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On Pseudo-harmonic Oscillations of Third Order

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Zin-ich NAGUMO*

Abstract

Nonlinear oscillations have so far primarily been considered in the case of second order differential equations. However, actual problems of nonlinear oscillations frequently lead to differential equations of higher order. The present paper is concerned with *pseudo-harmonic oscillations* which are represented by periodic solutions of third order differential equations of a special type. The periodic solutions are determined by the method of Coddington and Levinson, and their stability is investigated. This method of analysis is applied to electronic oscillating circuits, self-sustained and synchronized Colpitts oscillator and Parametron circuit, obtaining some new results which will be useful for practical design.

I. Introduction

Nonlinear oscillations have so far primarily been considered in the case of second order differential equations. However, actual problems of nonlinear oscillations frequently lead to differential equations of higher order. The present paper is concerned with a special type of nonlinear oscillations, so-called "*pseudo-harmonic oscillations*", which are represented by periodic solutions of third order differential equations of a particular type.

We will begin with giving the following three examples.

Ex. 1 Self-sustained oscillation of the Colpitts Oscillator

In Fig. 1, the only nonlinear element in the circuit is the static (=resistive) 3-pole (for example, vacuum-tube, transistor) which determines the currents I_1, I_2 as analytic functions of the voltages V_1, V_2

$$I_1=f(V_1, V_2), \quad I_2=g(V_1, V_2). \quad (1\text{-}1)$$

The differential equation** of the circuit is

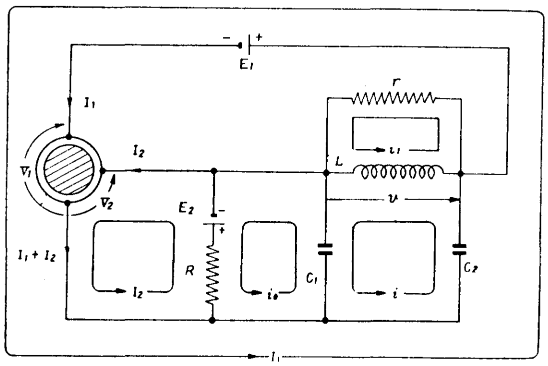

Fig. 1.

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*) 南雲仁一 Dr. of Eng., Assistant Prof. at Keio University.

**) In the following we shall simply say "equation" instead of "system of equations."

$$\begin{cases} \dfrac{dV_1}{dt}=-\dfrac{1}{C_2}i-\dfrac{1}{C_2}f(V_1,\ V_2) \\ \dfrac{dV_2}{dt}=\dfrac{1}{C_1}i-\dfrac{1}{RC_1}V_2-\dfrac{1}{RC_1}E_2-\dfrac{1}{C_1}g(V_1,\ V_2) \\ \dfrac{di}{dt}=\dfrac{1}{L}(V_1+E_1)-\dfrac{1}{L}V_2-\dfrac{1}{rC}i+\dfrac{1}{rRC_1}V_2+\dfrac{1}{rRC_1}E_2 \\ \qquad -\dfrac{1}{rC_2}f(V_1,\ V_2)+\dfrac{1}{rC_1}g(V_1,\ V_2). \end{cases}$$

Setting
$$\frac{1}{C}=\frac{1}{C_1}+\frac{1}{C_2},\quad \omega^2=\frac{1}{LC},\quad \tau=\omega t,$$

$$x=\frac{\omega C_2(V_1+E_1)}{I},\quad y=\frac{\omega C_1V_2}{I},\quad z=\frac{i}{I},$$

$$a=\frac{C}{C_2}\quad (0<a<1),\quad k=\frac{r}{R},$$

$$\mu F(x,\ y)=\frac{f(V_1,\ V_2)}{I},\quad \mu G(x,\ y)=\frac{g(V_1,\ V_2)}{I},$$

$$\mu b=\frac{1}{\omega rC},\quad \mu q=\frac{E_2}{RI},\qquad I: \text{unit current}$$

and assuming that $0<\mu\ll1$, we have

$$\begin{cases} \dfrac{dx}{d\tau}=-z-\mu F(x,\ y) \\ \dfrac{dy}{d\tau}=z-\mu b(1-a)ky-\mu q-\mu G(x,\ y) \\ \dfrac{dz}{d\tau}=ax-(1-a)y-\mu bz+O_2(\mu), \end{cases} \tag{1-2}$$

where $O_2(\mu)$ means terms of the order $\geqq 2$ in μ.

By a further transformation of variables:

$$\begin{cases} x=x_1+(1-a)x_3 \\ y=-x_1+ax_3 \\ z=x_2 \end{cases}\qquad \left(\frac{\partial(x,\ y,\ z)}{\partial(x_1,\ x_2,\ x_3)}=-1\right),$$

we obtain

$$\begin{cases} \dfrac{dx_1}{d\tau}=-x_2+\mu f_1(x_1,\ x_3) \\ \dfrac{dx_2}{d\tau}=x_1+\mu f_2(x_1,\ x_2,\ x_3,\ \mu) \\ \dfrac{dx_3}{d\tau}=\mu f_3(x_1,\ x_3), \end{cases} \tag{1-3}$$

where

$$\begin{cases} f_1=-b(1-a)^2kx_1+ab(1-a)^2kx_3-aF+(1-a)G+(1-a)q \\ f_2=-bx_2+O(\mu) \\ f_3=b(1-a)kx_1-ab(1-a)kx_3-F-G-q \end{cases}$$

and

$$F = F(x_1 + (1-a)x_3\,,\ -x_1 + ax_3)$$
$$G = G(x_1 + (1-a)x_3\,,\ -x_1 + ax_3).$$

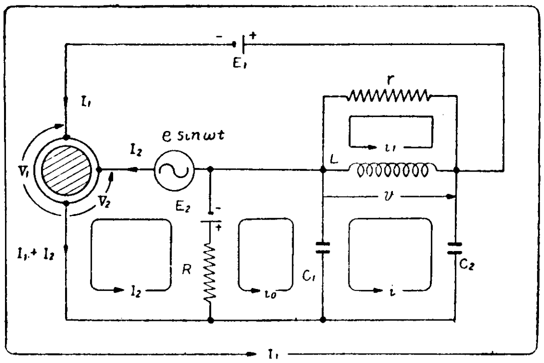

Fig. 2.

