Title	On vibration of an elastic bar immersed in the water region lying between the two parallel plane walls
Sub Title	
Author	鬼頭, 史城(Kito, Fumiki)
Publisher	慶応義塾大学藤原記念工学部
Publication year	1957
Jtitle	Proceedings of the Fujihara Memorial Faculty of Engineering Keio University Vol.10, No.39 (1957.) ,p.77(1)- 87(11)
JaLC DOI	
Abstract	A circular bar with the supported ends is supposed to be immersed in a water region which lies between two parallel rigid plane walls. The author has made a theoretical study on the effect of the surrounding water upon the natural frequency of vibration of the bar. Firstly, the effect of the surrounding water is estimated by the hydraulic force which acts upon the bar. Secondly, it is estimated by the amount of total kinetic energy of vibratory motion of the surrounding water.
Notes	
Genre	Departmental Bulletin Paper
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=KO50001004-00100039-0001

On Vibration of an Elastic Bar immersed in the Water Region lying between the two Parallel Plane Walls

(Received Dec. 8, 1958)

Fumiki KITO*

Abstract

A circular bar with the supported ends is supposed to be immersed in a water region which lies between two parallel rigid plane walls. The author has made a theoretical study on the effect of the surrounding water upon the natural frequency of vibration of the bar. Firstly, the effect of the surrounding water is estimated by the hydraulic force which acts upon the bar. Secondly, it is estimated by the amount of total kinetic energy of vibratory motion of the surrounding water.

I. Introduction

The author previously made a study on vibration of an elastic bar immersed in a cylindrical water tank, and gave the result of his researches in this Proceedings.(1) In that case, he investigated the effect of the surrounding fluid upon the natural frequency of vibration of an elastic circular bar with the supported ends. The center line of the bar at rest was assummed to coincide with the center line of the cylindrical water tank. The researches were made for the various values of the ratio b/a, in which a is the radius of the bar and b is the inner radius of the water tank. (Fig. 1) Naturally, there would arise a question how it comes when the radius b of the tank becomes infinitely large. In the present paper, the author is going to give the result of the researches about the case in which $b \to \infty$, with the intention of supplementing the previous report.

If we make b tend to infinity, the configuration will become as shown in Fig. 2. In this case shown in Fig. 2, an elastic bar of circular cross-section (with radius a and length l) is assumed to exist in the water region lying between the two parallel plane walls (which is at a distance l apart). The bar is assumed, for the sake of simplicity, to be held in the state of supported ends. The author has obtained analytical expression for the motion set up in the surrounding fluid, when the bar is vibrating freely, and thence made expression for the value of the hydraulic force which thereby acts on the bar. Also the amount of total kinetic energy T_w for the whole water region was

* 鬼 頭 史 城 : Dr. Eng., Professor at Keio University

(1) F. KITO, On vibration of an elastic bar immersed in a cylindrical water tank, this Proceedings, Vol. 10, No. 39 (1957)

evaluated. From these results of the theoretical calculation, the author can propose an approximate formula which gives the natural frequency of the bar immersed in the water region as shown in Fig 2.

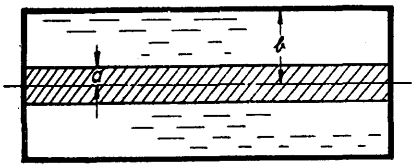

Fig. 1. An elastic bar immersed in a cylindrical water tank

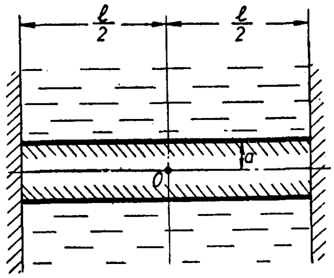

Fig. 2. An elastic bar immersed in a water region lying between the two parallel rigid walls

Throughout the present paper, the fluid is assumed to be incompressible and nonviscous, and the vibration to be of infinitesimal amplitude.

