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Conduction of Heat in a Circular Cylinder due to the Source in form of a Circular Ring moving axially on the Surface of the Cylinder with Constant Speed

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Abstract

In this paper is treated theoretically the problem of conduction of heat in a circular cylinder caused by the circular ring source which is moving axially, contacting with the surface of the cylinder. This problem corresponds approximately to the case where a metal bar of circular cross section is being turned and cut on a lathe. By the use of the Green's function, the author has obtained both steady and non-steady solutions of the heat conduction in a solid cylinder due to the circular ring source moving with a constant speed on the surface, at which the radiation is taking place into the surrounding medium. These solutions will give the temperature distribution in such a cylinder and basic relations for the calculation of the thermal stresses and strains in it.

I. Introduction

The problem of the conduction of heat due to moving sources has rarely been discussed, though it has wide applications to the machining of metals and manufacturing of machines.

In this paper the author analyses theoretically the conduction of heat in a circular cylinder generated by the heat-source which is moving axially, contacting with the surface of the cylinder. This problem correponds approximately to the case where a metal bar in form of circular cylinder is being turned and cut on a lathe, thus generating heat along the surface of the cylinder. The present analysis by the author will give the temperature distribution in such a cylinder and basic relations for the calculation of the thermal stresses and strains in it.

II. Fundamental Equations

Let (x', y', z') be the Cartesian coordinates of an instantaneous heat-source appearing in an infinite solid at the time t', and v the temperature at the time t in the point (x, y, z). Then the conduction of heat is governed by the following equation:——

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$$\frac{\partial v}{\partial t}-\kappa \nabla^2 v=\delta(x-x')\,\delta(y-y')\,\delta(z-z')\,\delta(t-t') \ , \tag{1}$$

where $\kappa=K/c\rho$ and K, c, ρ denote the thermal conductivity, the specific heat and the density respectively, and $\delta(x)$, the Dirac's delta function.

It is well known that the solution of equation (1) is given by

$$v=\frac{Q}{(2\sqrt{\pi\kappa(t-t')})^3}exp\left[-\frac{(x-x')^2+(y-y')^2+(z-z')^2}{4\kappa(t-t')}\right] \ , \tag{2}$$

where Q is the strength of heat-source at the point (x, y, z) at the time t. The equation (2) expresses the temperature due to an instantaneous point source of strength Q, generated at the point (x', y', z') at the time t', the solid being initially at zero temperature and the surface being kept at temperature zero. And it is none other than the Green's function to the equation (1).

By the transformation of the Cartesian coordinates into the cylindrical ones, (Fig. 1), namely

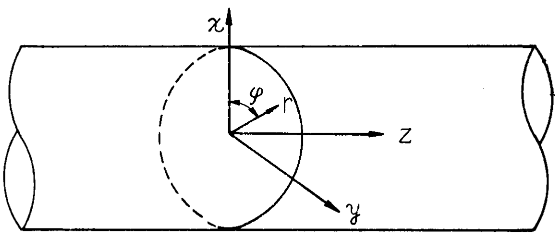

Fig. 1.

$$\begin{cases} x=r\cos\varphi \\ y=r\sin\varphi \\ z=z \end{cases} \qquad \begin{cases} x'=r'\cos\varphi' \\ y'=r'\sin\varphi' \\ z'=z' \end{cases} \ , \tag{3}$$

the equations (1) and (2) are rewritten as follows:

$$\frac{\partial v}{\partial t}-\kappa\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial\varphi^2}+\frac{\partial^2 v}{\partial z^2}\right)=\delta(r-r')\,\delta(\varphi-\varphi')\,\delta(z-z')\,\delta(t-t') \tag{4}$$

$$v=\frac{Q}{(2\sqrt{\pi\kappa(t-t')})^3}exp\left[-\frac{r^2+r'^2-2rr'\cos(\varphi-\varphi')+(z-z')^2}{4\kappa(t-t')}\right] . \tag{5}$$

By the use of the Weber's integral formula

$$\int_0^\infty J_0(at)exp(-p^2t^2)tdt=\frac{1}{2p^2}exp\left(-\frac{a^2}{4p^2}\right), \tag{6}$$

the expression (5) may be reduced to

$$v=\frac{Q}{2\sqrt{\pi\kappa(t-t')}}exp\left[-\frac{(z-z')^2}{4\kappa(t-t')}\right]\cdot\frac{1}{2\pi}\int_0^\infty \alpha\, exp\left[-\kappa\alpha(t-t')\right]J_0(\alpha R)d\alpha, \quad (7)$$

where $R^2=r^2+r'^2-2rr'\cos(\varphi-\varphi')$.