Ex. 2 Synchronized osillation of the order n/m of the Colpitts Oscillator

In Fig. 2, choosing three variables: V_1, $V_2' = V_2 + e\sin\omega t$ and i, and setting $C^{-1} = C_1^{-1} + C_2^{-1}$, we get

$$\begin{cases} \dfrac{dV_1}{dt} = -\dfrac{1}{C_2} i - \dfrac{1}{C_2} f(V_1, V_2' - e\sin\omega t) \\ \dfrac{dV_2'}{dt} = \dfrac{1}{C_1} i - \dfrac{1}{RC_1} V_2' - \dfrac{1}{RC_1} E_2 - \dfrac{1}{C_1} g(V_1, V_2' - e\sin\omega t) \\ \dfrac{di}{dt} = \dfrac{1}{L}(V_1 + E_1) - \dfrac{1}{L} V_2' - \dfrac{1}{rC} i + \dfrac{1}{rRC_1} V_2' + \dfrac{1}{rRC_1} E_2 \end{cases}$$
$$-\frac{1}{rC_2} f(V_1, V_2' - e\sin\omega t) + \frac{1}{rC_1} g(V_1, V_2' - e\sin\omega t).$$

If there exist two positive integers m and n, which are prime one another and such that

$$\omega \div \frac{1}{\sqrt{LC}} \doteqdot m \div n,$$

then letting $0 < \mu \ll 1$, we set as follows.

$$\mu p = n^2 - \frac{m^2}{\omega^2 LC}, \qquad \omega t = m\tau, \qquad \alpha = \frac{m^2}{\omega^2 LC_2},$$

$$x = \frac{\omega C_2(V_1 + E_1)}{Im}, \qquad y = \frac{\omega C_1 V_2'}{Im}, \qquad z = \frac{i}{I},$$

$$a = \frac{C}{C_2}\ (0 < a < 1), \qquad k = \frac{r}{R}, \qquad \gamma = \frac{\omega C_1 e}{Im},$$

$$\mu F(x, y - \gamma\sin m\tau) = f(V_1, V_2' - e\sin\omega t)/I,$$

$$\mu G(x, y - \gamma\sin m\tau) = g(V_1, V_2' - e\sin\omega t)/I,$$

$$\mu b = \frac{m}{\omega rC}, \qquad \mu q = \frac{E_2}{RI}, \qquad I:\ \text{unit current}.$$

It is apparent that the case: $m = n = 1$ corresponds to the fundamental synchronization, the case: $n = 1$ corresponds to the subharmonic synchronization of m-th order and the case: $m=1$ corresponds to the higher-harmonic synchronization of n-th order.

The differential equation takes the form

$$\begin{cases} \dfrac{dx}{d\tau}=-z-\mu F(x,\,y-\gamma\sin m\tau) \\ \dfrac{dy}{d\tau}=z-\mu b(1-a)ky-\mu q-\mu G(x,\,y-\gamma\sin m\tau) \\ \dfrac{dz}{d\tau}=\alpha x-(n^2-\alpha)y+\mu py-\mu bz+O_2(\mu)\,. \end{cases}$$

By a further transformation of the variables:

$$\begin{cases} x=x_1+(n^2-\alpha)x_3 \\ y=-x_1+\alpha x_3 \\ z=nx_2 \end{cases} \qquad \left(\frac{\partial(x,\,y,\,z)}{\partial(x_1,\,x_2,\,x_3)}=-n^3\right),$$

we have

$$\begin{cases} \dfrac{dx_1}{d\tau}=-nx_2+\mu f_1(x_1,\,x_3,\,\tau) \\ \dfrac{dx_2}{d\tau}=nx_1+\mu f_2(x_1,\,x_2,\,x_3,\,\tau,\,\mu) \\ \dfrac{dx_3}{d\tau}=\mu f_3(x_1,\,x_3,\,\tau) \end{cases} \tag{1-4}$$

where

$$\begin{cases} f_1=-b\left(1-\dfrac{\alpha}{n^2}\right)(1-a)kx_1+b\alpha\left(1-\dfrac{\alpha}{n^2}\right)(1-a)kx_3 \\ \qquad -\dfrac{\alpha}{n^2}F+\left(1-\dfrac{\alpha}{n^2}\right)G+\left(1-\dfrac{\alpha}{n^2}\right)q \\ f_2=-\dfrac{p}{n}x_1-bx_2+\dfrac{\alpha p}{n}x_3+O(\mu) \\ f_3=\dfrac{b}{n^2}(1-a)kx_1-\dfrac{\alpha b}{n^2}(1-a)kx_3-\dfrac{1}{n^2}F-\dfrac{1}{n^2}G-\dfrac{q}{n^2} \end{cases}$$

and

$$F=F(x_1+(n^2-\alpha)x_3,-x_1+\alpha x_3-\gamma\sin m\tau)$$

$$G=G(x_1+(n^2-\alpha)x_3,-x_1+\alpha x_3-\gamma\sin m\tau).$$

Note that f_i's are periodic functions of τ with the least period $2\pi/m$.

Ex. 3 Parametron circuit

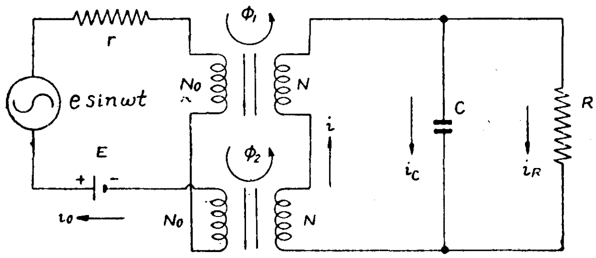

Fig. 3.

In Fig. 3, let us suppose first that the nonlinear characteristic (magnetization curve) of the two iron-cored transformers are identical and that it is represented by a single-valued analytic function f.

Then we have

$$\begin{cases} N\dfrac{d\phi_1}{dt}+N\dfrac{d\phi_2}{dt}=-\dfrac{1}{C}\displaystyle\int i_C dt=-Ri_R \\ i=i_C+i_R \\ N_0\dfrac{d\phi_1}{dt}-N_0\dfrac{d\phi_2}{dt}+ri_0=E+e\sin\omega t \\ Ni+N_0 i_0=f(\phi_1) \\ Ni-N_0 i_0=f(\phi_2) \end{cases}$$

where N_0 is the number of turns in the primary winding and N is that of the secondary winding of each transformer, ϕ is the flux.