II. Vibration of a Continuous Bar immersed in Water

Let me, first of all consider the case of the vibration of a continuous bar, which consists of a row of bars of span-length l, as shown in Fig. 3. Denoting by ϕ_1 the velocity potential of the flow of an ideal fluid, it must satisfy the Laplace's equation $\nabla^2\phi_1=0$, together with the boundary conditions appropriate to the problem now at hand. Assuming that the vibration is of infinitesimal mgnitude, the fluid pressure p_1 is given approximately by

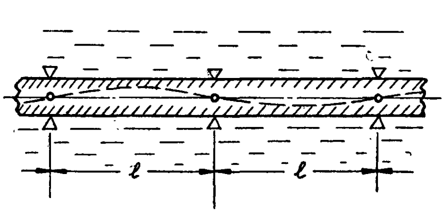

Fig. 3. Vibration of a continuous bar

$$p_1=\rho_w\frac{\partial\phi_1}{\partial t}$$

When the bar is vibrating with an angular frequency ω, the transverse displacement w of the center line of the bar in vibration is expressed by

$$w=W\cos kx\cos\omega t \qquad (1)$$

where $k=\pi/l$. The transverse velocity corresponding to it is given by

$$\frac{dw}{dt}=-\omega W\cos kx\sin\omega t$$

The solution of the Laplace's equation $\nabla^2\phi_1=0$ which satisfies the boundary conditions in which on the surface of the bar the normal velocities of the bar and the fluid coincide, together with the condition in which at an infinite distance from the bar the fluid is at rest, is given by

$$\phi_1 = AF(r)\cos kx \sin\theta \sin\omega t \tag{2}$$

where

$$F(r)=K_1(kr), \qquad A=-\frac{\omega W}{kK_1'(ka)}$$

K_1 being the modified Bessel function of the second kind, and cylindrical coordinates (r, θ, x) being used.

It was already shown, in the previous paper, how much the frequency of natural vibration of the bar is affected by the presence of surrounding water.

III. Vibration of an Elastic Bar immersed in the Water Region lying between the two Parallel Plane Walls

If we assume that the velocity potential ϕ_1 can apply also to the case of Fig. 2, there remains some velocity of fluid v_x transverse to the plane-walls. In order to take into account the existence of the rigid plane walls, we take $\phi=\phi_1+\phi_2$ instead of ϕ_1. ϕ_1 is to take the value given by Eq. (2), while ϕ_2 is an additional term. The new velocity potential ϕ must satisfy the equation $\nabla^2\phi=0$, together with the boundary conditions which is to be stated below:

(a) on the surface of the bar, the normal velocities coincide, thus by virtue of

$$V_r=\frac{\partial\phi}{\partial r}=\frac{\partial\phi_1}{\partial r}+\frac{\partial\phi_2}{\partial r}$$

we must have, for $r=a$,

$$\frac{\partial\phi_2}{\partial r}=0 \tag{3}$$

(b) at an infinite distance from the bar, the fluid velocity is null.

(c) at the two parallel walls, the normal velocity of fluid is zero., viz., at $x=\pm\frac{1}{2}l$

$$V_x=\frac{\partial\phi_1}{\partial x}+\frac{\partial\phi_2}{\partial x}=0 \tag{3a}$$

As a form of velocity potential ϕ_2 which satisfies these conditions we put

$$\phi_2=\int_0^\infty sf(s)\,S_1(s,r)\cosh sx\,ds\cdot\sin\theta\sin\omega t \tag{4}$$

where

$$S_1(s,r)=J_1(sr)Y_1'(sa)-Y_1(sr)J_1'(sa),$$

and $f(s)$ is a function of a real variable s, the form of which is to be determined later. J_1 is the Bessel function of first kind, while Y_1 is the Bessel function of second kind (in Weber's form), each of order 1. By assumption stated in Eq. (4), the above two conditions (a) and (b) are already satisfied. Moreover, in order to satisfy the third condition (c), we must have

$$kAK_1(kr)=\int_0^\infty sf(s)\,S_1(s,r)\,s\sinh\frac{sl}{2}ds$$

or, changing notation for variables;

$$kAK_1(k\lambda)=\int_0^\infty \rho f(\rho)S_1(\rho,\lambda)\,\rho\sinh\frac{\rho l}{2}d\rho \tag{5}$$

So that, if by any means, we could find out the value of function $f(\rho)$ which satisfies the Eq. (5), our problem is solved.

