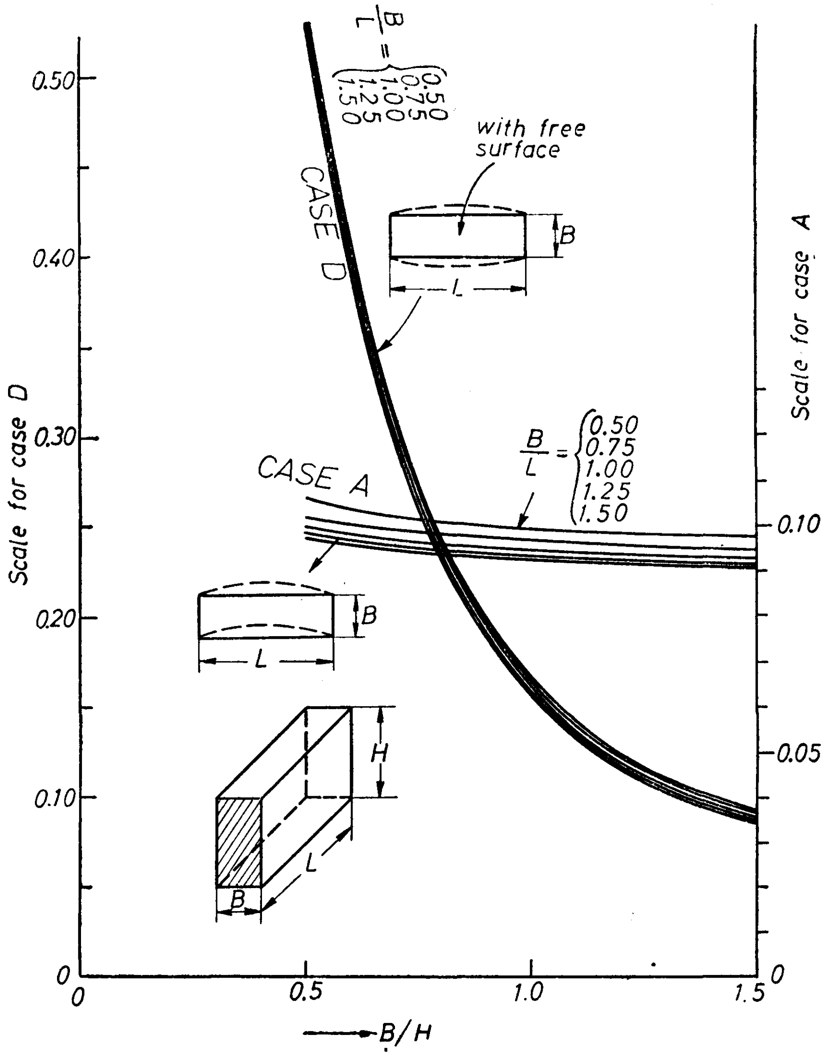

Fig. 12. Value of the numerical factor M for Cases A and D

According to the expression (34) for T_w, the vibrational mass of water (which is usually called the virtual mass) M_{wv} will be given by

$$M_{wv} = \rho_w VM$$

where $V = LBH$, the volume of water. So that, knowing the value of the numerical factor M, the virtual mass M_{wv} may easily be estimated.

Let f be the natural frequency of vibration of a rectangular elastic plate, when it is vibrating in the air (or, in the vacuum) with a given mode. When the same plate is vibrating, with its face in contact with water, and with the same mode as before, its natural frequency f' would be given by the formula

$$f' = \frac{f}{\sqrt{1+\varepsilon}} \quad \text{where} \quad \varepsilon = \frac{M_{wv}}{M_{mv}}$$

But, it must be remembered that this formula gives only an approximate value of f', because the mode of vibration may be altered due to the fact that its face is in contact with water.

It is to be mentioned here that the similar problem of the virtual mass of water for the various arrangements of the side-walls has been, and is now being, investigated by Dr. T. Kumai etc. of Applied Mechanics Research Institute of Kyushu University.

The authour is much indebted to Dr. T. Kumai for his valuable suggestions given to the author.