By substituting the Neuman's expansion

$$J_0(\alpha R)=J_0(\alpha r)J_0(\alpha r')+2\sum_{n=1}^{\infty}J_n(\alpha r)J_n(\alpha r')\cos n(\varphi-\varphi')$$

into the equation (7) for $J_0(\alpha R)$ and interchanging the order of summation and integration, the following expression is obtained,

$$v=\frac{Q}{4\pi^{3/2}\sqrt{\kappa(t-t')}}exp\left[-\frac{(z-z')^2}{4\kappa(t-t')}\right]\cdot\sum_{-\infty}^{+\infty}\cos n(\varphi-\varphi')\int_0^\infty \alpha exp\left[-\kappa\alpha(t-t')\right]J_0(\alpha r)J_n(\alpha r')d\alpha. \quad (8)$$

Then the use of the relation

$$J_n(x)=\frac{H_n^{(1)}(x)-e^{ni\pi}H_n^{(1)}(-x)}{2}$$

{where $H_n^{(1)}(x)=J_n(x)+iY_n(x)$}

will rewrite the integral in the right hand side of the above equation as follows,

$$\int_0^\infty \alpha exp[-\kappa\alpha^2 t]J_n(\alpha r')J_n(\alpha r)d\alpha=\frac{1}{2}\int_0^\infty \alpha exp[-k\alpha^2 t]J_n(\alpha r')\left\{H_n^{(1)}(\alpha r)-e^{ni\pi}H_n^{(1)}(-\alpha r)\right\}d\alpha$$

$$=\frac{1}{2}\int_{-\infty}^{+\infty}\alpha e^{-\kappa\alpha^2 t}J_n(\alpha r')H_n^{(1)}(\alpha r)d\alpha.$$

In order to calculate the value of this integral, we may consider the complex integral over the closed path Γ in α-plane shown in Fig. 2, which is composed

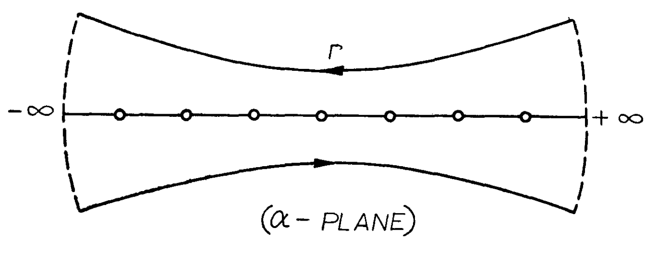

Fig. 2.

of the real axis and a curve, being chosen so that at infinity on the right the argument of α lies between 0 and $1/4\cdot\pi$, and on the left between $3/4\cdot\pi$ and π, and circular arcs, joining the curve to the points $\pm\infty$ on the real axis. Having no poles inside this closed circuit, it follows from Cauchy's theorem that

$$\int_0^\infty \alpha\, exp\left[-\kappa\alpha^2(t-t')\right] J_n(\alpha r') J_n(\alpha r) d\alpha = \frac{1}{2}\int \alpha\, exp\left[-\kappa\alpha^2(t-t')\right] J_n(\alpha r') H_n^{(1)}(\alpha r)\, d\alpha \qquad (r>r')$$

$$= -\frac{1}{2}\int \alpha exp\left[-\kappa\alpha^2(t-t')\right] J_n(\alpha r) H_n^{(1)}(\alpha r')\, d\alpha \qquad (r<r').$$

Thus,

$$v = -\frac{Q}{8\pi^{3/2}\kappa^{1/2}(t-t')^{1/2}} exp\left[-\frac{(z-z')^2}{4\kappa(t-t')}\right]\cdot\sum_{-\infty}^{+\infty}\cos n(\varphi-\varphi')\int \alpha\, exp\left[-\kappa\alpha^2(t-t')\right]\times$$

$$\times J_n(\alpha r') H_n^{(1)}(\alpha r)\, d\alpha \qquad (r>r') \qquad (9)$$

and r, r' being interchanged, when $r<r'$.

