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Thermal Stress in a Semi-Infinite Solid and a Thick Plate under Steady Distribution of Temperature *

(Recieved March 13, 1957)

Rokurō MUKI **

Abstract

This paper contains the exact and general solutions for the thermal stress in a semi-infinite solid and a thick plate under steady distribution of temperature. The approach used rests on the method of Hankel transforms in the three dimensional theory of elasticity which is introduced into the axisymmetric case by Harding and Sneddon and generalized to the unsymmetric case by the present author. It is found that the stresses in the direction normal to the plane surface, that is, σ_z, $\tau_{\theta z}$ and τ_{zr} vanish everywhere in a semi-infinite solid and a plate with infiinite extent when the distribution of temperature is steady. The general solution is then used to solve some particular problems of a thick plate. Numerical calculation is carried out in detail and the result is compared with the corresponding solution for a thin plate.

Nomenclatures

The following nomenclatures are used in this paper:

μ = Modulus of rigidity

ν = Poisson's ratio

ε = linear thermal expansion coefficient

T = distribution of temperature in the medium

$u,\ v,\ w;\ \sigma_r,\ \sigma_\theta,\ \cdots\cdots$ = components of displacements and stresses, respectively.

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla_m^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{m^2}{r^2} + \frac{\partial^2}{\partial z^2}$$

Introduction

Although considerable attention has been paid to the thermal stress in a body due to an inclusion of different material in it, comparatively little is known of the thermal stress in a body with three dimensional distribution of temperature varying

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from place to place. The solution of the latter problem have been obtained for a sphere by E. Almansi,[1] for a circular cylinder by T. Suhara,[2] by E. Melan[3] and by T. Tsubouchi[4] and for a spheroid by the present author.[5]

In this paper, the problem of the thermal stress in a semi-infinite solid and a thick plate with steady distribution of temperature is considered. In the first part of the paper is introduced the general expression in form of the Hankel tranforms for the particular solution of the thrmo-displacement equations which can be applied to any (steady or unsteady) state of temperature distribution. The procedure used is similar to what was adopted to obtain the transforms of the general solution of the displacement-equilibrium equations in the previous papers[6] [7] which dealt with the generalization of Sneddon's method[8] [9] to the unsymmetric case. The general solution for the thermal stress in a semi-infinite solid and a thick plate with the steady distribution of temperature are then obtained by the aid of the foregoing method of solution by Hankel transforms.

The Transformation of a Particular Solntion of the Thermo-displacement Equations

If we employ the cylindrical coordinates (r, θ, z), a particular solution of the thermo-displacement equations can be taken as[10]

$$u=\frac{1+\nu}{1-\nu}\,\varepsilon\frac{\partial\Omega}{\partial r},\qquad v=\frac{1+\nu}{1-\nu}\,\varepsilon\frac{\partial\Omega}{r\partial\theta},\qquad w=\frac{1+\nu}{1-\nu}\,\varepsilon\frac{\partial\Omega}{\partial z},\tag{1}$$

and the corresponding stress components are

$$\sigma_r/2\mu=\frac{1+\nu}{1-\nu}\,\varepsilon\Big[\frac{\partial^2\Omega}{\partial r^2}-T\Big],$$

$$\sigma_\theta/2\mu=\frac{1+\nu}{1-\nu}\,\varepsilon\Big[\frac{1}{r}\frac{\partial\Omega}{\partial\theta}+\frac{1}{r^2}\frac{\partial^2\Omega}{\partial\theta^2}-T\Big],$$

$$\sigma_z/2\mu=\frac{1+\nu}{1-\nu}\,\varepsilon\Big[\frac{\partial^2\Omega}{\partial z^2}-T\Big],$$

1) S. Timoshenko, & J.N. Goodier, Theory of Elasticity, McGraw-Hill. p. 433 (1951)
2) T. Suhara, read at the Congress of the Japan Soc. Mech. Eng. Nov. 11th, (1951)
3) E. Melan, & H. Parkus, Wärmespannungen, Springer Verlag, Wien, (1953)
4) T. Tsubouchi, Trans. Jap. Soc. Mech. Eng., Vol. 19, No.83, p. 35 (1953)
5) R. Muki, Proc. Fac. Keio Univ. Vol. 6, No. 20, p. 10, (1953)
6) R. Muki, Proc. Fac. Eng. Keio Univ. Vol. 8, No. 30, p. 8, (1955)
7) R. Muki, Proc. 5th Japan National Congress for Applied Mechanics. p. 119, (1955)
8) J. W. Harding and I. N. Sneddon, Proc. Camb. Phil. Soc., Vol. 41, p. 16, (1945)
9) I. N. Sneddon, Fourier Transforms, McGraw-Hill, (1951)
10) See (1), p. 433. The expressions employed here differ from Timoshenko's in the multiplier $\frac{1+\nu}{1-\nu}$ which is introduced for the sake of convenience.

$$\left.\begin{aligned}\tau_{\theta z}/2\mu &= \frac{1+\nu}{1-\nu}\,\varepsilon\frac{\partial^2\Omega}{r\partial\theta\partial z},\\ \tau_{zr}/2\mu &= \frac{1+\nu}{1-\nu}\,\varepsilon\frac{\partial^2\Omega}{\partial r\partial z},\\ \tau_{r\theta}/2\mu &= \frac{1+\nu}{1-\nu}\,\varepsilon\frac{\partial}{r\partial\theta}\left[\frac{\partial\Omega}{\partial r}-\frac{\Omega}{r}\right],\end{aligned}\right\} \tag{2}$$

where

$$\nabla^2\Omega = T. \tag{3}$$

In the derivation of (1), it is assumed that the inertia terms in the displacement equations are so small that they can be neglected in comparison with the other terms.

We may write Ω and T in the following forms;

$$\left.\begin{aligned}\Omega(r,\theta,z,t) &= \sum_{m=0}^{\infty}[\Omega_m(r,z,t)\cos m\theta + \bar{\Omega}_m(r,z,t)\sin m\theta]\\ T(r,\theta,z,t) &= \sum_{m=0}^{\infty}[T_m(r,z,t)\cos m\theta + \bar{T}_m(r,z,t)\sin m\theta]\end{aligned}\right\} \tag{4}$$

For the sake of simplicity, we put $\bar{\Omega}_m = \bar{T}_m = 0$ [11] and consider only a single value of m without loss in generality.

Substituting (4) in (3), the relation between Ω_m and T_m is obtained as

$$\nabla_m{}^2\Omega_m = \left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{m^2}{r^2}+\frac{\partial^2}{\partial z^2}\right)\Omega_m = T_m\,. \tag{5}$$

Using the formulas of the Hankel transforms,[12] it can be shown that

$$\left(\frac{d^2}{dz^2}-\xi^2\right)L_m = M_m \tag{6}$$

where

$$L_m = \int_0^\infty r\Omega_m J_m(\xi r)\,dr, \qquad M_m = \int_0^\infty rT_m J_m(\xi r)\,dr\,. \tag{7}$$

Now, we obtain the result that if the temperature distribution is prescribed in the medium, then the Hankel transforms of the corresponding particular solution of the thermo-displacement equations is given as a particular solution of the ordinary differential equation (6).

Next, we consider the transformation of the expressions for the displacement and stress components into relations involving L_m, M_m and their derivatives. Substituting one term of (4) into the expression of w in (1) we have

$$w = \frac{1+\nu}{1-\nu}\,\varepsilon\frac{\partial\Omega_m}{\partial z}\cos m\theta\,.$$

If we multiply both sides of the above equation by $rJ_m(\xi r)$ and integrate it with

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11) The solution for $\bar{T}_m$ is readily obtained if the one for T_m is given.