Letting $x=\phi_1+\phi_2$, $y=\phi_1-\phi_2$, $f(\phi)=\alpha\phi+\mu k(\phi)$ ($\alpha>0$, $\mu>0$, $k(\phi)$ is a power series of ϕ which does not contain the first order term),

$$g(x, y)=k(\phi_1)+k(\phi_2), \quad h(x, y)=k(\phi_1)-k(\phi_2),$$

we get

$$\begin{cases} 2CN^2\dfrac{d^2x}{dt^2}+\dfrac{2N^2}{R}\dfrac{dx}{dt}+\alpha x+\mu g(x,y)=0 \\ 2N_0{}^2\dfrac{dy}{dt}+r\alpha y+\mu rh(x,y)=2N_0E+2N_0 e\sin\omega t. \end{cases} \tag{1-5}$$

Introducing new variables τ and z by

$$\omega t+\theta=m\tau \qquad (m: \text{positive integer})$$
$$y=\bar{y}+y_0\cos(\omega t+\theta)+z$$
$$\bar{y}=\frac{2N_0E}{r\alpha}, \qquad y_0=\frac{2N_0e}{\sqrt{4N_0{}^4\omega^2+r^2\alpha^2}}$$

we have

$$\begin{cases} \dfrac{d^2x}{d\tau^2}+\dfrac{m}{CR\omega}\dfrac{dx}{d\tau}+\dfrac{m^2\alpha}{2CN^2\omega^2}x+\dfrac{\mu m^2}{2CN^2\omega^2}g=0 \\ \dfrac{dz}{d\tau}+\dfrac{mr\alpha}{2N_0{}^2\omega}z+\dfrac{\mu rm}{2N_0{}^2\omega}h=0 \end{cases}$$

where $g=g(x, \bar{y}+y_0\cos m\tau+z)$, $h=h(x, \bar{y}+y_0\cos m\tau+z)$.

If there exists a positive integer n, which is prime with m and such that

$$\omega\div\sqrt{\frac{\alpha}{2CN^2}}\fallingdotseq m\div n,$$

then letting $0<\mu\ll 1$, we set as follows.

$$\frac{m^2\alpha}{2CN^2\omega^2n}=n+\mu a, \qquad \frac{m}{CR\omega}=\mu b,$$

$$\frac{mr\alpha}{2N_0{}^2\omega}=-\sigma, \qquad x=x_1, \qquad \frac{dx}{d\tau}=-nx_2, \qquad z=x_3.$$

The differential equation takes the form

$$\begin{cases} \dfrac{dx_1}{d\tau}=-nx_2 \\ \dfrac{dx_2}{d\tau}=nx_1+\mu f_2(x_1,\, x_2,\, x_3\ \tau) \\ \dfrac{dx_3}{d\tau}=\sigma x_3+\mu f_3(x_1,\, x_3,\, \tau) \end{cases} \tag{1-6}$$

where

$$f_2=ax_1-bx_2+\frac{m^2}{2CN^2\omega^2 n}g$$

$$f_3=-\frac{rm}{2N_0{}^2\omega}h$$

and $g=g(x_1,\, \bar{y}+y_0\cos m\tau+x_3)$, $\quad h=h(x_1,\, \bar{y}+y_0\cos m\tau+x_3)$.

It is to be noted that $\sigma<0$ and f_i's are periodic functions of τ with the least period $2\pi/m$.

These examples lead to the general consideration of which we shall investigate in the following.

For the linear differential equation:

$$\dot{\boldsymbol{y}}=B\boldsymbol{y} \qquad (\dot{\boldsymbol{y}}=d\boldsymbol{y}/d\tau) \tag{1-7}$$

where $\boldsymbol{y}$ is a 3-vector and B is a 3×3 constant real matrix, assume that there exists a real periodic solution with period 2π. This is equivalent to the fact that B has a pair of characteristic root of the form $\pm in$ where n is a positive integer. We shall be interested in the purturbed differential equation:

$$\dot{\boldsymbol{y}}=B\boldsymbol{y}+\mu\boldsymbol{g}(\boldsymbol{y},\, \tau,\, \mu) \tag{1-8}$$

where μ is a small positive parameter, $\boldsymbol{g}$ is a real 3-vecter, the components of which are real analytic functions of $(\boldsymbol{y},\, \tau,\, \mu)$ and periodic of period 2π in τ. (The case where $\boldsymbol{g}$ does not contain τ explicitely is not excluded.)

Setting $\boldsymbol{y}=P\boldsymbol{x}$ where P is a real nonsingular constant 3×3 matrix, the differential equation (1-8) can be replaced by a differential equation for $\boldsymbol{x}$

$$\dot{\boldsymbol{x}}=A\boldsymbol{x}+\mu\boldsymbol{f}(\boldsymbol{x},\, \tau,\, \mu) \qquad (0<\mu\ll1) \tag{1-9}$$

where $A=P^{-1}BP$ is in real canonical form, $\boldsymbol{f}(\boldsymbol{x},\, \tau,\, \mu)=P^{-1}\boldsymbol{g}(P\boldsymbol{x},\, \tau,\, \mu)$. Moreover, this new differential equation satisfies the same assumptions as (1-8).

It is obvious that A takes the form

$$A=\begin{pmatrix} 0 & -n & 0 \\ n & 0 & 0 \\ 0 & 0 & \sigma \end{pmatrix}$$

where σ is a real number.

II. Periodic solution

We now proceed to investigate (1-9). Periodic solutions of (1-9), which are almost sinusoidal and analytic in μ for small μ, can be determined by the method of

Coddington and Levinson.*

We shall classify the type of the equation (1-9) in four cases which arise from special choices of the function $\boldsymbol{f}$ and the coefficient matrix A.

To begin with we devide these cases into two main groups:
(1) the nonautonomous case for which $\boldsymbol{f}$ contains τ explicitely, and (2) the autonomous case in which $\boldsymbol{f}$ does not depend explicitely on τ.

(1) **Nonautonomous case**

For small μ, (1-9) has an almost sinusoidal periodic solution $\boldsymbol{x}(\tau, \mu)$ with period 2π, if the approximate (with respect to μ) periodic solution $\boldsymbol{x}^{(0)}(\tau)$ can be decided and $J_1 \neq 0$.

