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On the Steady Motion of a Viscous Fluid through Double Pipes

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Abstract

It is pointed out by Greenhill that the Navier-Stokes equation of an incompressible viscous fluid flowing in a pipe has the same form as the equation which is satisfied by the "*torsion function*" in the theory of elasticity under some conditions. In this paper, by using this theory, we obtain the velocity distributions and the discharges per unit time over the pipe sections, boundary curves of which are confocal ellipses and eccentric circles. Futheremore, for a special case, the flow in confocal elliptic pipes is numerically estimated and compared with that in the corresponding singl elliptic one. Similary, comparisons are made between the flow in pipes of eccentric circles and that of concentric ones.

I. Introduction

There are many cases in which the exact solutions of laminar flows of viscous fluids through straight pipes can be obtained. These cases, however, are almost limited to the single straight pipes of certain cross-sections. If the cross-section of a pipe is composed of doubly connected domain, the exact solution of laminar flow through such a pipe has scarecely been obtained, except the trivial case of concentric circles.

We have solved two cases of steady laminar flows of viscous fluids through straight uniform pipes. The one is the case where the boundary curves of the cross-section of the pipe are confocal ellipses; the other is the case of eccentric circles.

In this paper, first, it is shown that the solution of the Navier-Stokes equation of pipe flow is reduced to the "*torsion function*" of the torsion problem for the same cylinder as the pipe under some appropriate conditions. Second, by applying the above theory, the general expressions of the velocity distributions and the discharges per unit time are obtained over the pipe sections, boundary curves of which are confocal ellipses and eccentric circles. Futheremore, for a special case, the velocity distribution and the discharge of confocal elliptic pipe are numerically obtained and compared with those of the correspondent single elliptic one. Similary, comparisons are made between the pipe of eccentric circles and that of concentric ones.

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II. Fundamental Theory

We shall describe a motion of a viscous fluid by the Cartesian coordinate x, y and z. By denoting the velocity components along x-, y-, z- directions by u, v, w respectively; the components of the body force by X, Y, Z; the density of the fluid by ρ; the viscosity by μ; and the pressure by p, the Navier-Stokes equations of a incompressible viscous fluid are expressed as follows,

$$\rho\frac{Du}{Dt}=\rho X-\frac{\partial p}{\partial x}+\mu\nabla^2 u$$
$$\rho\frac{Dv}{Dt}=\rho Y-\frac{\partial p}{\rho y}+\mu\nabla^2 v$$
$$\rho\frac{Dw}{Dt}=\rho Z-\frac{\partial p}{\partial z}+\mu\nabla^2 w\ , \tag{1}$$

where Du/Dt etc. are

$$\frac{Du}{Dt}=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\quad \text{etc.}$$

and $\nabla^2 u$ etc. are

$$\nabla^2 u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\quad \text{etc.}$$

Let us consider the case where a fluid flow in a straight pipe with a uniform cross-section. Take the z- axis in the direction of the pipe length, and assume the following four conditions;

i) No body force are exerted,

$$\text{i.e.}\quad X=Y=Z=0.$$

ii) w does not depend on z but x and y,

$$\text{i.e.}\quad \frac{\partial w}{\partial z}=0.$$

iii) The flow of the fluid is laminar,

$$\text{i.e.}\quad u=v=0.$$

iv) The flow is steady,

$$\text{i.e.}\quad \frac{\partial u}{\partial t}=\frac{\partial v}{\partial t}=\frac{\partial w}{\partial t}=0.$$

Under these conditions, the equations (1) are reduced to

$$\frac{\partial p}{\partial x}=0\ ,\quad \frac{\partial p}{\partial y}=0 \tag{2}$$

and

$$\mu\left(\frac{\partial^2 w}{\partial x^2}\right)+\left(\frac{\partial^2 w}{\partial y^2}\right)=\frac{\partial p}{\partial z}\ . \tag{3}$$

It follows that the pressure is a function of z alone and that the pressure gradient along the direction of the flow is constant. Let the constant pressure gradient denote by P, the equation (3) is reduced to

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=-\frac{P}{\mu} \quad . \tag{4}$$

By the boundary condition, it is required that $w=0$ at the boundary.