The solution (9) expresses the temperature at any point (x, y, z) in the cylinder due to the instantaneous heat-source of strength Q, appearing at (r', φ', z') in the solid at the time t'.

III. Circular Heat-source on the Cylinder

To obtain the temperature distribution in the solid due to the instantaneous circular heat-source at the surface of the cylinder, its radius being a, we may replace $dQ=qad\varphi'$ for Q in the equation (9) and integrate it with φ' from 0 to 2π.

Therefore, it follows that

$$v = -\frac{qa}{4\sqrt{\pi\kappa}(t-t')} exp\left[-\frac{(z-z')^2}{4\kappa(t-t')}\right]\int \lambda\, exp\left[-\kappa\lambda^2(t-t')\right] J_0(\lambda r) H_0^{(1)}(\lambda a)\, d\lambda. \qquad (10)$$

The equation (10) expresses the temperature in an infinite solid due to the heat-source appearing instantaneously in the circular ring shape, its radius being a, at the time t'.

Assuming that there is thermal radiation at the surface of the cylinder into a medium at zero temperature, which is consistent with the present problem, the boundary condition is written as Newton's law that

$$\frac{\partial v}{\partial r} + hv = 0 \qquad \text{at} \qquad r=a, \qquad (11)$$

where $h=H/K$ and H denotes the emissivity. To satisfy the boundary condition, we write again v_0 for v and associate with v_0 another solution v_1 in the form of the integral taken over the same contour path, where

$$v_1=\frac{qa}{4\sqrt{\pi\kappa(t-t')}}exp\Big[-\frac{(z-z')^2}{4\kappa(t-t')}\Big]\cdot\int A\lambda\, exp\,[-\kappa\lambda^2(t-t')]J_0(\lambda r)J_0(\lambda a(\,d\lambda\ .$$

Determining the value of A so that the solution $v=v_0+v_1$ satisfies the boundary condition,

$$v_1=\frac{qa}{4\sqrt{\pi\kappa(t-t')}}exp\Big[-\frac{(z-z')^2}{4\kappa(t-t')}\Big]\cdot\int\lambda\, exp\,[-\kappa\lambda^2(t-t')]J_0(\lambda r)J_0(\lambda a)\times$$

$$\times\Big\{\lambda\frac{d}{d(\lambda a)}\overset{(1)}{H_0}(\lambda a)+h\overset{(1)}{H_0}(\lambda a)\Big\}\cdot\Big\{\lambda J_0'(\lambda a)+hJ_0(\lambda a)\Big\}^{-1}d\lambda\ . \qquad (12)$$

From the process of making the solution, it is evident that the solution v satisfies the initial condition.

Therefore,

$$\begin{aligned}v&=v_0+v_1\\&=-\frac{qa}{4\sqrt{\pi\kappa(t-t')}}exp\Big[-\frac{(z-z')^2}{4\kappa(t-t')}\Big]\cdot\int\lambda exp\Big[-\kappa\lambda^2(t-t')\Big]J_0(\lambda r)\times\end{aligned}$$

$$\times\Big[\overset{(1)}{H_0}(\lambda a)\Big\{\lambda J_0'(\lambda a)+hJ_0(\lambda a)\Big\}-J_0(\lambda a)\Big\{\lambda \overset{(1)}{H_0}(\lambda a)+h\overset{(1)}{H_0}(\lambda a)\Big\}\Big]\cdot\Big\{\lambda J_0'(\lambda a)+hJ_0(\lambda a)\Big\}^{-1}d\lambda. \qquad (13)$$

Since λ_s, the roots of the equation

$$\lambda J_0'(\lambda a)+hJ_0(\lambda a)=0, \qquad (14)$$

are all real and not repeated, the value of the integration in equation (13) is expressed by $2\pi i$ times the sum of the residues at singular points on the real axis.