IV. A modified Form of Fourier-Bessel Integral Theorem

The author has, in a previous occation, shown the following theorem, which may be called a modified form of Fourier-Bessel integral theorem. Namely, when

$$f(\lambda)=\int_p^q \rho\,\phi(\rho)S_n(\rho,\lambda)\,d\rho \tag{6}$$

then it follows that, for $p<r<q$,

$$\int_a^\infty \lambda f(\lambda)S_n(r,\lambda)d\lambda=\tfrac{1}{2}M(ra)[\phi(r+0)+\phi(r-0)] \tag{7}$$

while for $0<r<p$ or for $q<r$,

$$\int_a^\infty \lambda f(\lambda)S_n(r,\lambda)d\lambda=0,$$

p, q being two real constants such that $p<q$. $\phi(\rho)$ is a function of a real variable ρ, such that the integral

$$\int_p^q \rho\,|\phi(\rho)|\,d\rho$$

exists, and also such that it satisfies the so called condition of Dirichlet. $S_n(\rho,\lambda)$ is a function of ρ and λ defined by

$$S_n(\rho,\lambda)=J_n(\rho\lambda)Y_n'(\rho a)-Y_n(\rho\lambda)J_n'(\rho a).$$

$M(ra)$ is a factor which, after author's estimation, is given by

$$M(ra)=\{J_n'(ra)\}^2+\{Y_n'(ra)\}^2$$

In these expressions r appears as a real parameter, and must not be confused with the radius r.

Putting, in the above theorem, $p=0$, $q=\infty$ and $n=1$, and furthermore, assuming that $\phi(\rho)$ is a continuous function, we can find the form of function $f(\rho)$ which satisfies the Eq. (5).

Thus we have, by comparing Eqs. (5) and (6):——

$$M(ra)\left[f(r)r\sinh\frac{rl}{2}\right]=\int_a^\infty \lambda kAK_1(k\lambda)S_1(r,\lambda)d\lambda$$

or, changing the notation for independent variable

$$M(sa)\left[sf(s)\sinh\frac{sl}{2}\right]=\int_a^\infty r k A K_1(kr) S_1(s,r)dr \tag{8}$$

V. Force exerted on the Elastic Bar due to vibratory Motion of the surrounding Water

Since the value of function $f(s)$ was thus determined, the function ϕ_2 can be obtained by putting into Eq. (4) the value of $f(s)$. The hydraulic pressure p_2, corresponding to the vibratory motion represented by ϕ_2, is given by

$$p_2=\rho_w\frac{\partial\phi_2}{\partial t}=\rho_w\omega\int_0^\infty sf(s)S_1(s,r)\cosh sx\,ds.\ \sin\theta\cos\omega t \tag{9}$$

Especially, the pressure p_2 on the surface of the bar is obtained by putting $r=a$ into Eq. (9). Also it is to be noted that we have $S_1(s,a)=-1/(sa)$.

Thus the resultant of hydraulic pressure p_2 acting on the surface of the bar $(r=a)$ becomes a force P_2 expressed by

$$P_2=\rho_w\pi\omega\int_0^\infty \rho(s)\cosh sx\,ds\cdot\cos\omega t \tag{10}$$

On the other hand, we have, by Eq. (8),

$$f(s)=\frac{kAF_1(s)}{M(sa)s\sinh(\frac{1}{2}sl)} \tag{11}$$

where

$$F_1(s)=\int_a^\infty rK_1(kr)S_1(s,r)\,dr \tag{12}$$

so that Eq. (10) can also be rewritten in the form ;

$$P_2=\rho_w\omega\pi kA\cos\omega t\int_0^\infty\frac{\cosh sx F_1(s)}{M(sa)s\sinh(\frac{1}{2}sl)}ds \tag{13}$$