Hence, if we put

$$w=\psi-\frac{P}{4\mu}(x^2+y^2) \qquad \text{with} \quad \nabla^2\psi=0 \, , \tag{5}$$

we can show that w satisfies the relation (4), and the boundary conditions turns out

$$\psi=\frac{P}{4\mu}(x^2+y^2) \tag{6}$$

at the boundary. Since the equation (4) is a linear equation, we can treat the problem more easily.

As be pointed out by Greenhill, the equation (4) is the same form as the equation which is satisfied with the "*torsion function*" in the theory of elasticity, and our problem is equivalent to seek for the torsion function of the cylinder of the corresponding cross-section.

III. The Case where the Boundaries are Confocal Ellipses

Using the previous results, we shall discuss the flow between a double pipe, of which outher and inner boundaries are confocal ellipses.

Introducing the elliptic coordinate ξ, η by the relation

$$\begin{aligned} & x+iy=\mathrm{c}\cdot\cosh\,(\xi+i\eta) \\ \text{i.e.}\quad & x=\mathrm{c}\cdot\cosh\xi\cdot\cos\eta \\ & y=\mathrm{c}\cdot\sinh\xi\cdot\sin\eta, \end{aligned} \tag{7}$$

it is immediately shown that the relation

$$\frac{x^2}{(\mathrm{c}\cdot\cosh\xi)^2}+\frac{y^2}{(\mathrm{c}\cdot\sinh\xi)^2}=1$$

is satisfied. Hence, $x-y$ curves corresponding $\xi=\xi_0$ and $\xi=\xi_1$ become confocal ellipses. (See Fig. 1)

Substituting the expressions (7) for x and y into the first equation of (5), we obtain the expression

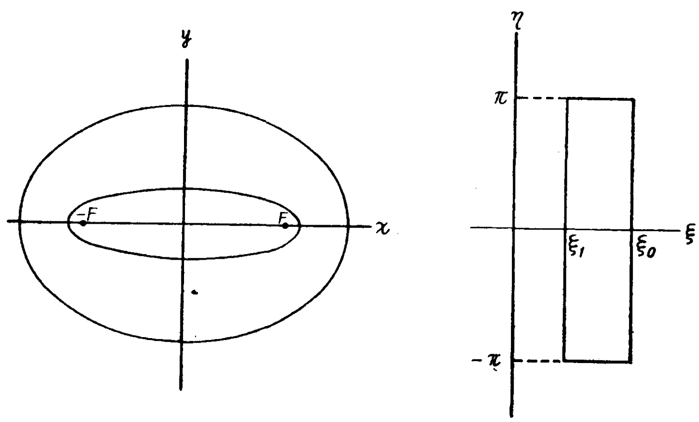

Figure 1

$$w=\psi-\frac{Pc^2}{8\mu}(\cosh 2\xi+\cos 2\eta) \tag{8}$$

Then, the boundary conditions become as follows,

$$\psi=\frac{Pc^2}{8\mu}(\cosh 2\xi_0+\cos 2\eta) \qquad (at \quad \xi=\xi_0)$$

and

$$\psi=\frac{Pc^2}{8\mu}(\cosh 2\xi_1+\cos 2\eta) \qquad (at \quad \xi=\xi_1). \tag{9}$$

Since the functions

$$\psi_n=\left\{\frac{A_n \sinh n(\xi_0-\xi)+B_n \sinh n(\xi-\xi_1)}{\sinh n(\xi_0-\xi_1)}\right\}\cos n\eta \quad (n=0,1,2,\cdots\cdots)$$

satisfy the relations $\nabla^2\psi_n=0$, ψ can be put in the form,

$$\psi=\sum_{n=0}^{\infty}\frac{A_n \sinh n(\xi_0-\xi)+B_n \sinh n(\xi-\xi_1)}{\sinh n(\xi_0-\xi_1)}\cdot\cos n\eta\,. \tag{10}$$