Case i) $\sigma=0$

$$\begin{cases} x_1^{(0)}(\tau)=a_1\cos n\tau-a_2\sin n\tau \\ x_2^{(0)}(\tau)=a_1\sin n\tau+a_2\cos n\tau \\ x_3^{(0)}(\tau)=a_3 \end{cases}$$

where a_1, a_2 and a_3 are given by

$$\begin{cases} H_1\equiv\int_0^{2\pi}([f_1]\cos ns+[f_2]\sin ns)\,ds=0 \\ H_2\equiv\int_0^{2\pi}(-[f_1]\sin ns+[f_2]\cos ns)\,ds=0 \\ H_3\equiv\int_0^{2\pi}[f_3]\,ds=0 \end{cases}$$

$$[f_i]=f_i(\boldsymbol{x}^{(0)}(s),\ s,\ 0),$$

and

$$J_1=\frac{\partial(H_1,\ H_2,\ H_3)}{\partial(a_1,\ a_2,\ a_3)}.$$

Case ii) $\sigma\neq 0$

$$\begin{cases} x_1^{(0)}(\tau)=a_1\cos n\tau-a_2\sin n\tau \\ x_2^{(0)}(\tau)=a_1\sin n\tau+a_2\cos n\tau \\ x_3^{(0)}(\tau)=0 \end{cases}$$

where a_1 and a_2 are given by

* E. A. Coddington and N. Levinson, Contributions to the Theory of Nonlinear Oscillations (II), Annals of Mathematics Studies, No. 29 (1952), Princeton Univ. Press or E. A. Coddington and N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill, New York (1955), Chap. 14.

$$H_1 \equiv \int_0^{2\pi} ([f_1] \cos ns + [f_2] \sin ns)\, ds = 0$$

$$H_2 \equiv \int_0^{2\pi} (-[f_1] \sin ns + [f_2] \cos ns)\, ds = 0$$

$$[f_i] = f_i(\boldsymbol{x}^{(0)}(s),\ s,\ 0),$$

and

$$J_1 = \frac{\partial(H_1,\ H_2)}{\partial(a_1,\ a_2)}.$$

(2) Autonomous case

In this case we may assume $n=1$ without loss of generality. For small μ, (1-9) has an almost sinusoidal periodic solution $\boldsymbol{x}(\tau,\mu)$ with period $T(\mu)$, if the approximate periodic solution $\boldsymbol{x}^{(0)}(\tau)$ and the approximate period $T^{(1)}$ can be decided and $J_2 \neq 0$.

Case i) $\sigma=0$

$$x_1^{(0)}(\tau) = a_1 \cos \tau$$

$$x_2^{(0)}(\tau) = a_1 \sin \tau$$

$$x_3^{(0)}(\tau) = a_3$$

$$T^{(1)} = 2\pi + \mu\nu$$

where a_1, a_3 and ν are given by

$$H_1 \equiv \int_0^{2\pi} ([f_1] \cos s + [f_2] \sin s)\, ds = 0$$

$$H_2 \equiv \nu a_1 + \int_0^{2\pi} (-[f_1] \sin s + [f_2] \cos s)\, ds = 0$$

$$H_3 \equiv \int_0^{2\pi} [f_3]\, ds = 0$$

$$[f_i] = f_i(\boldsymbol{x}^{(0)}(s),\ 0) \quad [\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x},\ \mu)],$$

and

$$J_2 = \frac{\partial(H_1,\ H_2,\ H_3)}{\partial(a_1,\ a_3,\ \nu)} = a_1 \frac{\partial(H_1,\ H_3)}{\partial(a_1,\ a_3)}.$$

Case ii) $\sigma \neq 0$

$$x_1^{(0)}(\tau) = a_1 \cos \tau$$

$$x_2^{(0)}(\tau) = a_1 \sin \tau$$

$$x_3^{(0)}(\tau) = 0$$

$$T^{(1)}=2\pi+\mu\nu$$

where a_1 and ν are given by

$$\begin{cases} H_1\equiv\int_0^{2\pi}([f_1]\cos s+[f_2]\sin s)\,ds=0 \\ H_2\equiv\nu a_1+\int_0^{2\pi}(-[f_1]\sin s+[f_2]\cos s)\,ds=0 \end{cases}$$

$$[f_i]=f_i(\boldsymbol{x}^{(0)}(s),\ 0)\,,$$

and

$$J_2=\frac{\partial(H_1,\ H_2)}{\partial(a_1,\ \nu)}=a_1\frac{\partial H_1}{\partial a_1}\,.$$

III. Stability

We now investigate the stability of the periodic solution $\boldsymbol{x}(\tau,\ \mu)$, the existence of which has been guaranteed and the approximate periodic solution $\boldsymbol{x}^{(0)}(\tau)$ has been calculated as in II.

The variation equation of (1-9) with respect to the periodic solution $\boldsymbol{x}(\tau,\ \mu)$ is

$$\dot{\xi}=\{A+\mu D(\boldsymbol{x}(\tau,\mu),\ \mu)\}\xi,\quad D_{ij}=\frac{\partial f_i}{\partial x_j} \tag{3-1}$$

where ξ is a 3-vector. Corresponding to this variation equation, we consider the matrix equation:

$$\dot{\Xi}=\{A+\mu D(\boldsymbol{x}(\tau,\mu),\mu)\}\Xi \tag{3-2}$$

where Ξ is a 3×3 matrix. The first task then is to investigate the value of the solution $\Xi(\tau,\mu)$, with the initial condition $\Xi(0,\mu)=E$ (3×3 unit matrix), at $\tau=T(\mu)$ where $T(\mu)$ is the period of the periodic solution $\boldsymbol{x}(\tau,\mu)$. (In nonautonomous case $T(\mu)=2\pi$.)

To carry out this program, we expand $\boldsymbol{x}(\tau,\mu)$, $D(\boldsymbol{x},\mu)$ and $\Xi(\tau,\mu)$ in power series of μ

$$\boldsymbol{x}(\tau,\mu)=\boldsymbol{x}^{(0)}(\tau)+\mu\boldsymbol{x}^{(1)}(\tau)+\cdots\cdots$$

$$D(\boldsymbol{x},\mu)=D^{(0)}(\boldsymbol{x})+\mu D^{(1)}(\boldsymbol{x})+\cdots\cdots$$

$$\Xi(\tau,\mu)=\Xi^{(0)}(\tau)+\mu\Xi^{(1)}(\tau)+\cdots\cdots.$$

Substituting these into (3-2) and identifying the coefficients of the powers of μ on both sides, we obtain

$$\dot{\Xi}^{(0)}=A\Xi^{(0)} \tag{3-3}$$

$$\dot{\Xi}^{(1)}=A\Xi^{(1)}+D^{(0)}(\boldsymbol{x}^{(0)}(\tau))\Xi^{(0)} \tag{3-4}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots.$$

The solution of (3-3), with the initial condition: $\Xi^{(0)}(0)=E$ is $\Xi^{(0)}(\tau)=\exp(\tau A)$.