Now, the right hand side of Eq. (12), being an aggregate of integrals of products of two Bessel functions, can be evaluated by applying the following relations which are obtainable in a similar way as in the case of what is called *Lommel's* formula.

$$\int_a^\infty(k^2+s^2)J_n(sr)K_n(kr)r\,dr=\left|r\left\{J_n(sr)\frac{d}{dr}K_n(kr)-K_n(kr)\frac{d}{dr}J_n(sr)\right\}\right|_a^\infty$$

$$\int_a^\infty(k^2+s^2)Y_n(sr)K_n(kr)r\,dr=\left|r\left\{Y_n(sr)\frac{d}{dr}K_n(kr)-K_n(kr)\frac{d}{dr}Y_n(sr)\right\}\right|_a^\infty$$

And we arrive at the result that

$$F_1(s) = -\frac{kK_1'(ka)}{s(k^2+s^2)}.$$

Putting this value into Eq. (13), we have

$$P_2 = -\rho_w \pi \omega A(ka)^2 K_1'(ka) \cos \omega t \cdot \int_0^\infty \frac{\cosh sx}{\sinh(\frac{1}{2}sl)} \frac{ds}{M(sa)(sa)^2(k^2+s^2)} \tag{14}$$

The resultant force P_1 which acts on the bar due to the portion ϕ_1 of velocity potential ϕ is given by

$$P_1 = \rho_w \omega^2 \pi a^2 G \cos kx \cdot W \cos \omega t \tag{15}$$

where

$$G = \frac{K_1(ka)}{K_1(ka) + kaK_0(ka)}$$

If we put, in accordance with Eq. (15),

$$P_2 = \rho_w \pi \omega^2 a^2 (ka) \cdot U \cdot W \cdot \cos \omega t \tag{16}$$

we shall have (using new variable $\xi = sa$),

$$U = \int_0^\infty \frac{\cosh\left(\frac{x}{a}\xi\right)}{\sinh\left(\frac{l}{2a}\xi\right)} \frac{d\xi}{\xi^2\{(ka)^2+\xi^2\} M(\xi)} \tag{17}$$

$$M(\xi) = \{Y_1'(\xi)\}^2 + \{J_1'(\xi)\}^2$$

so that the problem may be said to be solved if the numerical value of U is determined.

In the configuration as shown in Fig. 2, the amount of the hydraulic force acting at a section x of the bar, is equal to $P = P_1 + P_2$ per unit length. The maximum value of force P_1 occurs at the position $x = 0$ (mid-point) of the bar, and amounts to $P_{10} = \rho_w \omega^2 \pi a^2 GW$. The ratio of amplitude P_{2a} of P_2 to P_{10} is

$$P_{2a}/P_{10} = (ka)U/G \tag{18}$$

VI. Discussion about Nnmerical Values.

In order to discuss about numerical values, we must first obtain the numerical values of the factor U as given by Eq. (17). In the expression of Eq. (17) for U, there are involved three kinds of parameters, namely, $l/(2a)$, x/a and (ka). But, since $ka = \pi a/l$, and $x/a = (2x/l) \cdot (l/2a)$, only two parameters $l/2a$ and $2x/l$ are independent. As a representative example, we shall take up the case in which $ka = 0.16$ (for which we have $l/2a = 9.8$). This means that we take the case in which the length of the bar is about ten times the length of its diameter. The value of x/a, being indicative of position of cross-section, we shall take four cases $2x/l = 1$, 0.6, 0.4 and 0. The author has carried out numerical calculation of integrand in the integral expression for U, the result of which is shown by the graphs in Fig. 4. Here the values of $M(\xi)$ were obtained from the formula