The boundary conditions (9) are described as follows,

$$\sum_{n=0}^{\infty} B_n \cos n\eta=\frac{Pc^2}{8\mu}(\cosh 2\xi_0+\cos 2\eta)$$

and

$$\sum_{n=0}^{\infty} A_n \cos n\eta=\frac{Pc^2}{8\mu}(\cosh 2\xi_1+\cos 2\eta)\,.$$

Comparing the coefficients of cos $n\eta$ of the right hand side with those of the left hand side, we obtain

$$B_0=\frac{Pc^2}{8\mu}\cosh 2\xi_0\,,$$

$$B_2=\frac{Pc^2}{8\mu}\,,$$

$$A_0=\frac{Pc^2}{8\mu}\cosh 2\xi_1\,,$$

$$A_2=\frac{Pc^2}{8\mu}$$

and

$$A_n=B_n=0 \qquad \text{for} \quad n\neq 0 \quad \text{or} \quad 2.$$

Substituting these coefficients into the equation (10), after a few calculations we obtain

$$\psi=\frac{Pc^2}{8\mu}\left\{\left(\frac{\xi_0-\xi}{\xi_0-\xi_1}\right)\cosh 2\xi_1+\left(\frac{\xi-\xi_1}{\xi_0-\xi_1}\right)\cosh 2\xi_0 + \frac{\sinh 2(\xi_0-\xi)+\sinh 2(\xi-\xi_1)}{\sinh 2(\xi_0-\xi_1)}\cdot\cos 2\eta\right\}. \tag{11}$$

Hence, the velocity w is expressed as follows

$$w=\frac{Pc^2}{8\mu}\left\{\left(\frac{\xi_0-\xi}{\xi_0-\xi_1}\right)\cosh 2\xi_1+\left(\frac{\xi-\xi_1}{\xi_0-\xi_1}\right)\cosh 2\xi_0\right.$$
$$\left.+\frac{\sinh 2(\xi_0-\xi)+\sinh 2(\xi-\xi_1)}{\sinh 2(\xi_0-\xi_1)}\cos 2\eta-(\cosh 2\xi+\cos 2\eta)\right\}. \tag{12}$$

Discharge per unit time Q is obtained by the double integration of the velocity over the sectional area S of the pipe and expressed as follows

$$Q=\iint_S w(x,y)\,dx\,dy=\iint_{S'} w(\xi,\eta)\cdot\begin{vmatrix}\frac{\partial x}{\partial\xi} & \frac{\partial x}{\partial\eta}\\ \frac{\partial y}{\partial\xi} & \frac{\partial y}{\partial\eta}\end{vmatrix}\,d\xi\,d\eta\ , \tag{13}$$

where S' is the domain of the $\xi-\eta$ plane corresponding to the pipe section S. Since the Jacobian is easily calculated as $c^2/2\cdot(\cosh 2\xi-\cos 2\eta)$, the equation (13) is expressed as follows;

$$Q=\frac{Pc^4}{16\mu}\int_{\xi_1}^{\xi_0}\int_{-\pi}^{\pi}\left\{\frac{(\xi-\xi_1)\cosh 2\xi_0+(\xi_0-\xi)\cosh 2\xi_1}{\xi_0-\xi_1}+\frac{\sinh 2(\xi_0-\xi)+\sinh 2(\xi-\xi_1)}{\sinh 2(\xi_0-\xi_1)}\cos 2\eta\right.$$
$$\left.-(\cosh 2\xi+\cos 2\eta)\right\}(\cosh 2\xi-\cos 2\eta)\,d\eta\,d\xi$$
$$=\frac{Pc^4}{16\mu}\int_{\xi_1}^{\xi_0}\int_{-\pi}^{\pi}\left\{\frac{\xi_0\cosh 2\xi_1-\xi_1\cosh 2\xi_0}{\xi_0-\xi_1}\cdot\cosh 2\xi+\frac{\cosh 2\xi_0-\cosh 2\xi_1}{\xi_0-\xi_1}\cdot\xi\cosh 2\xi\right.$$
$$\left.-\frac{\sinh 2(\xi_0-\xi)+\sinh 2(\xi-\xi_1)}{\sinh 2(\xi_0-\xi_1)}\cos^2 2\eta-(\cosh^2 2\xi-\cos^2 2\eta)\right\}d\eta\,d\xi\ . \tag{14}$$