12) See (9), p. 61.

respect to r over the range 0, ∞, we obtain

$$\int_0^\infty w\, r J_m(\xi r)\, dr = \frac{1+\nu}{1-\nu}\, \varepsilon \frac{dL_m}{dz} \cos m\theta\,.$$

Inverting the result by the Hankel trnsform theorem [13] we have

$$w = \frac{1+\nu}{1-\nu}\, \varepsilon \int_0^\infty \xi \frac{dL_m}{dz} J_m(\xi r) \cos m\theta d\xi\,. \tag{8}$$

By a similar procedure, the expression for σ_z can be obtained. A single expression of the remaining components for displacement and stress, however, does not permit the transformation in terms of L_m and M_m. Constructing the following pairs of the components and carrying out similar calculations, we have

$$\left.\begin{aligned}
&u/\cos m\theta + v/\sin m\theta = -\frac{1+\nu}{1-\nu}\, \varepsilon \int_0^\infty \xi^2 L_m J_{m+1}(\xi r) d\xi\,,\\
&u/\cos m\theta - v/\sin m\theta = \frac{1+\nu}{1-\nu}\, \varepsilon \int_0^\infty \xi^2 L_m J_{m-1}(\xi r) d\xi\,,\\
&\sigma_r/\cos m\theta + \sigma_\theta/\sin m\theta = -2\mu \frac{1+\nu}{1-\nu}\, \varepsilon \int_0^\infty \left[\frac{d^2 L_m}{dz^2} + M_m\right] \xi J_m d\xi\,,\\
&\tau_{\theta z}/\sin m\theta + \tau_{zr}/\cos m\theta = -2\mu \frac{1+\nu}{1-\nu}\, \varepsilon \int_0^\infty \xi^2 \frac{dL_m}{dz} J_{m+1} d\xi\,,\\
&\tau_{\theta z}/\sin m\theta - \tau_{zr}/\cos m\theta = -2\mu \frac{1+\nu}{1-\nu}\, \varepsilon \int_0^\infty \xi^2 \frac{dL_m}{dz} J_{m-1} d\xi\,,\\
&\sigma_r/\cos m\theta + 2\mu u/r \cos m\theta + 2\mu m v/r \sin m\theta\\
&\qquad\qquad = -2\mu \frac{1+\nu}{1-\nu}\, \varepsilon \int_0^\infty \xi \frac{d^2 L_m}{dz^2} J_m(\xi r) d\xi\,,\\
&\tau_{r\theta}/\sin m\theta + 2\mu m u/r \cos m\theta + 2\mu v/r \sin m\theta = 0\,.
\end{aligned}\right\} \tag{9}$$

Solving these equations, we can find the expressions for the displacement and stress components in terms of L_m, M_m and their derivatives. Summing them up with respect to m, the expressions for displacement and stress component due to the particular solution of the thermo-displacement equations are obtained as follows.

$$u = -\frac{1+\nu}{1-\nu} \frac{\varepsilon}{2} \sum_{m=0}^\infty \left[\int_0^\infty \xi^2 L_m J_{m+1} d\xi - \int_0^\infty \xi^2 L_m J_{m-1} d\xi\right] \cos m\theta\,,$$

13) See (2), p. 48.

$$\left.\begin{aligned} v &= -\frac{1+\nu}{1-\nu}\frac{\varepsilon}{2}\sum_{m=0}^{\infty}\left[\int_0^\infty \xi^2 L_m J_{m+1}\, d\xi + \int_0^\infty \xi^2 L_m J_{m-1}\, d\xi\right]\sin m\theta\,, \\ w &= \frac{1+\nu}{1-\nu}\,\varepsilon\sum_{m=0}^{\infty}\left[\int_0^\infty \xi\frac{dL_m}{dz} J_m\, d\xi\right]\cos m\theta\,, \end{aligned}\right\} \tag{10}$$

$$\left.\begin{aligned} \frac{\sigma_r}{2\mu} &= \frac{1+\nu}{1-\nu}\,\varepsilon\sum_{m=0}^{\infty}\Big[-\int_0^\infty \xi\frac{d^2L_m}{dz^2}J_m\, d\xi + \frac{1}{2r}(m+1)\int_0^\infty \xi^2 L_m J_{m+1}\, d\xi \\ &\qquad + \frac{1}{2r}(m-1)\int_0^\infty \xi^2 L_m J_{m-1}\, d\xi\Big]\cos m\theta\,, \\ \frac{\sigma_\theta}{2\mu} &= \frac{1+\nu}{1-\nu}\,\varepsilon\sum_{m=0}^{\infty}\Big[-\int_0^\infty \xi M_m J_m\, d\xi - \frac{1}{2r}(m+1)\int_0^\infty \xi^2 L_m J_{m+1}\, d\xi \\ &\qquad - \frac{1}{2r}(m-1)\int_0^\infty \xi^2 L_m J_{m-1}\, d\xi\Big]\cos m\theta\,, \\ \frac{\sigma_z}{2\mu} &= \frac{1+\nu}{1-\nu}\,\varepsilon\sum_{m=0}^{\infty}\left[\int_0^\infty \xi^3 L_m J_m\, d\xi\right]\cos m\theta\,, \\ \frac{\tau_{\theta z}}{2\mu} &= -\frac{1+\nu}{1-\nu}\frac{\varepsilon}{2}\sum_{m=0}^{\infty}\left[\int_0^\infty \xi^2\frac{dL_m}{dz}J_{m+1}\, d\xi + \int_0^\infty \xi^2\frac{dL_m}{dz}J_{m-1}\, d\xi\right]\sin m\theta\,, \\ \frac{\tau_{zr}}{2\mu} &= -\frac{1+\nu}{1-\nu}\frac{\varepsilon}{2}\sum_{m=0}^{\infty}\left[\int_0^\infty \xi^2\frac{dL_m}{dz}J_{m+1}\, d\xi - \int_0^\infty \xi^2\frac{dL_m}{dz}J_{m-1}\, d\xi\right]\cos m\theta\,, \\ \frac{\tau_{r\theta}}{2\mu} &= \frac{1+\nu}{1-\nu}\frac{\varepsilon}{2}\sum_{m=0}^{\infty}\left[\frac{m+1}{r}\int_0^\infty \xi^2 L_m J_{m+1}\, d\xi - \frac{m-1}{r}\int_0^\infty \xi^2 L_m J_{m-1}\, d\xi\right]\sin m\theta\,. \end{aligned}\right\} \tag{11}$$

Up to this point, the theory is applicable to any state of temperature, that is, steady or unsteady. Now, we shall confine our discussion to the steady state of temperature and assume that heat is not generated in the solid. Then, it follows from (4) that T_m is the solution of the following differential equation

$$\nabla_m^2 T_m = 0\,. \tag{12}$$

After the operation of Hankel transforms, we have, in view of (7), that

$$\left(\frac{d^2}{dz^2} - \xi^2\right)M_m = 0\,. \tag{13}$$

The general solution of (12) is

$$M_m = a_m e^{\xi z} + b_m e^{-\xi z} \tag{14}$$

or

$$M_m = a_m \cosh \xi z + b_m \sinh \xi z\,, \tag{15}$$

where a_m, b_m are constants to be determined from the boundary conditions for temperature. When these constants have been determined, the expression for T_m may be obtained directly from (14) or (15) by means of the Hankel transform theorem

$$T_m = \int_0^\infty M_m \xi J_m(\xi_r) d\xi\,. \tag{16}$$

Furthermore, in view of (6) (14) and (15), L_m is easily found to be

$$L_m = \frac{z}{2\xi}\left[a_m e^{\xi z} - b_m e^{-\xi z}\right], \tag{17}$$

or

$$L_m = \frac{z}{2\xi}\left[a_m \sinh \xi z + b_m \cosh \xi z\right]. \tag{18}$$

The Transformation of the General Solution of the Equations of Equilibrium

In the previous paper [6) 7)] the expressions for displacement and stress components which satisfy the equations of equilibrium of an isotropic medium have been shown. For the sake of completeness the results are summarized here.