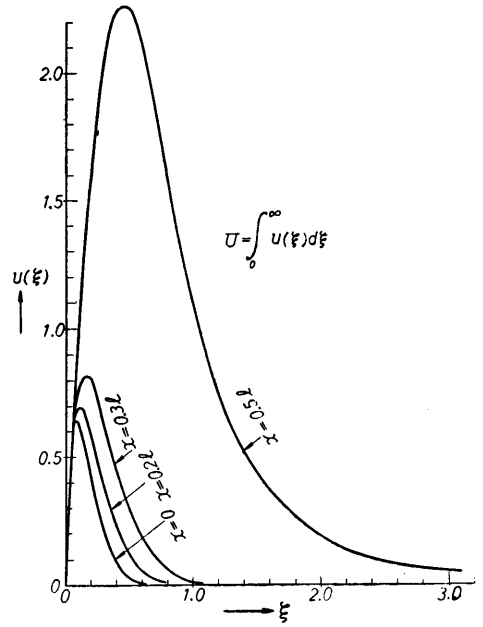

Fig. 4. Graph of numerical integration of a factor U

$$M(\xi)=\left\{J_0(\xi)-\frac{1}{\xi}J_1(\xi)\right\}^2+\left\{Y_0(\xi)-\frac{1}{\xi}Y_1(\xi)\right\}^2$$

and using the table of Bessel functions. Integrating numerically these curves, the following values were obtained.

$x=0$	$0.2l$	$0.3l$	$0.5l$
$U=0.15$	0.23	0.36	2.3

Thus it was possible to find the value of P_2 from Eq. (16). In Fig. 5, the relative

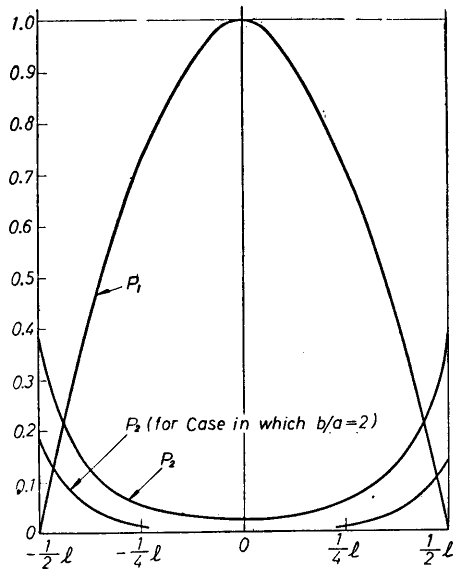

Fig. 5. Relative distribution of hydraulic forces P_1 and P_2 along the length of the bar

values of P_2 (its amplitude) and P_1 (its amplitude) are shown. In the same figure, the values of P_2 in the same scale, for the case in which the outer radius b is finite and such that $b/a=2$, are also shown. Comparing these curves, we infer that the effect of the plane walls in the case of $b\to\infty$ is twice as much as that in the case of $b/a=2$. But, in any way, although there appears a considerable value of P_2 at the immediate vicinity of both ends of the bar, it has very small values for points not so near to the ends. Henceforth, we infer that, so far as we are concerned with practical use, we can neglect the term P_2 in comparison with P_1. Thus a practical formula may be obtained by taking the term P_1 only, which means that, practically, the natural frequency of vibration is the same as for "continuous bars" as shown in Fig. 3. Thus we are led to the following formula for natural frequency of vibration;

$$\omega^2=\frac{EIk^4}{\rho_m\pi a^2}\div\left[1+\frac{\rho_w}{\rho_m}G\right] \tag{19}$$

which has already been given in the author's previous paper. Herein ω is the angular frequency of natural vibration, EI is the flexural rigidity of cross-section of the bar, ρ_m its density, while k is a constant equal to π/l.

VII. The amount of Kinetic Energy of the surrounding Water——Virtual Mass

When a fluid is in motion with velocity potential ϕ_1, the amount of kinetic energy for the whole water region is given by

$$T_w=\frac{\rho_w}{2}\int\int\phi\frac{\partial\phi}{\partial n}ds \tag{20}$$

where the double (surface) integral is to be taken over the boundary surface of the water region. In this expression, $\partial/\partial n$ means differentiation taken in direction normal to the boundary surface. In the present case, the boundary surface consists of four parts, namely the two parallel rigid walls, the surface of the bar, and the points at infinity. Among them, the two parallel walls and the points at infinity contribute nothing to the integral, since $\partial\phi/\partial n$ is null on them. So that the integration of Eq. (20) needs only to be taken over the surface of the bar. But we have on the surface of the bar

$$\frac{\partial\phi}{\partial n}=\frac{\partial\phi}{\partial r}=\omega W\sin\theta\cos kx\sin\omega t.$$