According to the elementary integral calculus,

$$\int_{\xi_1}^{\xi_0}\int_{-\pi}^{\pi}\cosh 2\xi\,d\eta\,d\xi=\pi(\sinh 2\xi_0-\sinh 2\xi_1)\ ,$$

$$\int_{\xi_1}^{\xi_0}\int_{-\pi}^{\pi}\xi\cosh 2\xi\,d\eta\,d\xi=\pi(\xi_0\sinh 2\xi_0-\xi_1\sinh 2\xi_1)-\frac{\pi}{2}(\cosh 2\xi_0-\cosh 2\xi_1)\ ,$$

$$\int_{\xi}^{\xi_0}\int_{-\pi}^{\pi}\{\sinh 2(\xi_0-\xi)+\sinh 2(\xi-\xi_1)\}\cosh^2 2\eta\,d\eta\,d\xi=\pi\{\cosh 2(\xi_0-\xi_1)-1\}$$

and

$$\int_{\xi_1}^{\xi_0}\int_{-\pi}^{\pi}(\cosh^2 2\xi-\cos^2 2\eta)\,d\eta\,d\xi=\frac{\pi}{4}\{\sinh 4\xi_0-\sinh 4\xi_1\}\ .$$

Putting these results in the expression (14), Q is reduced to

$$Q=\frac{P\pi c^4}{16\mu}\left\{\frac{\sinh 4\xi_0-\sinh 4\xi_1}{4}-\frac{(\cosh 2\xi_0-\cosh 2\xi_1)^2}{2(\xi_0-\xi_1)}-\tanh(\xi_0-\xi_1)\right\}\ . \tag{15}$$

As a special example, we have performed numerical calculations for the case where $\xi_0=1$, $\xi_1=0$ and $c=1$. The velocity distribution is shown in Fig. 2, and the discharge per unit time is $0.4408P/\mu$. For the single elliptic pipe of the same outer boundary, we can see in Fig. 3, and the flow quantity is $0.8838P/\mu$.