Making use of this result, the solution of (3-4), with the initial condition: $\Xi^{(1)}(0)=O$, becomes

$$\Xi^{(1)}(\tau)=\int_0^\tau \exp\{(\tau-s)A\}\cdot D^{(0)}(\boldsymbol{x}^{(0)}(s))\cdot\exp(sA)ds\,.$$

Hence

$$\begin{aligned}L\equiv\Xi(T(\mu),\mu)&=\Xi^{(0)}(T(\mu))+\mu\Xi^{(1)}(T(\mu))+O_2(\mu)\\&=\Xi^{(0)}(T(\mu))+\mu\Xi^{(1)}(2\pi)+O_2(\mu)=\exp(2\pi A)\cdot(E+\mu\pi K)\end{aligned}\tag{3-5}$$

where

$$K=K^0+O(\mu)$$

and

$$\begin{cases}K^0=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi}\exp(-sA)\cdot D^{(0)}(\boldsymbol{x}^{(0)}(s))\cdot\exp(sA)ds & \text{in nonautonomous case,}\\ K^0=\dfrac{\nu A}{\pi}+\dfrac{1}{\pi}\displaystyle\int_0^{2\pi}\exp(-sA)\cdot D^{(0)}(\boldsymbol{x}^{(0)}(s))\cdot\exp(sA)ds & \text{in autonomous case.}\end{cases}$$

Now let the characteristic value of L be ρ, that is $\det(L-\rho E)=0$. If we put $\rho=1+\mu\pi\lambda$, then λ is a root of $\det(L-E-\mu\pi\lambda E)=0$ or

$$\lambda^3+U\lambda^2+V\lambda+W=0\tag{3-6}$$

where

$$\begin{aligned}U&=\frac{1-\exp(2\pi\sigma)}{\mu\pi}-\{t_rM+\exp(2\pi\sigma)\cdot K_{33}\}\\V&=-\frac{1-\exp(2\pi\sigma)}{\mu\pi}t_rM+\det M+\exp(2\pi\sigma)\\&\quad\times\{(K_{11}K_{33}-K_{13}K_{31})+(K_{22}K_{33}-K_{23}K_{32})\}\\W&=\frac{1-\exp(2\pi\sigma)}{\mu\pi}\det M-\exp(2\pi\sigma)\cdot\det K\end{aligned}$$

and

$$M=\begin{pmatrix}K_{11} & K_{12}\\ K_{21} & K_{22}\end{pmatrix}.$$

If we set $\lambda=\alpha+i\beta$, $|\rho|^2=1+2\mu\pi\alpha+\mu^2\pi^2(\alpha^2+\beta^2)$ and hence $|\rho|<1$ is equivalent to $\alpha<0$ or to R. P. $(\lambda)<0$.

(1) Nonautonomous case

Let the three roots of (3-6) be λ_1, λ_2 and λ_3. We will say the periodic solution under consideration is "stable" if and only if $|\rho_i|<1$, i. e., R. P. $(\lambda_i)<0$ $(i=1, 2, 3)$.

Then the condition that the periodic solution is stable is given by (Routh-Hurwitz criterion)

$$U>0,\quad V>0,\quad W>0,\quad UV>W\,.\tag{3-7}$$

case i) $\sigma \neq 0$

Since μ is a small quantity, it follows immediately that (3-7) becomes

$$\sigma<0, \quad t_r M^0<0, \quad \det M^0>0$$

where

$$M^0=\begin{pmatrix} K_{11}^0 & K_{12}^0 \\ K_{21}^0 & K_{22}^0 \end{pmatrix}.$$

Further it is obvious that the following relations hold.

$$t_r M^0=\frac{1}{\pi}\int_0^{2\pi} t_r G^0(\boldsymbol{x}^{(0)}(s))ds, \quad J_1=\pi^2 \det M^0$$

where

$$G^0=\begin{pmatrix} D_{11}^{(0)} & D_{21}^{(0)} \\ D_{12}^{(0)} & D_{22}^{(0)} \end{pmatrix}.$$

case ii) $\sigma=0$

It is easily observed that (3-7) reduces to

$$U_0>0, \quad V_0>0, \quad W_0>0, \quad U_0 V_0>W_0$$

where

$$U_0=-t_r K^0, \quad V_0=\sum_{i<j}(K_{ii}^0 K_{jj}^0-K_{ij}^0 K_{ji}^0),$$

$$W_0=-\det K^0.$$

Further it is apparent that the following relations hold.

$$U_0=-\frac{1}{\pi}\int_0^{2\pi} t_r D^{(0)}(\boldsymbol{x}^{(0)}(s)), \quad J_1=-\pi^3 W_0.$$

(2) Autonomous case

In this case, one of the ρ_i's is equal to 1 (Poincaré) and hence one of the λ_i's is equal to 0. The other two roots (say λ_1, λ_2) are decided from

$$\lambda^2+U\lambda+V=0. \tag{3-8}$$

We will say the periodic solution under consideration is "stable" if and only if R. P. $(\lambda_i)<0$ $(i=1, 2)$.

Then the condition that the periodic solution is stable is given by

$$U>0, \quad V>0. \tag{3-9}$$

case i) $\sigma \neq 0$

It is obvious that (3-9) becomes

$$\sigma<0, \quad t_r M^0<0$$

where

$$t_r M^0 = \frac{1}{\pi}\int_0^{2\pi} t_r G^0(x^{(0)}(s))ds\,.$$

case ii) $\sigma=0$

It is easily seen that (3-9) becomes

$$U_0>0,\quad V_0>0$$

where

$$U_0 = -t_r K^0 = -\frac{1}{\pi}\int_0^{2\pi} t_r D^{(0)}(x^{(0)}(s))ds\,,$$

$$V_0 = \sum_{i<j}(K^0_{ii}K^0_{jj} - K^0_{ij}K^0_{ji})\,.$$

IV. Pseudo-harmonic Oscillations of the Third Order

From the discussion mentioned above we know that the *pseudo-harmonic oscillations* of the third order are classified into four types corresponding to the classification of the differential equations which describe the oscillations.

f \ σ	$\sigma=0$	$\sigma\neq0$
f contains τ explicitely	tpye B	type C
f does not contain τ explicitely	type A	type D

An example of the *pseudo-harmonic oscillation* of type A is the self-sustained oscillation of the Colpitts oscillator (Ex. 1 in I.).