Also we have $\phi=\phi_1+\phi_2$. Thus the kinetic energy T_w is (referring to Fig. 2)

$$T_w=\tfrac{1}{2}\int_{x=-l/2}^{x=l/2}dx\int_{\theta=0}^{2\pi}\left\{\omega W\cos kx\sin\theta\cos\omega t\right\}$$

$$\times\left\{AK_1(ka)\cos kx\sin\theta\sin\omega t+\int_0^\infty sf(s)S_1(s,a)\cosh sx\,ds\cdot\sin\theta\sin\omega t\right\}$$

$$=\tfrac{1}{2}\omega W\cos^2\omega t\,.\,\pi a\left[AK_1(ka)\int_{x=-l/2}^{x=l/2}\cos^2kx\,dx\right.$$

$$+\int_0^\infty sf(s)\, S_1(s,a)ds \int_{x=-l/2}^{x=l/2} \cosh sx \cos kx\, dx\Big]$$

Carrying out the integration with respect to x,

$$T_w = \frac{\pi}{2}\omega Wa \cos^2\omega t\, [E + \tfrac{1}{2}lAK_1(ka)] \tag{21}$$

where we write for shortness,

$$E = 2(ka)^3 AK_1'(ka)\cdot aH$$

$$H = \int_0^\infty \frac{\coth\left(\frac{l\xi}{2a}\right)d\xi}{M(sa)\xi^2[(ka)^2+\xi^2]^2}$$

Further, by rearranging it, we have

$$T_w = \tfrac{1}{2}\rho_w\Big[\frac{\pi}{2}a^2 l\Big]\Big[\omega W \cos\omega t\Big]^2\Big[G + \frac{4}{\pi}(ka)^3 H\Big] \tag{22}$$

Hence we see that the kinetic energy of fluid motion is increased by the factor

$$1 + \frac{4}{\pi}(ka)^3 \frac{H}{G} = 1 + \varepsilon \tag{23}$$

when there exist the tow parallel plane walls, in comparison with the case in which there are no plane walls.

We have obtained the numerical values $h(\xi)$ of integrand in integral expression for H, with regard to three cases, namely,

(*a*) $ka = 0.08$ $l/(2a) = 19.6$
(*b*) $ka = 0.16$ $l/(2a) = 9.8$
(*c*) $ka = 0.24$ $l/(2a) = 6.55$

The result of numerical calculation is shown by the graph in Fig. 6. Also in this case, the values of H were obtained by numerical calculation of Fig. 6, and the result is shown in Fig. 7, From this figure, we infer that the amount of additional kinetic energy due to existence of the plane walls, as given by Eq. (23), is at most of order of ten percent.

We saw that, when an elastic bar of circular cross-section is immersed in water, the effect of the surrounding water upon natural frequency of vibration of the bar is expressed by the factor G. Furthermore, the effect of existence of the tow parallel plane walls is expressed by the factor ε [in Eq. (23)]. Therefore, a formula for natural frequency of vibration of the bar (whose both ends are "supported"), wherein the above-mentioned tow effects are taken into account, may be given in the form

$$\omega^2 = \frac{EIk^4}{\rho_m \pi a^4} \div [1 + \frac{\rho_w}{\rho_m} G(1+\varepsilon)] \tag{24}$$