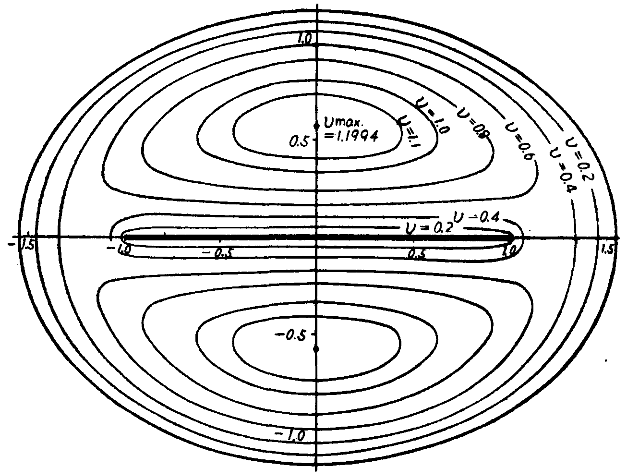

Figure 2

$$v = w \times \frac{8\mu}{P}$$

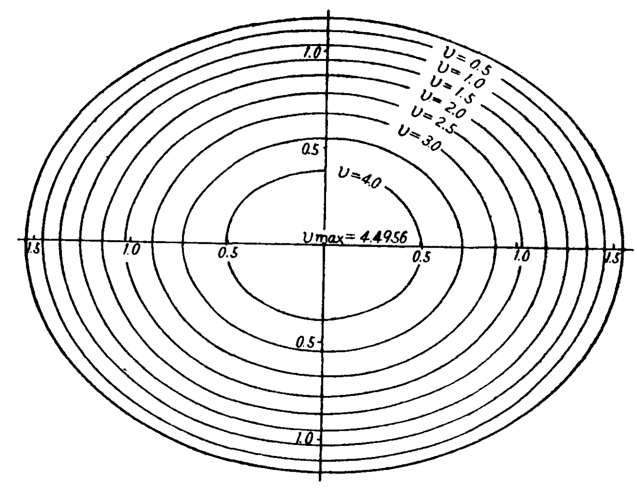

Figure 3

$$v = w \times \frac{8\mu}{P}$$

IV. The Case where the Boundaries are Eccentric Circles

As another example, we shall consider the case where the inner and outer boundaries are circles which are not concentric. If the bipolar coordinate ξ, η is introduced by the relation

$$x + iy = a \tan\frac{1}{2}(\xi + i\eta)$$

$$i.\ e.\quad x = \frac{a\sin\xi}{\cos\xi + \cosh\xi} \tag{16}$$

$$y = \frac{a\sinh\eta}{\cos\xi + \cosh\eta},$$

the expression,

$$x^2+(y-a\coth\eta)^2=a^2\mathrm{cosech}^2\eta$$

shows that $x-y$ curves corresponding to $\eta=\alpha$ and $\eta=\beta$ produce eccentric circles. (See Fig. 4) And, inversely, if the radii of two circles r_1, r_2 and the distance between them d, are arbitrarily given, the necessary parameters a, α, β for these eccentric circles are easily obtained by the following relations,

$$\alpha/r_1=\coth\alpha-\sqrt{\left(\frac{r_2}{r_1}\right)^2-\sinh^2\alpha}$$

$$a=r_1\sinh\alpha$$

$$\beta=\sinh^{-1}\frac{a}{r_2}\ .$$

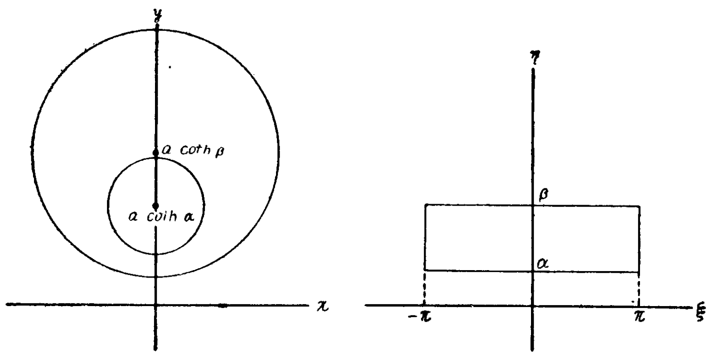

Figure 4

Substituting the equation (16) into the equation (6), we obtain

$$w=\psi-\frac{Pa^2}{4\mu}\left(\frac{\cosh\eta-\cos\xi}{\cosh\eta+\cos\xi}\right)\ . \qquad (17)$$

The boundary conditions are

$$\eta=\alpha:\quad \psi=\frac{Pa^2}{4\mu}\left(\frac{\cosh\alpha-\cos\xi}{\cosh\alpha+\cos\xi}\right)$$

and

$$\eta=\beta:\quad \psi=\frac{Pa^2}{4\mu}\left(\frac{\cosh\beta-\cos\xi}{\cosh\beta+\cos\xi}\right)\ .$$

