Title	On a modified form of Fourier-Bessel integral theorem
Sub Title	
Author	鬼頭, 史城(Kito, Fumiki)
Publisher	慶応義塾大学藤原記念工学部
Publication year	1957
Jtitle	Proceedings of the Fujihara Memorial Faculty of Engineering Keio University Vol.10, No.37 (1957.) ,p.44(18)- 47(21)
JaLC DOI	
Abstract	In this short note, the author gives a modified form of Fourier-Bessel integral theorem, using the Bessel functions of second kind Yn(Z). The theorem relates to functions of the form Tn(r,λ)=Jn(rλ)Yn(ra)-Yn(rλ)Jn(ra) which is practically the same as used by Sneddon, but it is thought that the form given by the author may be more convenient for practical computation. The theorem is also extended to the case of functions of the form Sn(r,λ)=Jn(rλ)Yn'(ra)-Yn(rλ)Jn'(ra)
Notes	
Genre	Departmental Bulletin Paper
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=KO50001004-00100037-0018

On a Modified Form of Fourier-Bessel Integral Theorem

(Received Oct. 8, 1958)

Fumiki KITO*

Abstract

In this short note, the author gives a modified form of Fourier-Bessel integral theorem, using the Bessel functions of second kind $Y_n(Z)$. The theorem relates to functions of the form $T_n(r,\lambda)=J_n(r\lambda)Y_n(ra)-Y_n(r\lambda)J_n(ra)$ which is practically the same as used by Sneddon,[1] but it is thought that the form given by the author may be more convenient for practical computation. The theorem is also extended to the case of functions of the form
$S_n(r,\lambda)=J_n(r\lambda)Y_n'(ra)-Y_n(r\lambda)J_n'(ra)$

I. Fourier-Bessel Integral Theorem for $T_n(r,\lambda)$

Let

$$T_n(r,\lambda)=J_n(r\lambda)Y_n(ra)-Y_n(r\lambda)J_n(ra) \tag{1}$$

where $J_n(x)$ is the Bessel function of first kind of order n. $Y_n(x)$ is the (*Weber's*) Bessel function of second kind of order n, defined by

$$Y_n(Z)=\frac{\cos n\pi\, J_n(Z)-J_{-n}(Z)}{\sin n\pi}$$

[it corresponds to $Y_n(Z)$ as given in Watson's Treatise on Bessel Functions. It also corresponds to $N_n(Z)$ of Jahnke-Emde's Table of Functions]

a is a real positive constant. λ, r and ρ are real positive variables. n is a real positive number, but in practical applications n is usually taken as positive integer.

If we write

$$I=(r^2-\rho^2)\int_a^h T_n(r,\lambda)T_n(\rho,\lambda)\lambda d\lambda \tag{2}$$

and transform this integral exprssion by means of the Lommel's integral formula, we have

---

* 鬼頭史城 : Dr. Eng., Professor at Keio University

(1) I. N. Sneddon ; Fourier Transforms (1951)

$$I = Y_n(ra)Y_n(\rho a)[\rho\lambda J_n(r\lambda)J_n'(\rho\lambda) - r\lambda J_n'(r\lambda)J_n(\rho\lambda)]_a^h$$

$$-Y_n(ra)J_n(\rho a)[\rho\lambda J_n(r\lambda)Y_n'(\rho\lambda) - r\lambda J_n'(r\lambda)Y_n(\rho\lambda)]_a^h$$

$$-Y_n(\rho a)J_n(ra)[\rho\lambda Y_n(r\lambda)J_n'(\rho\lambda) - r\lambda Y_n'(r\lambda)J_n(\rho\lambda)]_a^h$$

$$+J_n(ra)J_n(\rho a)[\rho\lambda Y_n(r\lambda)Y_n'(\rho\lambda) - r\lambda Y_n'(r\lambda)Y_n(\rho\lambda)]_a^h \tag{3}$$

Collecting terms pertaining to lower limit of integration $\lambda = a$, we find that the sum is null.

The expression (3) may also be written :——

$$I = Y_n(ra)Y_n(\rho a)[-\rho\lambda J_n(r\lambda)Y_{n+1}(\rho\lambda) + r\lambda Y_n(\rho\lambda)J_{n+1}(r\lambda)]_a^h$$

$$-Y_n(ra)J_n(\rho a)[-\rho\lambda J_n(r\lambda)Y_{n+1}(\rho\lambda) + r\lambda Y_n(\rho\lambda)J_{n+1}(r\lambda)]_a^h$$

$$-Y_n(\rho a)J_n(ra)[-\rho\lambda Y_n(r\lambda)J_{n+1}(\rho\lambda) + r\lambda J_n(\rho\lambda)Y_{n+1}(r\lambda)]_a^h$$

$$+J_n(ra)J_n(\rho a)[-\rho\lambda Y_n(r\lambda)Y_{n+1}(\rho\lambda) + r\lambda Y_n(\rho\lambda)Y_{n+1}(r\lambda)]_a^h \tag{4}$$

