Title	A three-dimensional problem of a semi-infinite elastic solid under the compressive action of a rigid body
Sub Title	
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Publisher	慶應義塾大学藤原記念工学部
Publication year	1955
Jtitle	Proceedings of the Fujihara Memorial Faculty of Engineering Keio University Vol.8, No.30 (1955.) ,p.68(8)- 80(20)
JaLC DOI	
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Notes	折り込み図
Genre	Departmental Bulletin Paper
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=KO50001004-00080030-0008

A Three-Dimensional Problem of a Semi-Infinite Elastic Solid under the Compressive Action of a Rigid Body*

(Received May 19, 1956)

Rokurō MUKI**

Abstract

In this paper is given the general solution for a three-dimensional problem in elasticity, which consists in finding the stress-distribution of a semi-infinite elastic solid under the compressive action of a rigid body of arbitrary (non-symmetrical) shape. Further, the solution has been applied to a partibular case with the numerical calculations carried out in detail. It is assumed, in the analysis, that the contact area between a semi-infinite elastid solid and a rigid body be kept in a circular form.

Introduction

The problem to determine the distribution of stress in a semi-infinite elastic solid under the compressive action of a rigid body has been considered by a number of authors.[1] The work by Muskelishvili[2] for the two dimensional case and the one by Sneddon[3] for the three-dimensional axisymmetric case are most notable, since they have developed the general theories for the punch of an arbitrary shape. The solution for the distribution of stress in an infinite solid under the compressive action of an elastic plane is obtained for the two dimensional case by H. Okubo.[4]

In a previous paper by the author,[5] the method of solution for the axisymmetric problem of elasticity, which was introduced and used by Sneddon[6] for

*Part of this work has been published in Japanese on the Trans. Jap. Soc. Mech., Eng., Vol. 21, No. 111, 1955, p. 767

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1) For further reference, readers are referred to (2) and (3) for the two and three dimesional case, respectively.

2) Muskelishbili (translated from the Russian by Radok) Some Basic Problems of the Math. Theory of Elasticity. P. Noordhoff Ltd. p. 457 (1953)

3) Harding and Sneddon, Cambridge Phil. Soc., Proc. Vol. 41, p. 61 (1945)

4) H. Okubo, Trans. Jap. Soc. Mech., Vol. 18, No. 65, p. 58 (1952)

5) R. Muki, "On the Sneddon's Method by Hankel Transforms for the Three Dimensional Problem of Elasticity Theory." This paper will appear in the Proceedings of the 5th Japan Natioanal Congress for Applied Mechanics (1955)

6) Sneddon, "Fourier Transforms" Chap. 10, McGraw-Hill (1951)

various interesting problems, was extended to the general nonsymmetrical case. This method is used to obtain the general solution for the displacement and stress fields in a semi-infinite elastic solid under the compressive action of a rigid body with an arbitrary shape. It is assumed, in the analysis, that the contact area between the solid and a rigid body be kept in a circular form. The solution is then applied to a special problem where the flat ended circular cylinder, which at first instance was indented normally to the plane surface of the elastic solid, is inclined by a small angle $\Delta\theta$ due to the moment working on the cylinder. Numerical calculation is carried out to show the influence of the inclnation of the cylinder on the distribtion of stress.

The Expressions by Hankel Transforms for the Displacement and Stress Components.

The derivation [7] of the expressions by Hankel transforms for the displacement and stress components has been shown before, but, for the sake of completeness, this will be recorded here.

If we employ the cylindrical coordinates (r, θ, z), the equations of equilibrium in terms of the displacements u, v, w, in r, θ and z directions are

$$\left.\begin{aligned} &\nabla^2 u+\frac{1}{1-2\nu}\frac{\partial\Delta}{\partial r}-\frac{1}{r}\left(2\frac{\partial v}{r\partial\theta}+\frac{u}{r}\right)=0,\\ &\nabla^2 v+\frac{1}{1-2\nu}\frac{\partial\Delta}{r\partial\theta}-\frac{1}{r}\left(\frac{v}{r}-2\frac{\partial u}{r\partial\theta}\right)=0,\\ &\nabla^2 w+\frac{1}{1-2\nu}\frac{\partial\Delta}{\partial z}=0. \end{aligned}\right\} \tag{1}$$

where ∇^2 and ν denote the Laplcian operator and Poissn's ratio, respectively, and Δ denotes the dilatation and is given by

$$\Delta=\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial v}{r\partial\theta}+\frac{\partial w}{\partial z}. \tag{2}$$

It may be verified by direct substitution that the equations of equilibrium (1) are satisfied if we take

$$\left.\begin{aligned} &u=-\frac{\partial^2\Phi}{\partial r\partial z}+\frac{2}{r}\frac{\partial\psi}{\partial\theta} \qquad v=-\frac{\partial^2\Phi}{r\partial\theta\partial z}-2\frac{\partial\psi}{\partial r},\\ &w=2(1-\nu)\nabla^2\Phi-\frac{\partial^2\Phi}{\partial z^2}, \end{aligned}\right\} \tag{3}$$

where $\qquad \nabla^4\Phi=0, \qquad \nabla^2\psi=0.$ (4)