An example of type B is the synchronized oscillation of the Colpitts oscillator (Ex. 2 in I.).

Finally, an example of type C is the periodic oscillation of the Parametron circuit (Ex. 3 in I.).

The remaining case —— the *pseudo-harmonic oscillation* of type D —— is essentially equivalent to the pseudo-harmonic autonomous oscillation of the second order and hence it will be unnecessary to give an example.

We now show some practical results which are obtained by applying above analysis to the Colpitts oscillator.

IV-A. Self-sustained Oscillation of the Colpitts Oscillator

In Fig. 1 the oscillating current $A\sin\Omega t$ through the LC circuit and the dc voltage V_{20} (dc component of the oscillating voltage V_2) are approximately determined by

$$\begin{cases} \dfrac{f_c}{A}\dfrac{1}{C_2} - \dfrac{g_c}{A}\dfrac{1}{C_1} = -\dfrac{1}{\Omega}\left(\dfrac{L\Omega^2}{rC} + \dfrac{1}{RC_1^2}\right) & \text{(4-1)} \\ \left(1+\dfrac{f_s}{A}\right)\dfrac{1}{C_2} + \left(1-\dfrac{g_s}{A}\right)\dfrac{1}{C_1} = L\Omega^2 & \text{(4-2)} \\ f_0 + g_0 = -\dfrac{2}{R}(V_{20}+E_2) & \text{(4-3)} \end{cases}$$

where

$$\begin{cases} f(V_{20}-E_1+\frac{A}{\Omega C_2}\cos s\,,\ V_{20}-\frac{A}{\Omega C_1}\cos s)=\frac{f_0}{2}+(f_c\cos s+f_s\sin s)+\cdots\cdots \\ g(V_{20}-E_1+\frac{A}{\Omega C_2}\cos s\,,\ V_{20}-\frac{A}{\Omega C_1}\cos s)=\frac{g_0}{2}+(g_c\cos s+g_s\sin s)+\cdots\cdots . \end{cases}$$

Especially if $|f_s|\ll A$, $|g_s|\ll A$, then $\Omega \fallingdotseq 1/\sqrt{LC}$, hence A and V_{20} are determined from

$$\begin{cases} \frac{f_c}{A}\frac{1}{C_2}-\frac{g_c}{A}\frac{1}{C_1}=-\sqrt{LC}\left(\frac{1}{rC^2}+\frac{1}{RC_1{}^2}\right) & (4\text{-}4) \\ f_0+g_0=-\frac{2}{R}(V_{20}+E_2) & (4\text{-}5) \end{cases}$$

where

$$\begin{cases} f\left(V_{20}-E_1+\frac{A\sqrt{LC}}{C_2}\cos s\,,\ V_{20}-\frac{A\sqrt{LC}}{C_1}\cos s\right)=\frac{f_0}{2}+(f_c\cos s+f_s\sin s)+\cdots\cdots \\ g\left(V_{20}-E_1+\frac{A\sqrt{LC}}{C_2}\cos s\,,\ V_{20}-\frac{A\sqrt{LC}}{C_1}\cos s\right)=\frac{g_0}{2}+(g_c\cos s+g_s\sin s)+\cdots\cdots . \end{cases}$$

(1) Transistor Colpitts Oscillator

Let the "in-the-large characteristic" of a p-n-p transistor be represented by*

$$\begin{cases} I_E=\frac{-I_{EO}}{1-\alpha_N\alpha_I}(e^{\gamma V_E}-1)+\frac{\alpha_I I_{CO}}{1-\alpha_N\alpha_I}(e^{\gamma V_C}-1) \\ I_C=\frac{\alpha_N I_{EO}}{1-\alpha_N\alpha_I}(e^{\gamma V_E}-1)-\frac{I_{CO}}{1-\alpha_N\alpha_I}(e^{\gamma V_C}-1) \end{cases}$$

where $\gamma=q/kT$, $\alpha_N I_{EO}=\alpha_I I_{CO}<0$.

The relations between the notations of the transistor in Fig. 1 and that in Fig. 4 are

$$\begin{cases} V_E=-V_2\,, & V_C=V_1-V_2 \\ I_E=-I_1-I_2\,, & I_C=I_1\,. \end{cases}$$

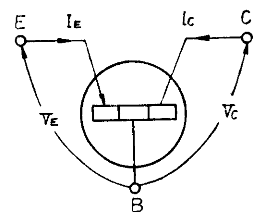

Fig. 4.

In this case we have $f_s=0$, $g_s=0$ and hence the period is approximately $2\pi/\sqrt{LC}$. The oscillating current through the LC circuit $A\sin(t/\sqrt{LC})$ and the dc voltage V_{20} (dc component of the oscillating voltage $V_{BE}=V_B-V_E$) are decided from

$$\begin{cases} \left(\frac{1}{C_2}+\frac{1-\alpha_I}{C_1}\right)\alpha_N e^{-\gamma E_1}I_1(\eta)-\left(\frac{\alpha_N}{C_2}-\frac{1-\alpha_N}{C_1}\right)\alpha_I e^{-\gamma V_{20}}I_1(\xi) \\ \qquad =\frac{\alpha_I(1-\alpha_N\alpha_I)}{I_{EO}}A\sqrt{LC}\left(\frac{1}{rC^2}+\frac{1}{RC_1{}^2}\right) \\ \alpha_N e^{-\gamma E_1}I_0(\eta)-e^{-\gamma V_{20}}I_0(\xi)=\frac{1-\alpha_N\alpha_I}{RI_{EO}}(V_{20}+E_2)-(1-\alpha_N) \end{cases}$$

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*) W. Shockley, M. Sparks aud G. K. Teal, Phys. Rev. 83 (1951) 151 or J. J. Ebers and J. K. Moll, Proc. I. R. E. 42 (1954) 1761.

where $\xi=\gamma A\sqrt{LC}/C_1$, $\eta=\gamma A\sqrt{LC}/C$; I_0, I_1 are modified Bessel functions.