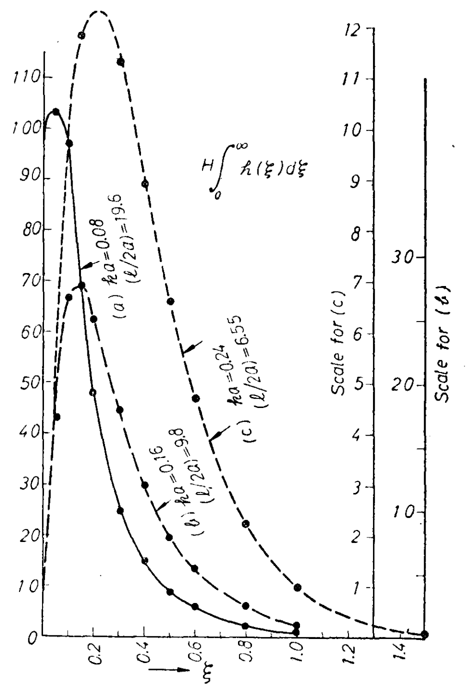

Fig. 6. Graph for numerical integration of a factor H

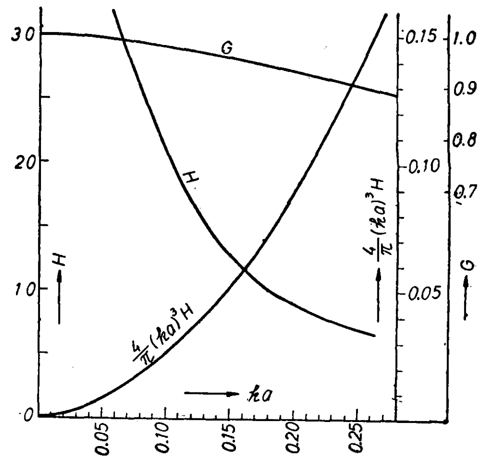

Fig. 7. Graphs for G and H

With regard to this Eq. (24), we may call attention to the two points, namely, (*a*) in the case of the continuous bar as shown in Fig. 3, the factor G which gives the effect of the surrounding water, becomes the same value whether we calculate it from action of the hydraulic pressure upon the bar, or from the amount of kinetic energy of the fluid region. (*b*) but in the case where there exist the parallel plane walls, we cannot expect to obtain the same value of $G(1+\varepsilon)$ by the above-mentioned two different methods of evaluation. This means that the above Eq. (24) is not the rigorous one, but is an approximate formula intended for practical use.

VIII. Numerical Example

The value of moment of inertia of the circular section of bar of radius a is $I=\pi a^4/4$, So that the Eq. (24) may also be written

$$\omega=\sqrt{\frac{Eg}{\gamma_m}}\,\frac{a\pi^2}{2l^2}\div\left[1+\frac{\rho_w}{\rho_m}G(1+\varepsilon)\right]^{1/2}$$

As an illustration of the above mentioned analysis, let me take case of a bar whose radius is $a=1$ cm, its length being $l=9.8\times 2a=19.6$. (both ends being supported). Assuming that the bar is of mild steel, we put $E=2\times10^6$ kg/cm^2 and $\gamma_m=8\times10^{-3}$ kg/cm^3. When the bar is vibrating freely in the air (or, rigorously in vacuo), the angular frequency ω_1 of vibration will be given by

$$\omega_1=\sqrt{\frac{2\times10^6\times980}{8\times10^{-3}}}\times\frac{\pi^2}{2\times(19.6)^2}=199 \text{ rad/sec.}$$

which corresponds to 31.7 cycles per second. When the same bar is vibrating in water, as a continuous bar (Fig. 3), we shall have

$$\omega_2=\omega_1\div[1+\frac{1}{8}\times0.95]^{1/2}=188 \text{ rad/sec.}$$

so that its natural frequency will be 30 cycles/sec. If there exist the two parallel plane walls (*ka* being equal to 0.16), we shall have

$$\omega_3=\omega_1\div[1+\frac{1}{8}(0.95+0,058)]^{1/2}=188 \text{ rad/sec.}$$

and there would be no practical difference between ω_2 and ω_3. In the example, the effect of water is considerably small. This is due to the fact that $\rho_m/\rho_w=8$ and the bar is solid. If the bar was hollow, or if the density ρ_m was smaller than that, there would be seen a considerable effect upon the natural frequency of vibration by the presense of the surrounding water.