Therefore, the integral in equation (13) becomes

$$-2\pi i\sum_{s=1}^{\infty}\lambda_s exp[-\kappa\lambda_s^2(t-t')]\,J_0(\lambda_s r)\,J_0(\lambda_s a)\Big\{\lambda_s\frac{d}{d(\lambda a)}\overset{(1)}{H_0}(\lambda_s a)+h\overset{(1)}{H_0}(\lambda_s a)\Big\}\times$$

$$\times\{\lambda_s a\overset{(1)}{J_0}(\lambda_s a)+(1+ha)J_0'(\lambda_s a)\}^{-1}, \qquad (15)$$

where $\sum$ means the smmation over all real roots λ_s.

But, since

$$J_0(\lambda a)\frac{d}{d(\lambda a)}\overset{(1)}{H_0}(\lambda a)-\overset{(1)}{H_0}(\lambda a)\frac{d}{d(\lambda a)}J_0(\lambda a)=\frac{2i}{\pi\lambda a}$$

and

$$\lambda J_0'(\lambda a)+hJ_0(\lambda a)=0$$

$$xJ_0''(x)+J_0'(x)+xJ_0(x)\equiv 0,$$

it follows that

$$\lambda\frac{d}{d(\lambda a)}\overset{(1)}{H_0}(\lambda a)+h\overset{(1)}{H_0}(\lambda a)=\frac{2i}{\pi aJ_0(\lambda a)} \qquad (16)$$

and

$$\lambda_s a J_0''(\lambda_s a)+(1+ha)J_0'(\lambda_s a)=-\frac{a}{\lambda}(h^2+\lambda_s^2)J_0(\lambda_s a)\,. \tag{17}$$

Using the relations (15), (16) and the equation (17) becomes then

$$-\frac{4}{a^2}\sum_{s=1}^{\infty}\lambda_s^2\, exp\Big[-\kappa\lambda_s^2(t-t')\Big]\frac{J_0(\lambda_s r)}{(h^2+\lambda_s^2)\,J_0(\lambda_s a)}.$$

And, therefore, we obtain

$$v=\frac{q}{a\sqrt{\pi\kappa(t-t')}}exp\Big[-\frac{(z-z')^2}{4\kappa(t-t')}\Big]\sum_{s=1}^{\infty}\lambda_s\, exp\Big[-\kappa\lambda_s^2(t-t')\Big]\frac{J_0(\lambda_s r)}{(h^2+\lambda_s^2)J_0(\lambda_s a)}\,. \tag{18}$$

It is known that the equation (18) expresses the temperature in the solid cylindrical bar, the radius being a, due to circular heat-source existing instantaneously on the surface of cylinder at the time t' when the radiation takes place at the surface into a surrounding medium.

IV. Moving Heat-source on the Surface of Cylinder

In the preceding articles, the heat-sources were at rest and instantaneous. In this section, we consider ones which are continuous and moving on the surface along the cylinder.

Consider the circular heat-source situated on ($r=a$, $z=z_0$) at the reference time, moving with constant speed c along the axis in the positive direction. Then, the position z' of the source at the arbitrary time t' is expressed that $z'=z_0+ct$.

In the case of a continuous heat-source with constant strength, we may only integrate the above temperature distribution with t' from the reference time to the time t. Therefore, in the equation (18), we place $dq=Qdz'=cQ\,dt'$ for q and integrate it from 0 to t.