$$\left.\begin{aligned}
u &= \frac{1}{2}\sum_{m=0}^{\infty}\Big[U_{m+1}(r,z) - V_{m-1}(r,z)\Big]\cos m\theta\,,\\
v &= \frac{1}{2}\sum_{m=0}^{\infty}\Big[U_{m+1}(r,z) + V_{m-1}(r,z)\Big]\sin m\theta\,,\\
w &= \sum_{m=0}^{\infty}\Big[\int_0^\infty \Big\{(1-2\nu)\frac{d^2G_m}{dz^2} - 2(1-\nu)\xi^2 G_m\Big\}\xi J_m(\xi r)d\xi\Big]\cos m\theta\,.
\end{aligned}\right\} \tag{19}$$

$$\begin{aligned}
\frac{\sigma_r}{2\mu} &= \sum_{m=0}^{\infty}\Big[\int_0^\infty \Big\{\nu\frac{d^3G_m}{dz^3} + (1-\nu)\xi^2\frac{dG_m}{dz}\Big\}\xi J_m(\xi r)d\xi\\
&\qquad - \frac{(m+1)}{2r}U_{m+1} - \frac{(m-1)}{2r}V_{m-1}\Big]\cos m\theta\,,\\
\frac{\sigma_\theta}{2\mu} &= \sum_{m=0}^{\infty}\Big[\nu\int_0^\infty \Big\{\frac{d^3G_m}{dz^3} - \xi^2\frac{dG_m}{dz}\Big\}\xi J_m(\xi r)d\xi\\
&\qquad + \frac{(m+1)}{2r}U_{m+1} + \frac{(m-1)}{2r}V_{m-1}\Big]\cos m\theta\,,\\
\frac{\sigma_z}{2\mu} &= \sum_{m=0}^{\infty}\Big[\int_0^\infty \Big\{(1-\nu)\frac{d^3G_m}{dz^3} - (2-\nu)\xi^2\frac{dG_m}{dz}\Big\}\xi J_m(\xi r)d\xi\Big]\cos m\theta\,,
\end{aligned}$$

$$\left.\begin{aligned}
\frac{\tau_{\theta z}}{2\mu}&=\frac{1}{2}\sum_{m=0}^{\infty}\Big[\int_0^\infty\Big\{\nu\frac{d^2G_m}{dz^2}+(1-\nu)\xi^2G_m+\frac{dH_m}{dz}\Big\}\xi^2J_{m+1}(\xi r)d\xi\\
&\quad+\int_0^\infty\Big\{\nu\frac{d^2G_m}{dz^2}+(1-\nu)\xi^2G_m-\frac{dH_m}{dz}\Big\}\xi^2J_{m-1}(\xi r)d\xi\Big]\sin m\theta\,,\\
\frac{\tau_{zr}}{2\mu}&=\frac{1}{2}\sum_{m=0}^{\infty}\Big[\int_0^\infty\Big\{\nu\frac{d^2G_m}{dz^2}+(1-\nu)\xi^2G_m+\frac{dH_m}{dz}\Big\}\xi^2J_{m+1}(\xi r)d\xi\\
&\quad-\int_0^\infty\Big\{\nu\frac{d^2G_m}{dz^2}+(1-\nu)\xi^2G_m-\frac{dH_m}{dz}\Big\}\xi^2J_{m-1}(\xi r)d\xi\Big]\cos m\theta\,,\\
\frac{\tau_{r\theta}}{2\mu}&=\sum_{m=0}^{\infty}\Big[\int_0^\infty H_m\xi^3J_m(\xi r)d\xi-\frac{(m+1)}{2r}U_{m+1}+\frac{(m-1)}{2r}V_{m-1}\Big]\sin m\theta\,,
\end{aligned}\right\}\tag{20}$$

where

$$\left.\begin{aligned}
U_{m+1}(r,z)&=\int_0^\infty\Big(\frac{dG_m}{dz}+2H_m\Big)\xi^2J_{m+1}(\xi r)d\xi\,,\\
V_{m-1}(r,z)&=\int_0^\infty\Big(\frac{dG_m}{dz}-2H_m\Big)\xi^2J_{m-1}(\xi r)d\xi\,,
\end{aligned}\right\}\tag{21}$$

and G_m and H_m are the solutions of the ordinary differential equations

$$\left.\begin{aligned}
\Big(\frac{d^2}{dz^2}-\xi^2\Big)^2G_m&=0\,,\\
\Big(\frac{d^2}{dz^2}-\xi^2\Big)H_m&=0\,.
\end{aligned}\right\}\tag{22}$$

Solution for a Semi - infinite Solid

Choose the z axis normal to the plane surface and pointing into the semi-infinite body. It will be supposed that the distribution of temperature is prescribed on the surface. The boundary condition for tenperature on $z=0$ then becomes

$$T(r,\theta,o)=\sum_{m=0}^{\infty}t_m(r)\cos m\theta\,.\tag{23}$$

Furthermore, we assume that the surface is free from external tractions which requires at $z=0$ that

$$\sigma_z=\tau_{\theta z}=\tau_{zr}=0\,.\tag{24}$$

For the time being, we shall consider only a single value for m.
From the requirement that temperature tends to zero as z tends to infinity, we assume the solution of (13) in the form.

$$M_m = \int_0^\infty T_m(r,z)\, r J_m(\xi r)\, dr = b_m e^{-\xi z}. \tag{25}$$

Putting $z=0$ and inserting the prescribed boundary condition (23) we obtain

$$b_m = \int_0^\infty t_m(r)\, r J_m(\xi r)\, dr. \tag{26}$$

In view of (17) and (25), we have

$$L_m = -\frac{b_m z}{2\xi} e^{-\xi z}. \tag{27}$$

Inverting σ_z in equation (11) by the Hankel transform theorem, we obtain

$$\left.\begin{aligned}
&\int_0^\infty [\sigma_z/2\mu\cos m\theta]\, r J_m(\xi r)\, dr = \frac{1+\nu}{1-\nu}\varepsilon\xi^2 L_m = -\frac{1+\nu}{1-\nu}\cdot\frac{\varepsilon b_m \xi z}{2} e^{-\xi z},\\
&\text{and combining } \tau_{\theta z} \text{ and } \tau_{zr}, \text{ we find}\\
&\int_0^\infty [\tau_{\theta z}/2\mu\sin m\theta + \tau_{zr}/2\mu\cos m\theta]\, r J_{m+1}(\xi r)\, dr = -\frac{1+\nu}{1-\nu}\varepsilon\xi\frac{dL_m}{dz}\\
&\qquad = \frac{1+\nu}{1-\nu}\frac{\varepsilon b_m}{2}(1-\xi z)e^{-\xi z},\\
&\int_0^\infty [\tau_{\theta z}/2\mu\sin m\theta - \tau_{zr}/2\mu\cos m\theta]\, r J_{m-1}(\xi r)\, dr = -\frac{1+\nu}{1-\nu}\varepsilon\xi\frac{dL_m}{dz}\\
&\qquad = \frac{1+\nu}{1-\nu}\frac{\varepsilon b_m}{2}(1-\xi z)e^{-\xi z}.
\end{aligned}\right\} \tag{28}$$

Since the stresses due to L_m do not satisfy the boundary conditions (24) at $z=0$, we employ the solution of (22) of the forms.

$$\left.\begin{aligned}
G_m(\xi,z) &= (C_m + D_m z)e^{-\xi z},\\
H_m(\xi,z) &= F_m e^{-\xi z}.
\end{aligned}\right\} \tag{29}$$

In similar procedures as to L_m, we obtain

$$\left.\begin{aligned}
&\int_0^\infty [\sigma_z/2\mu\cos m\theta]\, r J_m(\xi r)\, dr = \left[(1-\nu)\frac{d^3G_m}{dz^3} - (2-\nu)\xi^2\frac{dG_m}{dz}\right]\\
&\qquad = [\xi C_m + \{\xi z + (1-2\nu)\} D_m]\, e^{-\xi z},\\
&\int_0^\infty [\tau_{\theta z}/2\mu\sin m\theta + \tau_{zr}/2\mu\cos m\theta]\, r J_{m+1}(\xi r)\, dr\\
&\qquad = \xi\left[\nu\frac{d^2G_m}{dz^2} + (1-\nu)\xi^2 G_m + \frac{dH_m}{dz}\right]\\
&\qquad = \xi^2[\xi C_m + (\xi z - 2\nu)D_m - F_m]\, e^{-\xi z}.
\end{aligned}\right\} \tag{30}$$

$$\int_0^\infty [\tau_{\theta z}/2\mu \sin m\theta + \tau_{zr}/2\mu \cos m\theta]$$

$$= \xi\left[\nu \frac{d^2 G_m}{dz^2} + (1-\nu)\xi^2 G_m - \frac{dH_m}{dz}\right]$$

$$= \xi^2 [\xi C_m + (\xi z - 2\nu) D_m + F_m] e^{-\xi z} .$$

From the boundary conditions (24), the sum of the corresponding components of (28) and (30) must vanish for $z=0$. Solving these equations for C_m, D_m and F_m, and substituting them into (29), we have

$$\left.\begin{aligned} G_m &= \frac{1+\nu}{1-\nu}\varepsilon \frac{b_m}{2\xi^3}[-(1-2\nu)+\xi z]\, e^{-\xi z} , \\ H_m &= 0 . \end{aligned}\right\} \qquad (31)$$