(2) Vacuum-Tube Colppitts Oscillator

We shall consider the vauum-tube Colpitts oscillator of Fig. 5. Proceeding as before, we find that the oscillating current $A\sin\Omega t$ through the LC circuit and the grid bias voltage V_{20} are approximately determined by the followin relations.

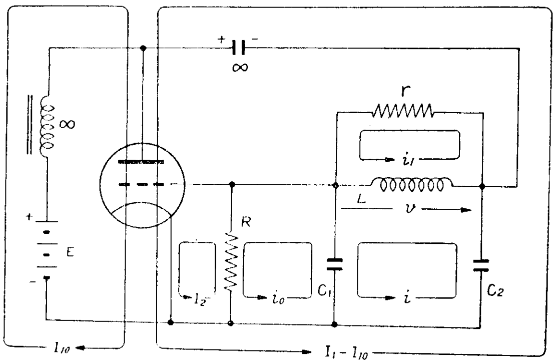

Fig. 5.

$$\begin{cases} \dfrac{f_c}{A}\dfrac{1}{C_2}-\dfrac{g_c}{A}\dfrac{1}{C_1}=-\dfrac{1}{\Omega}\left(\dfrac{L\Omega^2}{rC}+\dfrac{1}{RC_1^2}\right) \\ \left(1+\dfrac{f_s}{A}\right)\dfrac{1}{C_2}+\left(1-\dfrac{g_s}{A}\right)\dfrac{1}{C_1}=L\Omega^2 \\ g_0=-\dfrac{2}{R}V_{20} \end{cases}$$

where

$$\begin{cases} f\left(E+\dfrac{A}{\Omega C_2}\cos s,\ \ V_{20}-\dfrac{A}{\Omega C_1}\cos s\right)=\dfrac{f_0}{2}+(f_c\cos s+f_s\sin s)+\cdots\cdots \\ g\left(E+\dfrac{A}{\Omega C_2}\cos s,\ \ V_{20}-\dfrac{A}{\Omega C_1}\cos s\right)=\dfrac{g_0}{2}+(g_c\cos s+g_s\sin s)+\cdots\cdots. \end{cases}$$

Especially, if $|f_s|\ll A$, $|g_s|\ll A$, then $\Omega\doteqdot 1/\sqrt{LC}$, hence A and V_{20} are decided from

$$\begin{cases} \dfrac{f_c}{A}\dfrac{1}{C_2}-\dfrac{g_c}{A}\dfrac{1}{C_1}=-\sqrt{LC}\left(\dfrac{1}{rC^2}+\dfrac{1}{RC_1^2}\right) \\ g_0=-\dfrac{2}{R}V_{20} \end{cases}$$

where

$$\begin{cases} f\left(E+\dfrac{A\sqrt{LC}}{C_2}\cos s,\ \ V_{20}-\dfrac{A\sqrt{LC}}{C_1}\cos s\right)=\dfrac{f_0}{2}+(f_c\cos s+f_s\sin s)+\cdots\cdots \\ g\left(E+\dfrac{A\sqrt{LC}}{C_2}\cos s,\ \ V_{20}-\dfrac{A\sqrt{LC}}{C_1}\cos s\right)=\dfrac{g_0}{2}+(g_c\cos s+g_s\sin s)+\cdots\cdots. \end{cases}$$

These results are practically calculated by the analytical method making use of the analytical representation of the vacuum-tube characteristic,* or by the graphical method using the "constant-current characteristic curves" (See Fig. 6).

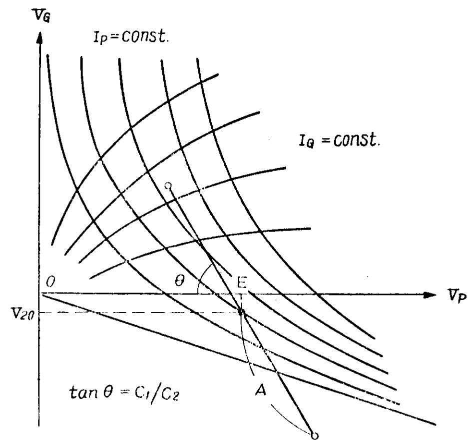

Fig. 6.

The efficiency η of this oscillator is easily obtained. Since $v \doteqdot (A/C\Omega)\cos\Omega t$, the average power dissipated at the load resistance r is $LA^2/2rC$. On the other hand, the averaage supplied power at E is $Ef_0/2$. Accordingly

$$\eta = \frac{LA^2}{ErCf_0}$$

where

$$f_0 = \frac{1}{\pi}\int_0^{2\pi} f\left(E + \frac{A\sqrt{LC}}{C_2}\cos s\,,\quad V_{20} - \frac{A\sqrt{LC}}{C_1}\cos s\right)ds\,.$$

(3) On the Representation of the Characteristic of the Nonlinear Element

For the time being, we have proceeded under the assumption that the nonlinear 3-pole is static (=resistive). However in some cases it will be necessary to characterize the nonlinear 3-pole by a dynamic (=non-resistive) representation. An example for the capasitive case is as follows.

$$\begin{cases} I_1 = f\left(V_1,\ V_2,\ \dfrac{dV_1}{dt},\ \dfrac{dV_2}{dt}\right) \\ I_2 = g\left(V_1,\ V_2,\ \dfrac{dV_1}{dt},\ \dfrac{dV_2}{dt}\right). \end{cases}$$

In this case we have the differential equation:

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*) To cite an example,

$I_g=0$ when $V_g+DV_p \leqq 0$ or $V_g \leqq 0$

$I_p=0$ when $V_g+DV_p \leqq 0$ or $V_p \leqq 0$

$I_p=0,\ I_g=\beta V_s^{\alpha}$ when $V_g+DV_p \geqq 0$ and $V_p \leqq 0$

$I_p=\beta V_s^{\alpha},\ I_g=0$ when $V_g+DV_p \geqq 0$ and $V_g \leqq 0$

$I_p=\beta V_s^{\alpha} V_p^{\alpha}/(V_g^{\alpha}+V_p^{\alpha}),\ I_g=\beta V_s^{\alpha} V_g^{\alpha}/(V_g^{\alpha}+V_p^{\alpha})$ when $V_g \geqq 0$ and $V_p \geqq 0$

where $V_s=\dfrac{V_g+DV_p}{1+D}$, D is Durchgriff; α, β are positive constant and $\alpha \doteqdot 1.5$.