If the uniform convergence of the series in equation (18) is assumed, it is found that

$$v=\frac{Q}{a\sqrt{\pi\kappa}}\sum_{s=1}^{\infty}\lambda_s^2\frac{J_0(\lambda_s r)}{(h^2+\lambda_s^2)\,J_0(\lambda_s a)}\int_{t_0'}^{t'}(t-t')^{-1/2}exp\Big[-\frac{(z-z_0-ct')^2}{4\kappa\,(t-t')}-\kappa\lambda_s^2(t-t')\Big]dt'\,. \tag{19}$$

Without loss of generality, we can choose $t'_0=0$, $z_0=0$. And the equation (19) is written that

$$v=\frac{Q}{a\sqrt{\pi\kappa}}\sum_{s=1}^{\infty}\lambda_s^2\frac{J_0(\lambda_s r)}{(h^2+\lambda_s^2)\,J_0(\lambda_s a)}\int_0^t(t-t')^{-1/2}\,exp\Big[-\frac{(z-z_0-ct')^2}{4\kappa(t-t')}-\kappa\lambda_s^2(t-t')\Big]dt'\,. \tag{20}$$

The solution (20) expresses the temperature in the cylinder at an arbitrary point (z, r) at the time t, when the circular heat-source, having appeared at the origin at $t=0$ is moving on the cylinder along the axis and the radiation takes place at the surface into a medium at zero temperature. It is, therefore, nothing but the solution of the present problem.

It is, however, very difficult to evaluate the integral in equation (20), and the numerical integration will be the only remaining alternative.

The phenomena of the heat conduction would show almost steady property for the large value of t. In other wards, the temperature distribution will uniquely be determined by the relative distance between the instantaneous position of heat-source and the point of observation.

Next we obtain the steady solution as limitting one when t_0' tends to infinity. For the sake of convenience, we may consider that the heat-source, having appeared at $t=-\infty$, has been moved to the origin at the present time t. Therefore, formally, we may only replace t' with $-\infty$ in the lower bound of the integration in expression (19).

Then,

$$v=\frac{Q}{a\sqrt{\pi\kappa}}\sum_{s=1}^{\infty}\lambda_s^2\frac{J_0(\lambda_s r)}{(h^2+\lambda_s^2)J_0(\lambda_s a)}\int_{-\infty}^{t}(t-t')^{-1/2}exp\Big[-\frac{(z-ct)^2}{4\kappa(t-t')}-\kappa\lambda_s^2(t-t')\Big]dt'$$

$$=\frac{Q}{a\sqrt{\pi\kappa}}\sum_{s=1}^{\infty}\lambda_s^2\frac{J_0(\lambda_s r)}{(h^2+\lambda_s^2)J_0(\lambda_s a)}\int_0^{\infty}t^{-1/2}exp\Big[-\frac{(z+ct)^2}{4\kappa t}-\kappa\lambda_s^2 t\Big]dt.$$

But,

$$\int_0^{\infty}t^{-1/2}exp\Big[-\frac{(z+ct)^2}{4\kappa t}-\kappa\lambda_s^2 t\Big]dt=\sqrt{\frac{\pi}{c^2/4\kappa+\kappa\lambda_s^2}}exp\Big[-\frac{zc}{2\kappa}-z\sqrt{\frac{c^2}{4\kappa^2}+\lambda_s^2}\Big].$$

Finally the equation (20) becomes

$$v=\frac{Q}{a\sqrt{\kappa}}exp\Big[-\frac{zc}{2\kappa}\Big]\cdot\sum_{s=1}^{\infty}\lambda_s^2\frac{J_0(\lambda_s r)}{(h^2+\lambda_s^2)J_0(\lambda_s a)}\cdot\Big(\frac{c^2}{4\kappa}+\kappa\lambda_s^2\Big)^{-1/2}exp\Big[-z\sqrt{\frac{c^2}{4\kappa^2}+\lambda_s^2}\Big],$$

which is the required steady solution.

V. Concluding remarks

The analysis has been made to the conduction of heat in a solid cylinder due to the circular source moving on the surface, at which the radiation takes place into a surrounding medium. Both the non-steady and steady solutions have been obtained by the use of Green's function and contour integration.

My first acknowledgment is to Professor Toyotaro Suhara, who initiated me into this problem and gave me incessant help and guidance throughout this work. I am also indebted to Dr. Shigeteru Niwa, the former dean of our department, and to Professor Fumiki Kito, who gave me always the inspiration and encouragement.

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