Calculating the components of displacement and stress due to L_m and G_m independently and then adding them up, we find the general solution for the thermal stress in a semi-infinite solid with steady distribution of temprature as follows;

$$T = \sum_{m=0}^\infty I_m{}^1(r, z) \cos m\theta \qquad (32)$$

$$\left.\begin{aligned} u &= (1+\nu)\frac{\varepsilon}{2}\sum_{m=0}^\infty [I_{m+1}{}^0 - I_{m-1}{}^0] \cos m\theta , \\ v &= (1+\nu)\frac{\varepsilon}{2}\sum_{m=0}^\infty [I_{m+1}{}^0 + I_{m-1}{}^0] \sin m\theta , \\ w &= -(1+\nu)\varepsilon \sum_{m=0}^\infty I_m{}^0 \cos m\theta . \end{aligned}\right\} \qquad (33)$$

$$\left.\begin{aligned} \frac{\sigma_r}{E} &= -\varepsilon \sum_{m=0}^\infty \left[\frac{m+1}{2r} I_{m+1}{}^0 + \frac{m-1}{2r} I_{m-1}{}^0\right] \cos m\theta , \\ \frac{\sigma_\theta}{E} &= \varepsilon \sum_{m=0}^\infty \left[-I_m{}^1 + \frac{m+1}{2r} I_{m+1}{}^0 + \frac{m-1}{2r} I_{m-1}{}^0\right] \cos m\theta , \\ \sigma_z &= \tau_{\theta z} = \tau_{zr} = 0 , \\ \frac{\tau_{r\theta}}{E} &= -\frac{\varepsilon}{2}\sum_{m=0}^\infty \left[\frac{m+1}{r} I_{m+1}{}^0 - \frac{m-1}{r} I_{m-1}{}^0\right] \sin m\theta . \end{aligned}\right\} \qquad (34)$$

where

$$I^p{}_{m+q}(r,z) = \int_0^\infty b_m e^{-\xi z} J_{m+q}(\xi r) \xi^p d\xi \qquad (35)$$

and b_m is given by (26).

It is interesting to note that the steady distribution of temperature does not produce any stress in the direction normal to the plane surface. Moreover, the distribution of stress is not affected by Poisson's ratio ν. As will be seen later, these results are also true for a region bounded by two parallel planes.

Solution for a Thick Plate

Consider a thick plate bounded by two parallel planes $z=\pm d$ and with infiinite extent. We assume the boundary conditions for temperature in the forms

$$\left.\begin{aligned} T &= \sum_{m=0}^{\infty} t^1{}_m(r)\cos m\theta\,, && \text{at } z=+d \\ T &= \sum_{m=0}^{\infty} t^2{}_m(r)\cos m\theta\,, && \text{at } z=-d \end{aligned}\right\} \tag{36}$$

and for stresses

$$\sigma_z = \tau_{\theta z} = \tau_{zr} = 0\,. \qquad \text{at } z=\pm d \tag{37}$$

Assuming a solution of (13) in the form

$$M_m = \int_0^\infty T_m(r,z)\, r J_m(\xi r)\, dr = a_m \cosh \xi z + b_m \sinh \xi z \tag{38}$$

and inserting the boundary condition (36), we obtain

$$a_m = \frac{\bar{t}^1{}_m + \bar{t}^2{}_m}{2\cosh \xi d}\,, \qquad b_m = \frac{\bar{t}^1{}_m - \bar{t}^2{}_m}{2\sinh \xi d}. \tag{39}$$

where

$$\bar{t}^{1,2}{}_m = \int_0^\infty t^{1,2}{}_m(r) J_m(\xi r)\, dr\,. \tag{40}$$

From (18), we have

$$L_m = \frac{z}{2\xi}\left[a_m \sinh \xi z + b_m \cosh \xi z\right]. \tag{41}$$

We assume the solutions of (12) in the forms

$$\left.\begin{aligned} G_m &= [A_m + B_m z]\cosh \xi z + [C_m + D_m z]\sinh \xi z\,, \\ H_m &= [E_m \cosh \xi z + F_m \sinh \xi z]\,. \end{aligned}\right\} \tag{42}$$

Inserting L_m and H_m into (28) and (30) respectively, and considering the boundary conditions (37), we obtain six linear equations. Solving these equations for A_m, B_m, $\cdots F_m$, and substituting them into (42), we obtain finally

$$\left.\begin{aligned} G_m &= \frac{1+\nu}{1-\nu}\frac{\mathcal{E}}{4}(\bar{t}^1{}_m + \bar{t}^2{}_m)\cdot\frac{\{(1-2\nu)\sinh \xi z + \xi z \cosh \xi z\}}{\xi^3 \cosh \xi d} \\ &\quad + \frac{1+\nu}{1-\nu}\frac{\mathcal{E}}{4}(\bar{t}^1{}_m - \bar{t}^2{}_m)\cdot\frac{\{(1-2\nu)\cosh \xi z + \xi z \sinh \xi z\}}{\xi^3 \sinh \xi d}\,, \quad H_m = 0. \end{aligned}\right\} \tag{34}$$

Inserting L_m and G_m into (10) (11) and (19) (20) respectively, and adding them up, we can find the general solution in integral forms for the thermal stress of a thick plate.

Now, we shall consider the case where the distribution of temperature is symmetric about the plane $z=0$. In this case

$$t^1_m(r)=t^2_m(r)=t_m(r) \tag{44}$$

and L_m and G_m are reduced to

$$\left.\begin{aligned} L_m &= \frac{z\bar{t}_m}{2\xi}\frac{\sinh\xi z}{\cosh\xi d}\,, \\ G_m &= \frac{\varepsilon}{2}\frac{1+\nu}{1-\nu}\bar{t}_m\frac{\{(1-2\nu)\sinh\xi z+\xi z\cosh\xi z\}}{\xi^3\cosh\xi d}\,. \end{aligned}\right\} \tag{45}$$

Substituting L_m and G_m into (10) (11) (20) respectively, and then adding them up, we obtain

$$T=\sum_{m=0}^{\infty}\Big[\int_0^{\infty}\bar{t}_m\frac{\cosh\xi z}{\cosh\xi d}\xi J_m(\xi r)\,d\xi\Big]\cos m\theta\,, \tag{46}$$

$$\left.\begin{aligned} u &= (1+\nu)\frac{\varepsilon}{2}\sum_{m=0}^{\infty}[u_{m+1}(r,z)-v_{m-1}(r,z)]\cos m\theta\,, \\ v &= (1+\nu)\frac{\varepsilon}{2}\sum_{m=0}^{\infty}[u_{m+1}(r,z)+v_{m-1}(r,z)]\sin m\theta\,, \\ w &= (1+\nu)\varepsilon\sum_{m=0}^{\infty}\Big[\int_0^{\infty}\bar{t}_m\frac{\sinh\xi z}{\cosh\xi d}J_m(\xi r)\,d\xi\Big]\cos m\theta\,. \end{aligned}\right\} \tag{47}$$