This representation is not analytic in the whole (V_p, V_g)-plane but can he used in the former analysis since four conductances $\left(\dfrac{\partial I_p}{\partial V_p},\ \dfrac{\partial I_p}{\partial V_g},\ \dfrac{\partial I_g}{\partial V_p},\ \dfrac{\partial I_g}{\partial V_g}\right)$ are continuous functions of variables (V_p, V_g).

$$\begin{cases} \dfrac{dx}{d\tau} = -z+\mu F(x,\ y,-z,\ z)+O_2(\mu) \\ \dfrac{dy}{d\tau} = z-\mu b(1-a)ky-\mu q-\mu G(x,\ y,-z,\ z,)+O_2(\mu) \\ \dfrac{dz}{d\tau} = ax-(1-a)y-\mu l z+O_2(\mu) \end{cases}$$

instead of (1-2) since we may write

$$\begin{cases} \dfrac{f\left(V_1,\ V_2,\ \dfrac{dV_1}{dt},\ \dfrac{dV_2}{dt}\right)}{I} = \mu F\left(x,\ y,\ \dfrac{dx}{d\tau},\ \dfrac{dy}{d\tau}\right) = \mu F(x,\ y,-z,\ z)+O_2(\mu) \\ \dfrac{g\left(V_1,\ V_2,\ \dfrac{dV_1}{dt},\ \dfrac{dV_2}{dt}\right)}{I} = \mu G\left(x,\ y,\ \dfrac{dx}{d\tau},\ \dfrac{dy}{d\tau}\right) = \mu G(x,\ y,-z,\ z)+O_2(\mu). \end{cases}$$

This equation can be treated by the similar procedure as before.

IV-B. Synchronized Oscillation of the Colpitts Oscillator

In this section we shall consider the case of subharmonic synchronization ($n=1$) for practical importance.

The oscillating current through the LC circuit: $A\sin\frac{\omega}{m}t+B\sin\frac{\omega}{m}t$ (m-th order subharmonic of the synchronising signal) and the dc voltage V_{20} (dc component of the oscillating voltage V_{20}) are decided from

$$\begin{cases} \dfrac{f_c}{C_2} - \dfrac{g_c}{C_1} = B\left(\dfrac{\omega^2 L}{m^2} - \dfrac{1}{C}\right) - \dfrac{Am}{\omega Cr}\left(\dfrac{\omega^2 L}{m^2} + \dfrac{rC}{RC_1^2}\right) \\ \dfrac{f_s}{C_2} - \dfrac{g_s}{C_1} = A\left(\dfrac{\omega^2 L}{m^2} - \dfrac{1}{C}\right) + \dfrac{Bm}{\omega Cr}\left(\dfrac{\omega^2 L}{m^2} + \dfrac{rC}{RC_1^2}\right) \\ f_0+g_0 = -\dfrac{2}{R}(V_{20}+E_2) \end{cases}$$

where

$$\begin{cases} f\left(V_{20}-E_1+\dfrac{Am}{\omega C_1}\cos s-\dfrac{Bm}{\omega C_2}\sin s,\ V_{20}-\dfrac{Am}{\omega C_1}\cos s+\dfrac{Bm}{\omega C_2}\sin s-e\sin ms\right) \\ \qquad = \dfrac{f_0}{2}+(f_c\cos s+f_s\sin s)+\cdots\cdots \\ g\left(V_{20}-E_1+\dfrac{Am}{\omega C_1}\cos s-\dfrac{Bm}{\omega C_2}\sin s,\ V_{20}-\dfrac{Am}{\omega C_1}\cos s+\dfrac{Bm}{\omega C_2}\sin s-e\sin ms\right) \\ \qquad = \dfrac{g_0}{2}+(g_c\cos s+g_s\sin s)+\cdots\cdots. \end{cases}$$

If the sychronising signal is inserted as in Fig. 7, we have the differential equation:

$$\begin{cases} \dfrac{dV_1}{dt} = -\dfrac{1}{C_2} - \dfrac{1}{C_2}f(V_1,\ V_2) \\ \dfrac{dV_2}{dt} = \dfrac{1}{C_2}i - \dfrac{1}{RC_1}V_2 - \dfrac{1}{RC_1}E_2 - \dfrac{1}{C_1}g(V_1,\ V_2) \end{cases}$$

$$\left\{\begin{aligned} \frac{di}{dt} &= \frac{1}{L}(V_1+E_1)-\frac{1}{L}V_2+\frac{e}{L}\sin\omega t-\frac{1}{rC}i+\frac{1}{rRC_1}V_2 \\ &+\frac{1}{rRC_1}E_2-\frac{1}{rC_2}f(V_1, V_2)+\frac{1}{rC_1}g(V_1, V_2). \end{aligned}\right.$$

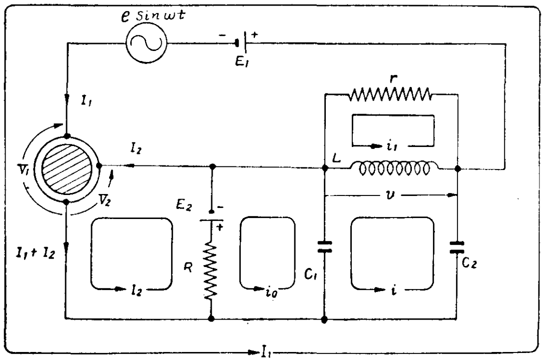

Fig. 7

In case of $|\omega^2LC-1|\ll 1$, we can treat as before provided $e\ll\omega LI$. If this is not the case, putting

$$\left\{\begin{aligned} V_1 &= V_1'+\frac{Ce}{(1-\omega^2LC)C_2}\sin\omega t \\ V_2 &= V_2'-\frac{Ce}{(1-\omega^2LC)C_1}\sin\omega t \\ i &= i'-\frac{\omega Ce}{1-\omega^2LC}\cos\omega t \end{aligned}\right.$$

and making use of new variables (V_1', V_2', i') instead of (V_1, V_2, i) we can proceed as before.

Acknowledgements

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