Since the functions

$$\psi_n=\left\{A_n\frac{\sinh n(\eta-\alpha)}{\sinh n(\beta-\alpha)}+B_n\frac{\sinh n(\beta-\eta)}{\sinh n(\beta-\alpha)}\right\}\cos n\xi \qquad (n=0,\ 1,\ 2,\cdots\cdots)$$

satisfy the equations $\nabla^2\psi_n=0$, it may be considered the function ψ is expanded as follows

$$\psi=\sum_{n=0}^{\infty}\left\{A_n\frac{\sinh n(\eta-\alpha)}{\sinh n(\beta-\alpha)}+B_n\frac{\sinh n(\beta-\eta)}{\sinh n(\beta-\alpha)}\right\}\cos n\xi \qquad (18)$$

to satisfy the boundary conditions

$$\eta=\alpha:\quad \sum_{n=0}^{\infty} B_n \cos n\xi=\frac{Pa^2}{4\mu}\cdot\frac{\cosh\alpha-\cos\xi}{\cosh\alpha+\cos\xi}$$

$$\eta=\beta:\quad \sum_{n=0}^{\infty} A_n \cos n\xi=\frac{Pa^2}{4\mu}\cdot\frac{\cosh\beta-\cos\xi}{\cosh\beta+\cos\xi} \qquad (19)$$

And the coefficients A_n, B_n can be determined as the coefficients of Fourier series. The results of calculations are

$$A_o=\frac{Pa^2}{4\mu}\frac{1}{\pi}\int_0^{\pi}\frac{\cosh\beta-\cos\xi}{\cosh\beta+\cos\xi}d\xi=\frac{Pa^2}{4\mu}(2\coth\beta-1)\ ,$$

$$B_o=\frac{Pa^2}{4\mu}\frac{1}{\pi}\int_0^{\pi}\frac{\cosh\alpha-\cos\xi}{\cosh\alpha+\cos\xi}d\xi=\frac{Pa^2}{4\mu}(2\coth\alpha-1)\ ,$$

$$A_n=\frac{Pa^2}{4\mu}\frac{2}{\pi}\int_0^{\pi}\frac{\cosh\beta-\cos\xi}{\cosh\beta+\cos\xi}\cos n\xi\, d\xi=\frac{Pa^2}{\mu}\cdot(-1)^n\coth\beta\cdot e^{-n\beta}$$

and

$$B_n=\frac{Pa^2}{4\mu}\frac{2}{\pi}\int_0^{\pi}\frac{\cosh\alpha-\cos\xi}{\cosh\alpha+\cos\xi}\cos n\xi\, d\xi=\frac{Pa^2}{\mu}\cdot(-1)^n\coth\alpha\cdot e^{-n\alpha}\ .$$

Then, we can obtain

$$\psi=\frac{Pa^2}{4\mu}\left\{\frac{\eta-\alpha}{\beta-\alpha}\cdot(2\coth\beta-1)+\frac{\beta-\eta}{\beta-\alpha}\cdot(2\coth\alpha-1)\right.$$
$$\left.+4\sum_{n=0}^{\infty}(-1)^n\frac{e^{-n\beta}\coth\beta\sinh n(\eta-\alpha)+e^{-n\alpha}\coth\alpha\sinh n(\beta-\alpha)}{\sinh n(\beta-\alpha)}\right\}\cos n\xi$$

and

$$w=\frac{Pa^2}{4\mu}\left\{2\cdot\frac{(\eta-\alpha)\coth\beta+(\beta-\eta)\coth\alpha}{\beta-\alpha}\right.$$

$$\left.+4\sum_{n=1}^{\infty}(-1)^n\frac{e^{-n\beta}\coth\beta\sinh n(\eta-\alpha)+e^{-n\alpha}\coth\alpha\sinh n(\beta-\eta)}{\sinh n(\beta-\alpha)}\cdot\cos n\xi-\frac{2\cosh\eta}{\cosh\eta+\cos\xi}\right\}.$$
$$\cdots\cdots(20)$$