$$\left.\begin{aligned} \frac{\sigma_r}{E} &= -\varepsilon\sum_{m=0}^{\infty}\Big[\frac{m+1}{2r}u_{m+1}+\frac{m-1}{2r}u_{m-1}\Big]\cos m\theta\,, \\ \frac{\sigma_\theta}{E} &= \varepsilon\sum_{m=0}^{\infty}\Big[-\int_0^{\infty}\bar{t}_m\frac{\cosh\xi z}{\cosh\xi d}\xi J_m(\xi r)\,d\xi+\frac{m+1}{2r}u_{m+1}+\frac{m-1}{2r}u_{m-1}\Big]\cos m\theta\,, \\ \sigma_z &= \tau_{\theta z}=\tau_{zr}=0\,, \\ \frac{\tau_{r\theta}}{E} &= \varepsilon\sum_{m=0}^{\infty}\Big[-\frac{m+1}{2r}u_{m+1}+\frac{m-1}{2r}u_{m-1}\Big]\sin m\theta\,, \end{aligned}\right\} \tag{48}$$

where

$$\left.\begin{aligned} u_{m+1} &= \int_0^{\infty}\bar{t}_m\frac{\cosh\xi z}{\cosh\xi d}J_{m+1}(\xi r)\,d\xi\,, \\ u_{m-1} &= \int_0^{\infty}\bar{t}_m\frac{\cosh\xi z}{\cosh\xi d}J_{m-1}(\xi r)\,d\xi\,, \\ \bar{t}_m &= \int_0^{\infty}t_m\,rJ_m(\xi r)\,dr\,. \end{aligned}\right\} \tag{49}$$

When the distribution of temperature is antisymmetric about the plane $z=0$, that is,

$$t^1{}_m(r) = -t^2{}_m(r) = t_m(r) \tag{50}$$

the expressions for the components of displacement and stress can readily be obtained by transcribing

$$\cosh \longrightarrow \sinh, \qquad \cosh \longrightarrow \sinh \tag{51}$$

in equations (46) (47) and (48).

We again arrive at the result, as in the semi-infinite solid, that every stress acting in the direction normal to the plane surface of a thick plate vanish identically throughout the plate when the distribution of temperature is steady and no heat is produced in the solid. It is also known from the general solution that the distribution of stress is not affected by the Poisson's ratio ν and the stresses on the surface are completely determined by the distribution of the surface temperature.

Example 1.

We shall consider the problem of a plate where the surface temperature are kept constant, say T_0, over the circles $r<a$ ane zero outsides of them. The boundary conditions then assume the forms;

$$\left.\begin{aligned} &T(r,+d) = T(r,-d) = t_0(r), \\ &t_0(r) = T_0 \qquad \text{for} \quad r<a \\ &\quad\;\; = 0 \qquad \text{for} \quad r>a \end{aligned}\right\} \tag{52}$$

From (41), we have

$$\bar{t}_0 = \int_0^\infty t_0 r J_0(\xi r)\,dr = T_0 \frac{aJ_1(\xi a)}{\xi}. \tag{53}$$

In view of (46) (47) (48) and (53), it is easily found to be

$$T = T_0 R_0(r,z) \tag{54}$$

$$\left.\begin{aligned} &u = (1+\nu)\alpha\varepsilon T_0 R_1, \qquad w = (1+\nu)\alpha\varepsilon T_0 R_2, \\ &\frac{\sigma_r}{E} = -\varepsilon T_0 \frac{a}{r} R_1, \qquad \frac{\sigma_\theta}{E} = \varepsilon T_0 \left\{ -R_0 + \frac{a}{r} R_1 \right\}, \\ &v = \sigma_z = \tau_{rz} = \tau_{z\theta} = \tau_{r\theta} = 0, \end{aligned}\right\} \tag{55}$$

where

$$\left.\begin{aligned} &R_0(r,z) = a\int_0^\infty \frac{\cosh \xi z}{\cosh \xi d} J_1(\xi a) J_0(\xi r)\,d\xi, \\ &R_1(r,z) = \int_0^\infty \frac{\cosh \xi z}{\xi \cosh \xi d} J_1(\xi a) J_1(\xi r)\,d\xi, \\ &R_2(r,z) = \int_0^\infty \frac{\sinh \xi z}{\xi \cosh \xi d} J_1(\xi a) J_0(\xi r)\,d\xi. \end{aligned}\right\} \tag{56}$$

To determine the distribution of stress on $z = \pm d$, we must calculate $R_0(r,d)$ and

$R_1(r,d)$ which are reduced to Weber-Schafheitlin discontinuous integrals[14]; the computations yield

$$\left.\begin{aligned} R_0(r,d) &= 1 \quad & r<a, \qquad R_1(r,d) &= \frac{1}{2}\frac{r}{a} \quad & r\leqslant a \\ &= 0, & r>a, \qquad &= \frac{1}{2}\frac{a}{r} & r\geqslant a \end{aligned}\right\} \tag{57}$$

In view of (55) and (57), we have on $z=\pm d$

$$\left.\begin{aligned} \sigma_r/\varepsilon E T_0 &= -\frac{1}{2} & 0\leqslant r\leqslant a \\ &= -\frac{1}{2}\frac{a^2}{r^2}, & r\geqslant a \\ \sigma_\theta/\varepsilon E T_0 &= -\frac{1}{2} & 0\leqslant r<a \\ &= \frac{1}{2}\frac{a^2}{r^2}, & r>a \end{aligned}\right\} \tag{58}$$

Eq. (58) coincide with the ones computed from the solution for a thin plate.

The integrals (56) for arbitrary values of z and r can be evaluated by the method of calculus of residues.[15] The convergency of the series thus obtained is rapid except the vicinity of $r=a$ where some devices are introduced to accelerate the convergence.

The results of the evaluations are as follows.

$$\left.\begin{aligned} R_0(r,z) &= 1-2\frac{a}{d}\sum_{n=0}^{\infty}(-1)^n K_1\left(\beta_n\frac{a}{d}\right)I_0\left(\beta_n\frac{r}{d}\right)\cos\beta_n\frac{z}{d} & r\leqslant a \\ &= \frac{1}{2}+2\frac{a}{d}\sum_{n=0}^{\infty}(-1)^n\left[K_1\left(\beta_n\frac{a}{d}\right)I_0\left(\beta_n\frac{a}{d}\right)-\frac{d}{2\beta_n a}\right]\cos\beta_n\frac{z}{d} & r=a \\ &= 2\frac{a}{d}\sum_{n=0}^{\infty}(-1)^n I_1\left(\beta_n\frac{a}{d}\right)K_0\left(\beta_n\frac{r}{d}\right)\cos\beta_n\frac{z}{d}, & r\geqslant a \\ R_1(r,z) &= \frac{1}{2}\frac{r}{a}-\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}K_1\left(\beta_n\frac{a}{d}\right)I_1\left(\beta_n\frac{r}{d}\right)\cos\beta_n\frac{z}{d} & r\leqslant a \\ &= \frac{1}{2}-\frac{4}{\pi^2}\frac{d}{a}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\cos\beta_n\frac{z}{d}-\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\Big[K_1\left(\beta_n\frac{a}{d}\right)I_1\left(\beta_n\frac{a}{d}\right) \\ &\qquad -\frac{d}{2\beta_n a}\Big]\cos\beta_n\frac{z}{d} & r=a \\ &= \frac{1}{2}\frac{a}{r}-\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}I_1\left(\beta_n\frac{a}{d}\right)K_1\left(\beta_n\frac{r}{d}\right)\cos\beta_n\frac{z}{d}, & r\geqslant a \\ R_2(r,z) &= \frac{z}{a}-\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}K_1\left(\beta_n\frac{a}{d}\right)I_0\left(\beta_n\frac{r}{d}\right)\sin\beta_n\frac{z}{d} & r\leqslant a \end{aligned}\right\} \tag{59}$$

14) Watson, "Theory of Bessel Functions" p. 398, (1922)

15) See the Appendix A.