As to terms pertaining to upper limit $\lambda = h$, of integration, we assume that h has a very large positive real value, and put for $J_n(\rho h)$, etc., their asymptotic expressions. Using the known result

$$J_n(Z) \doteqdot \sqrt{\frac{2}{\pi Z}}\cos\left(Z - \frac{\pi}{4} - \frac{n\pi}{2}\right)$$

$$Y_n(Z) \doteqdot \sqrt{\frac{2}{\pi Z}}\sin\left(Z - \frac{\pi}{4} - \frac{n\pi}{2}\right)$$

we find that

$$\frac{I}{(r^2-\rho^2)} \doteqdot Y_n(ra)Y_n(\rho a)\left[\frac{1}{\pi\sqrt{\rho r}}\frac{\sin(r-\rho)h}{r-\rho}\right] - Y_n(ra)J_n(\rho a)[0] - Y_n(\rho a)J_n(ra)[0]$$

$$+J_n(ra)J_n(\rho a)\left[\frac{1}{\pi\sqrt{\rho r}}\frac{\sin(r-\rho)h}{r-\rho}\right] + p(h)$$

$$\doteqdot \frac{1}{\pi\sqrt{\rho r}}[Y_n(ra)Y_n(\rho a) + J_n(ra)J_n(\rho a)]\cdot\frac{\sin(r-\rho)h}{r-\rho} + p(h) \tag{5}$$

The term denoted by $p(h)$ vanishes as $h \to \infty$. Thus we see that for a very large value of h ;

$$\int_a^h T_n(r,\lambda)T_n(\rho,\lambda)\lambda d\lambda \doteqdot \frac{1}{\pi\sqrt{\rho r}}M_1(r,\rho)\frac{\sin(r-\rho)h}{r-\rho} \tag{6}$$

where

$$M_1(\rho,r)=Y_n(ra)Y_n(\rho a)+J_n(ra)J_n(\rho a) \tag{7}$$

In order to apply this relation (6) in establishing a modified form of Fourier-Bessel integral theorem, let us put

$$f(\lambda)=\int_p^q \rho\phi(\rho)T_n(\rho,\lambda)d\rho \tag{8}$$

where p and q are positive real constants such that $o<p<q$. $\phi(\rho)$ is an arbitrary function of real variable ρ, which satisfies the Dirichlet's conditions. Also, by putting

$$U=\int_p^q \lambda f(\lambda)T_n(r,\lambda)d\lambda \tag{9}$$

we have, by (6) :——

$$U=\int_p^q \rho\phi(\rho)\frac{M_1(\rho,r)}{\pi\sqrt{\rho r}}\frac{\sin(r-\rho)h}{r-\rho}d\rho$$

Thus we see, by Dirichlet's theorem, that, for $h\to\infty$ we have,

$$U=\tfrac{1}{2}[\phi(r+0)+\phi(r-0)]M_1(r,r)$$

so long as $p<r<q$. The value of $M_1(r,r)$ is, by(7),

$$M_1(r,r)=[Y_n(ra)]^2+[J_n(ra)]^2$$

Thus we have the theorem :——

If

$$f(\lambda)=\int_p^q \rho\phi(\rho)T_n(\rho,\lambda)d\rho$$

then

$$\int_a^\infty \lambda f(\lambda)T_n(r,\lambda)d\lambda=\begin{cases}\tfrac{1}{2}M_1(r,r)[\phi(r+0)+\phi(r-0)] & p<r<q\,, \\ 0 & 0<r<p \quad or \quad r>q\end{cases}$$

II. Fourier-Bessel integral theorem for $S_n(r,\lambda)$

In some problems of hydrodynamics, the similar Fourier-Bessel integral theorem with respect to the functions

$$S_n(r,\lambda)=J_n(r\lambda)Y_n'(ra)-Y_n(r\lambda)J_n'(ra) \tag{10}$$

is required.[(2)(3)] Here, it will be shown that the similar inference as above can be made with respect to this system of functions (10).

In fact, if we calculate the integral

$$I=(r^2-\rho^2)\int_a^h S_n(r,\lambda)S_n(\rho,\lambda)\,\lambda\, d\lambda \tag{11}$$

in similar manner as above, we obtain first the expression of the form of (3), where we put $J_n'(ra)$, $J_n'(\rho a)$, $Y_n'(ra)$ and $Y_n'(\rho a)$ instead of $J_n(ra)$, $J_n(\rho a)$, $Y_n(ra)$ and $Y_n(\rho a)$ therein. Collecting terms pertaining to lower limit $\lambda=a$ of integration, we see again that the sum is null. In the same way, the integral (11) can be put into the form of (4), where we write $J_n'(ra)$ etc. instead of $J_n(ra)$ etc. Putting asymptotic expressions of $J_n(\rho h)$ etc., we obtain, for a very large value of h;

$$\int_a^h S_n(r,\lambda)S_n(\rho,\lambda)\lambda d\lambda \doteqdot \frac{1}{\pi\sqrt{\rho r}}M_2(r,\rho)\frac{\sin(r-\rho)h}{r-\rho}$$

where

$$M_2(r,\rho)=[Y_n'(ra)\,Y_n'(\rho a)+J_n'(ra)J_n'(\rho a)]$$

In this way, we obtain the following theorem ;——

If

$$f(\lambda)=\int_p^q \rho\phi(\rho)S_n(\rho,\lambda)d\rho$$

then

$$\int_a^\infty \lambda f(\lambda)S_n(r,\lambda)d\lambda=\begin{cases}\frac{1}{2}M_2(r,r)[\phi(r+0)+\phi(r-0)] & p<r<q\\ 0 & 0<r<p \quad or \quad r>q\end{cases}$$

where

$$M_2(r,r)=[Y_n'(ra)]^2+[J_n'(ra)]^2$$

(2) F. Kito, On Vibration of a Cylindrical Shell immersed in Water, This Proceedings, 5. 32 (1952)

(3) F. Kito, On a Fourier-Bessel Expansion of Special Kind, This Proceedings. 5. 41 (1952)