After easy but tedious calculations, the discharge per unit time is expressed as follows

$$Q=\iint_S w(x,y)\,dx\,dy=\int_\alpha^\beta\int_{-\pi}^{\pi} w(\xi,\eta)\cdot\frac{a^2}{(\cosh\eta+\cos\xi)^2}\,d\xi\,d\eta$$

$$=\frac{\pi Pa^2}{2\mu}\Big[\frac{\coth\alpha-1}{\sinh^2\alpha}-\frac{\coth\beta-1}{\sinh^2\beta}-\frac{(\coth\alpha-\coth\beta)^2}{\beta-\alpha}-\frac{3}{4}\Big(\frac{1}{\sinh^4\alpha}-\frac{1}{\sinh^4\beta}\Big)$$

$$+\sum_{n=1}^{\infty}\Big\{\frac{-e^{-n(\alpha+\beta)}(\coth\beta-\coth\alpha)^2(\coth\alpha+\coth\beta+2n)}{\sinh n(\beta-\alpha)}$$

$$+\frac{e^{-(\beta-\alpha)}\coth\beta-e^{n(\beta-\alpha)}\coth\alpha}{\sinh n(\beta-\alpha)}\cdot\Big(\frac{e^{-2n\beta}}{\sinh^2\beta}-\frac{e^{-2n\alpha}}{\sinh^2\alpha}\Big)\Big\}\Big]. \qquad (21)$$

For a special case ($\alpha=1$, $\beta=2$ and $a=1$), the velocity distribution is shown in Fig. 5 and the numerical comparisons of the discharge between the case the eccentric circles and that of concentric ones of the same dimention are seen in the following,

Eccentric circles ………………………………………… $Q=0.07369P/\mu$
Concentric circles………………………………………… $Q=0.05727P/\mu$.

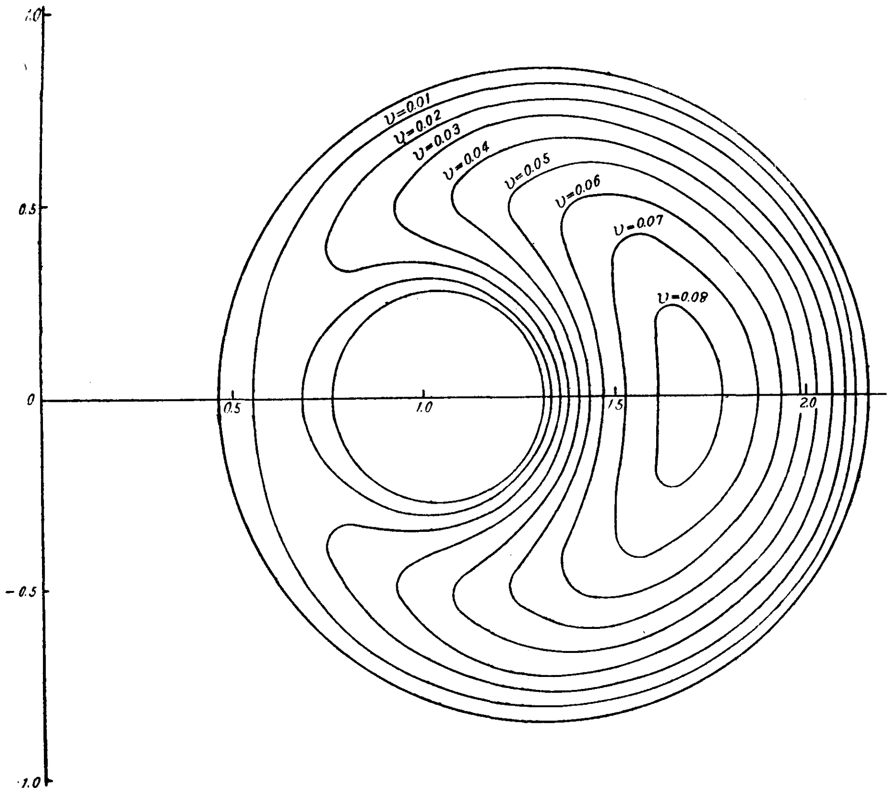

Figure 5 $v=w\times\dfrac{\mu}{Pa^2}$