$$= \frac{1}{2}\frac{z}{a} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left[K_1\left(\beta_n\frac{a}{d}\right)I_0\left(\beta_n\frac{a}{d}\right) - \frac{d}{2\beta_n a}\right]\sin\beta_n\frac{z}{d} \qquad r = a$$

$$= \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}I_1\left(\beta_n\frac{a}{d}\right)K_0\left(\beta_n\frac{r}{d}\right)\sin\beta_n\frac{z}{d}, \qquad r \geqslant a$$

where

$$\beta_n = \frac{2n+1}{2}\pi. \tag{60}$$

In view of (54) (55) and (59), temperature and the components of displacement and stress at any point in the plate are readily computed. Numerical calculations are carried out in detail for the case $a/d=1$. Fig. 1 and Fig. 2 show the variations of $\sigma_r/\varepsilon ET_0$ and $\sigma_\theta/\varepsilon ET_0$ respectively, for planes parallel to the faces of the plate. The contours of equal temperature and equal maximum shearing stress are shown in Fig. 3 and Fig. 4 respectively. The shape of curves in Fig. 4 shows clearly the

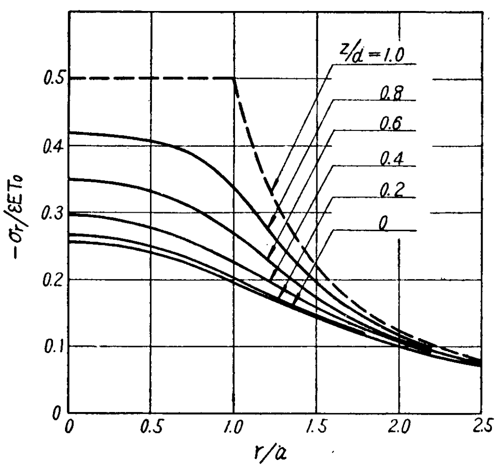

Fig. 1. Variation of $\sigma_r/\varepsilon ET_0$ with r and z fos $a/d=1$.

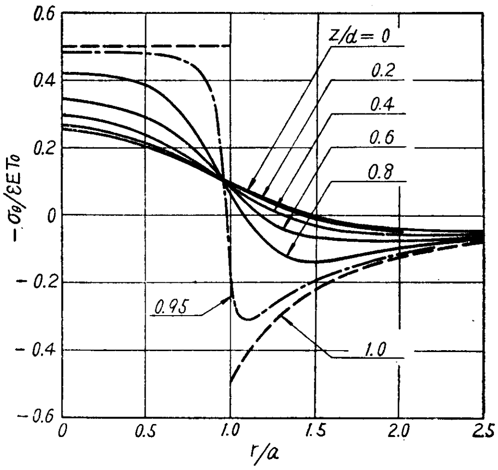

Fig. 2. Variation of $\sigma_\theta/\varepsilon ET_0$ with r and z for $a/d=1$.

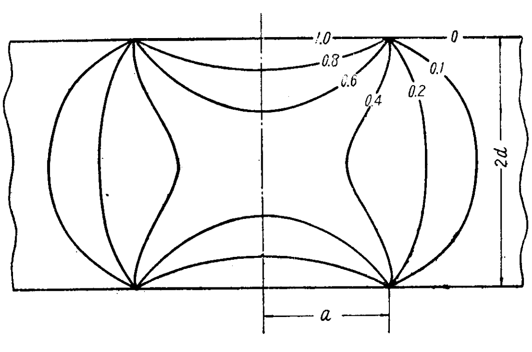

Fig. 3. Contours of Equal Temperature for $a/d=1$.

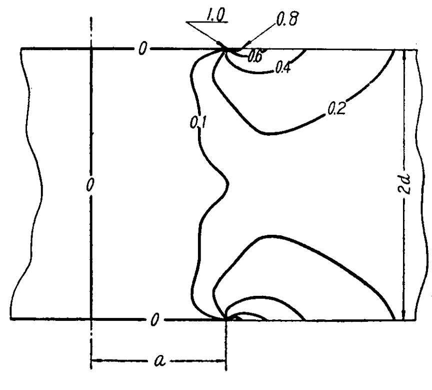

Fig. 4. Contours of Equal Maximum Shearing Stress for $a/d=1$.

concentration of stress in the neighbourfood of the point from which the step of temperature begins. Fig. 5 shows the displacements u and w on the upper and the middle planes of the plate. In Fig. 6, the thermal expansion of the plate at $r=0$ are plotted for the ratio between the thickness of the plate and the diameter of the circular area in which the temperature is kept constant, T_0. The chain line represents the thermal expansion of the plate when the temperature of the whole plate is raised to T_0, that is $d\varepsilon T_0$. It is interesting to note that the expansion at $r=0$ for a partly heated case is greater than the expansion for a completely heated case in a certain range of the ratio a/d.

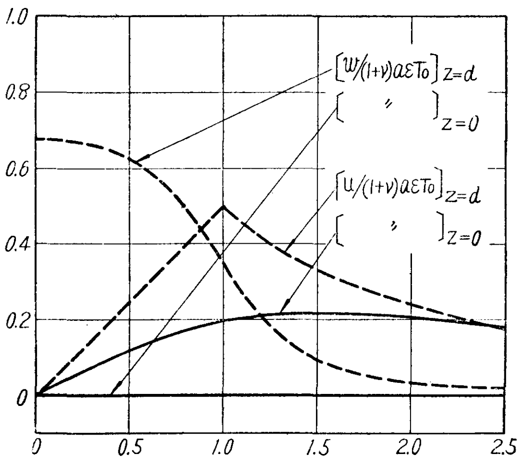

Fig. 5. The Variations ot u and w on the Upper and the Middle Planes of the Plate.

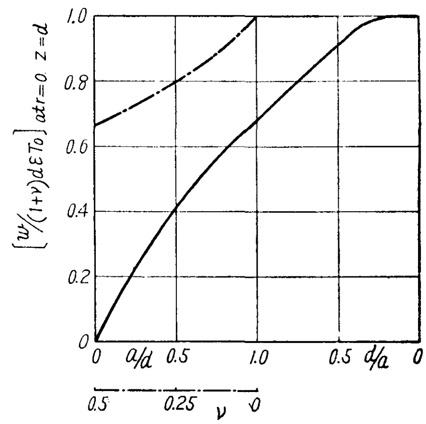

Fig. 6. The Variation of the Expansion of the Plate at $r=0$ with a/d.

Example 2.

Consider the problem of a plate where temperature is kept constant T_0 over the circle $r<a$, $z=d$ and $-T_0$ over the circle $r<a$, $z=-d$. The boundary conditions then assume the forms;

$$\left.\begin{aligned} &T(r,+d)=t_0(r), && T(r,-d)=-t_0(r). \\ \text{and}\quad &t_0(r)=T_0 \quad \text{for} && r<a \\ &=0 \quad \text{for} && r>a \end{aligned}\right\} \tag{61}$$

From (49), we have

$$\bar{t}_0 = T_0\frac{aJ_1(\xi a)}{\xi}. \tag{62}$$

As in the previous example, we have from (50) (51)

$$T = T_0 S_0(r,z) \tag{63}$$

$$\left.\begin{aligned} &u=(1+\nu)a\varepsilon T_0 S_1(r,z), && w=(1+\nu)a\varepsilon T_0 S_2(r,z), \\ &\sigma_r/E = -\varepsilon T_0\frac{a}{r}S_1(r,z), && \sigma_\theta/E=\varepsilon T_0\left\{-S_0+\frac{a}{r}S_1\right\}, \end{aligned}\right\} \tag{64}$$

$$v=\sigma_z=\tau_{rz}=0,$$

where

$$
\left.\begin{aligned}
S_0(r,z) &= a\int_0^\infty \frac{\sinh \xi z}{\sinh \xi d} J_1(\xi a) J_0(\xi r)\, d\xi\,,\\
S_1(r,z) &= a\int_0^\infty \frac{\cosh \xi z}{\xi \sinh \xi d} J_1(\xi a) J_1(\xi r)\, d\xi\,,\\
S_2(r,z) &= \int_0^\infty \frac{\cosh \xi z}{\xi \sinh \xi d} J_1(\xi a) J_0(\xi r)\, d\xi\,.
\end{aligned}\right\} \tag{65}
$$

From the fact that σ_z and τ_{zr} vanish everywhere in the plate, it is easily seen that (64) also gives the solution for the following probiem;

$$
\left.\begin{aligned}
&T(r,d) = t_0(r)\,, \qquad T(r,0) = 0\,,\\
\text{and}\quad &\sigma_z = \tau_{zr} = 0\,, \qquad \text{at} \quad z = d \text{ and } z = 0\,.
\end{aligned}\right\} \tag{66}
$$

In a similar procedure as in the previous example, the distribution of stress on $z=\pm d$ can be calculated. The variations of stresses on the plane $z=d$ are the same with (58) while the ones on $z=-d$ differ from (58) in the sign.
The integrals S_0 and S_1 for arbitrary values of z and r can be evaluated by the method of calculus of residues as in the previous example. The integral S_2 is, however, divergent and can not be evaluated. Considering the physical meaning of S_2, we introduce the following convergent integral[16];

$$
\frac{w'}{(1+\nu)\alpha\varepsilon T_0} = \frac{1}{(1+\nu)\alpha\varepsilon T_0}\left[\int_0^r \frac{\partial w}{\partial r} dr + w(0,z) - w(0,0)\right] = \int_0^r \frac{\partial S_2}{\partial r} dr + [S_2(0,z) - S_2(0,0)]
$$
$$
= S_2'(r,z)\,. \tag{67}
$$

The results of the evaluations are;

$$
\left.\begin{aligned}
S_0(r,z) &= \frac{z}{d} + \frac{2a}{d}\sum_{n=1}^\infty (-1)^n K_1\Big(n\pi\frac{a}{d}\Big) I_0\Big(n\pi\frac{r}{d}\Big)\sin n\pi\frac{z}{d} && r<a\\
&= \frac{z}{2d} + \frac{2a}{d}\sum_{n=1}^\infty (-1)^n\Big[K_1 I_0 - \frac{d}{2n\pi a}\Big]\sin n\pi\frac{z}{d} && r=a\\
&= -\frac{2a}{d}\sum_{n=1}^\infty (-1)^n I_1\Big(n\pi\frac{a}{d}\Big) K_0\Big(n\pi\frac{r}{d}\Big)\sin n\pi\frac{z}{d}\,. && r>a\\
S_1(r,z) &= \frac{1}{2}\frac{z\,r}{ad} + \frac{2}{\pi}\sum_{n=1}^\infty \frac{(-1)^n}{n} K_1\Big(n\pi\frac{a}{d}\Big) I_1\Big(n\pi\frac{r}{d}\Big)\sin n\pi\frac{z}{d} && r<a\\
&= \frac{1}{2}\frac{z}{d} + \frac{d}{\pi^2 a}\sum_{n=1}^\infty \frac{(-1)^n}{n^2}\sin n\pi\frac{z}{d} + \frac{2}{\pi}\sum_{n=1}^\infty (-1)^n \frac{1}{n}\Big[K_1 I_1 - \frac{d}{2n\pi a}\Big]\sin n\pi\frac{z}{d} && r=a\\
&= \frac{1}{2}\frac{za}{dr} + \frac{2}{\pi}\sum_{n=1}^\infty \frac{(-1)^n}{n} I_1\Big(n\pi\frac{a}{d}\Big) K_1\Big(n\pi\frac{r}{d}\Big)\sin n\pi\frac{z}{d}\,. && r\geq a
\end{aligned}\right\} \tag{68}
$$

16) For the evaluation of the integral, see Appendix B.

$$S_2'(r,z)=\frac{1}{2}\frac{z^2}{ad}-\frac{1}{4}\frac{r^2}{ad}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}K_1\left(n\pi\frac{a}{d}\right)$$

$$-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}K_1\left(n\pi\frac{a}{d}\right)I_0\left(n\pi\frac{r}{d}\right)\cos n\pi\frac{z}{d} \qquad r\leqslant a$$

$$=\frac{1}{4}\frac{z^2}{ad}+\frac{1}{12}\frac{d}{a}-\frac{1}{4}\frac{a}{d}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}K_1\left(n\pi\frac{a}{d}\right)$$

$$-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left[K_1I_0-\frac{1}{2n\pi}\frac{d}{a}\right]\cos n\pi\frac{z}{d} \qquad r=a$$

$$=-\frac{1}{4}\frac{a}{d}+\frac{1}{6}\frac{d}{a}-\frac{1}{2}\frac{a}{d}\log\frac{r}{a}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}K_1\left(n\pi\frac{a}{d}\right)$$

$$+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}I_1\left(n\pi\frac{a}{d}\right)K_0\left(n\pi\frac{r}{d}\right)\cos n\pi\frac{z}{d}. \qquad r\geqslant a$$

From (63) (64) (67) and (68), we can find temperature and the components of displacement and stress in the plate. Fig. 7 and Fig. 8 show the variations of $\sigma_r/\varepsilon ET_0$

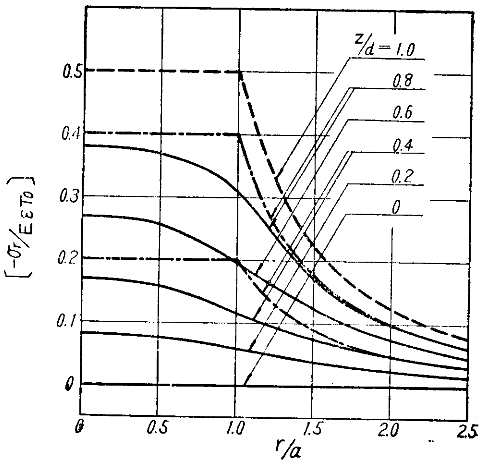

Fig. 7. Variation of $\sigma_r/\varepsilon ET_0$ with r and z for $a/d=1$.

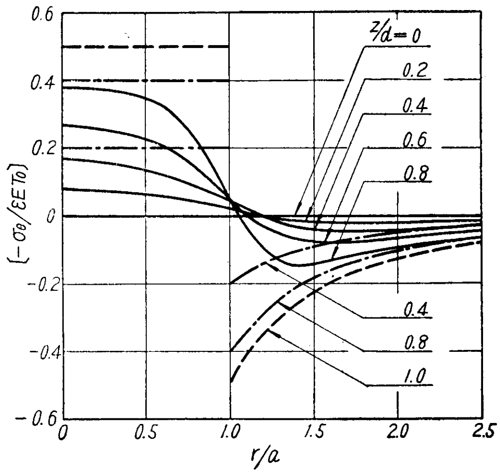

Fig. 8. Variation of $\sigma_\theta/\varepsilon ET_0$ with r and z for $a/d=1$.

and $\sigma_\theta/\varepsilon ET_0$ respectively for six planes parallel to the faces of the plate. The chain lines represent the corresponding stresses computed from the theory of thin plate obtained by Goldberg(17) which coincido with the stresses computed from the present solution on the free surfaces and give higher values in the interior of the plate. So we can conclude that the stresses computed from the theory of a thin plate are always on the safe side. The contours of equal temperature and equal maximum shearing stress are shown in Fig. 9 and Fig. 10. Fig. 11 shows the displacements u and w

17) J, E, Goldberg, J. Appl. Mech. vol. 22, p. 257, (1953)

on the upper and the middl: planes of the plate.

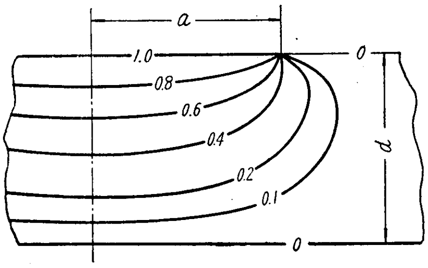

Fig. 9. Contours of Equal Tmperature for $a/d=1$.

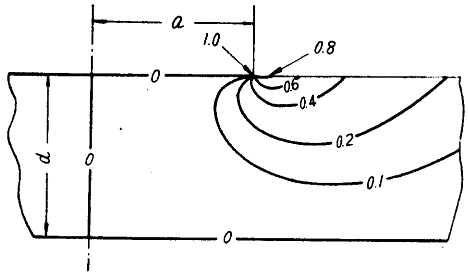

Fig. 10. Contours of Equal Maximum Shearing Stress for $a/d=1$.

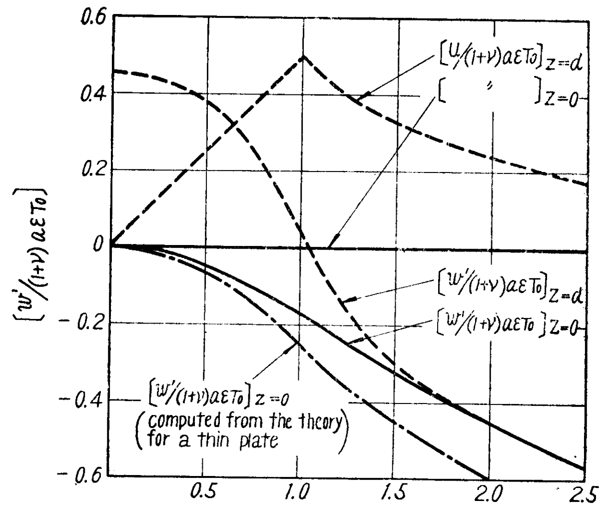

Fig. 11. The Variations of u and w on the Upper and the Middle Planes of the Plate.

Conclusions

The general solutions for the thermal stresses in a semi-infinite solid and a large thick plate with steady distribution of temperature are obtained and the following results are derived.

1. The stresses in the direction normal to the plane surface, that is, σ_z, $\tau_{\theta z}$ and τ_{zr} vanish everywhere in the medium.
2. Stresses on the surface are entirely determined by the distribution of surface temperature and coincide with the ones computed from the solution of a thin plate.
3. Every stress has δE as a linear factor and the pattern of the stress distribution is determined by the distribution of temperature and is not affected by the Poisson's ratio ν.

Acknowledgements

The author wishes to express his gratitude to Professor T. Suhara for his valuable guidance throughout the course of the work.
His thanks are also due to Professor E. Kiyooka for kindly reading the manuscript of the paper and to Miss T. Suzuki for the preparation of the diagrams in this paper.

Postscript

After the completion of the paper, the author found that, at approximately the same time of the publication of the Japanese issue [18] of this paper, the solution of the steady state thermoelastic problem for a semi-infinite solid has been presented by E. Sternberg and E. L. McDowell [19] and the one for a thick plate of infinite extent with axisymmetric distribution of temperature by B. Sharma [20] who, however, did not apply the solution to any particular example of a thick plate. Sternberg and McDowell have studied the problem by the use of the Green's functions, while Sharma directly integrated the thermo-elastic equations.

Appendix A.

The evaluation of the infinite integral $R_0 = a\int_0^\infty \frac{\cosh \xi z}{\cosh \xi d} J_1(\xi a) J_0(\xi r)\, d\xi$.

For $r \leqq a$, consider the integral

$$\frac{1}{2\pi i}\int \frac{\cosh wz}{\cosh wd} H_1^{(1)}(wa) J_0(wr)\, dw\,, \tag{69}$$

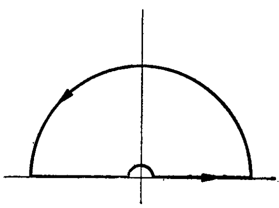

Fig. 12. The Contour

around the contour which is taken to be a large semicircle above the real axis with its center at the origin, together with that part of the real axis (indented at the origin) which joins the ends of the semicircle.

Then, we find that

$$\frac{1}{2\pi i}\int \frac{\cosh mz}{\cosh wd} H_1^{(1)}(wa) J_0(wr)\, dw$$

$$= \frac{1}{2\pi i} \lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \Big[\int_\varepsilon^R \frac{\cosh \xi z}{\cosh \xi d} \{H_1^{(1)}(a\xi) + H_1(\xi a e^{\pi i})\} J_0(\xi r)\, d\xi$$

$$- i\int_0^\pi \frac{\cosh \varepsilon z e^{i\theta}}{\cosh \varepsilon d e^{i\theta}} H_1^{(1)}(\varepsilon a e^{i\theta}) J_0(\varepsilon a e^{i\theta}) \varepsilon e^{i\theta}\, d\theta \Big]$$

18) See, note * on page 12.

19) E. Strenberg, E. L. McDowell, On the steady state thermoelastic problem for the half space, Quart. Appl. Math. vol. 14, No. 4, p. 382, (1957)

20) B. Sharma, Thermal stresses in infinite elastic diskes, J. Appl. Mech. Vol. 23, No. 4, p. 527, (1956)

$$= \frac{1}{\pi i}\left[\int_0^\infty \frac{\cosh \xi z}{\cosh \xi d} J_1(\xi a) J_1(\xi r) d\xi - \frac{1}{a}\right]$$

$$= \text{sum of residues at poles, } wd = \frac{2n+1}{2}\pi i. \tag{70}$$

Evaluating the residues at the poles, the infinite integral R_0 is then obtained as shown in (59).

Using the relation

$$K_m(x) I_n(x) \doteqdot \frac{1}{2x} \tag{71}$$

which is valid with less errors for larger value of x, we can accelerate the convergence of R_0 at $r=a$. From the expression of $R_0(r,z)$ for $r \leqq a$ in (59), we have

$$R_0(a,z) = 1 - \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\cos\beta_n\frac{z}{d} - 2\frac{a}{d}\sum_{n=0}^{\infty}(-1)^n\left[I_0\left(\beta_n\frac{a}{d}\right)K_1\left(\beta_n\frac{a}{d}\right) - \frac{a}{2\beta_n d}\right]\cos\beta_n\frac{z}{d}$$

$$= \frac{1}{2} - 2\frac{a}{d}\sum_{n=0}^{\infty}(-1)^n\left[K_1\left(\beta_n\frac{a}{d}\right)I_0\left(\beta_n\frac{a}{d}\right) - \frac{d}{2\beta_n a}\right]\cos\beta_n\frac{z}{d}. \tag{72}$$

in view of

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\cos\beta_n\frac{z}{d} = \frac{\pi}{4}, \qquad -1 < \frac{z}{d} < 1. \tag{73}$$

where $\beta_n = \frac{2n+1}{2}\pi$.

For $r \geqq a$, considering the integral

$$\frac{1}{2\pi i}\int \frac{\cosh wz}{\cosh wd} J_1(wa) H_0^{(1)}(wr)\, dw. \tag{74}$$

and carrying a similar calculation as to the case of $r \leqq a$, we obtain the result indicated in (59). The other integrals R_1, R_2, S_0 and S_1 can be evaluated in similar ways.

Appendix B.

The evaluation of the integral $S_2' = \int_0^r \frac{\partial S_2}{\partial r} dr + [S_2(0,z) - S_2(0,0)]$.

The integrals, $\frac{\partial S_2}{\partial r}$ and $S_2(0,z) - S_2(0,0)$, can be evaluated in a similar way as R_0. The integration of $\frac{\partial S_2}{\partial r}$ thus obtained with respect to r presents no particular difficulty. Then we obtain the result for $r \leqq a$ as shown in (68).

The result for $r \geqq a$ is

$$S'_2 = \frac{1}{2}\frac{z^2}{ad} - \frac{1}{4}\frac{a}{d} - \frac{1}{2}\frac{a}{d}\log\frac{r}{a} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}K_1\left(n\pi\frac{a}{d}\right)$$

$$+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}I_1\left(n\pi\frac{a}{d}\right)K_0\left(n\pi\frac{r}{d}\right)\cos n\pi\frac{z}{d}$$

$$-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}[I_0K_1+I_1K_0]\cos n\pi\frac{z}{d} \qquad (75)$$

which can be expressed in a neater form indicated in (68) by employing the following formulas.

$$\left.\begin{aligned} &I_{\gamma-1}(x)K_\gamma(x)+I_\gamma(x)K_{\gamma-1}(x)=\frac{1}{x}=\frac{d}{n\pi a}, \\ &\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi\frac{z}{d}=\frac{\pi^2}{4}\left[\left(\frac{z}{d}\right)^2-\frac{1}{3}\right], \qquad -1\leqq\frac{z}{d}\leqq 1 \end{aligned}\right\} \qquad (76)$$