Eq. (3) is the generalization of the Michell's stress function and if we take Φ to be independent of θ and $\psi=0$, eq. (3) is reduced to the Michell's stress function which is defined to the axisymmetric torsion free deformation. On the contrary,

[7] See (5)

if we take $\Phi=0$ and ψ being independent of θ, eq. (3) represents the state of pure torsion. Moreover, the generality of eq. (3) is assured, since a suitable choice of the biharmonic function Φ makes eq. (3) equall to the displacement field derived from Boussinesq's approach of which generality has been proved by H. Miyamoto.[8]

The stress field corresponding to the displacement field, eq. (3), is easily found to be

$$\left.\begin{aligned}
\sigma_r/2\mu &= \frac{\partial}{\partial z}\left(\nu\nabla^2\Phi-\frac{\partial^2\Phi}{\partial r^2}\right)+\frac{2}{r}\frac{\partial^2\psi}{\partial\theta\partial r}-\frac{2}{r^2}\frac{\partial\psi}{\partial\theta},\\
\sigma_\theta/2\mu &= \frac{\partial}{\partial z}\left(\nu\nabla^2\Phi-\frac{1}{r}\frac{\partial\Phi}{\partial r}-\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}\right)-\frac{2}{r}\frac{\partial^2\psi}{\partial\theta\partial r}+\frac{2}{r^2}\frac{\partial\psi}{\partial\theta},\\
\sigma_z/2\mu &= \frac{\partial}{\partial z}\left\{(2-\nu)\nabla^2\Phi-\frac{\partial^2\Phi}{\partial z^2}\right\},\\
\tau_{\theta z}/2\mu &= \frac{\partial}{r\partial\theta}\left\{(1-\nu)\nabla^2\Phi-\frac{\partial^2\Phi}{\partial z^2}\right\}-\frac{\partial^2\psi}{\partial r\partial z},\\
\tau_{zr}/2\mu &= \frac{\partial}{\partial r}\left\{(1-\nu)\nabla^2\Phi-\frac{\partial^2\Phi}{\partial z^2}\right\}+\frac{\partial^2\psi}{r\partial\theta\partial z},\\
\tau_{r\theta}/2\mu &= \frac{\partial^2}{r\partial\theta\partial z}\left(\frac{\Phi}{r}-\frac{\partial\Phi}{\partial r}\right)-2\frac{\partial^2\psi}{\partial r^2}-\frac{\partial^2\psi}{\partial z^2}.
\end{aligned}\right\} \quad (5)$$

We may write the biharmonic function Φ and the harmonic function ψ in the following forms

$$\left.\begin{aligned}
\Phi(r,\theta,z) &= \sum_{m=0}^{\infty}\left[\Phi_m(r,z)\cos m\theta+\bar{\Phi}_m(r,z)\sin m\theta\right],\\
\psi(r,\theta,z) &= \sum_{m=0}^{\infty}\left[\psi_m(r,z)\sin m\theta+\bar{\psi}_m(r,z)\cos m\theta\right].
\end{aligned}\right\} \quad (6)$$

We assume $\bar{\Phi}_m=\bar{\psi}_m=0$ and consider only a single term of m in eq. (6) without loss in generality. It is easily seen that Φ_m and ψ_m are the solutions of the following partial differential epuations

$$\nabla_m^4\Phi_m=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{m^2}{r^2}+\frac{\partial^2}{\partial z^2}\right)^2\Phi_m,=0\ , \quad (7)$$

$$\nabla_m^2\psi_m=\left(\qquad\qquad // \qquad\qquad\right)\psi_m=0\ . \quad (8)$$

Using the formulas of the Bessel functions, it can be shown that[9]

$$\int_0^\infty \nabla_m^4\Phi_m rJ_m(\xi r)dr=\left(\frac{d^2}{dz^2}-\xi^2\right)^2\int_0^\infty r\Phi_m J_m(\xi r)dr=0\ , \quad (9)$$

$$\int_0^\infty \nabla_m^2\psi_m rJ_m(\xi r)dr=\left(\frac{d^2}{dz^2}-\xi^2\right)\int_0^\infty r\psi_m J_m(\xi r)dr=0\ . \quad (10)$$

[8] This paper was read at tre 4th Japan National Congress for Applied Mechanics on Sept, 2nd, 1954.

[9] See (6) p. 61.

Now, if we write

$$G_m(\xi, z)=\int_0^\infty \Phi_m J_m(\xi r)dr, \tag{11}$$

$$H_m(\xi, z)=\int_0^\infty r\psi_m J_m(\xi r)dr. \tag{12}$$

we obtain the result that if Φ_m and ψ_m are the solutions of the partial differential eq. (7) and (8), then their Hankel transforms G_m and H_m must be the solutions of the ordinary differential equations

$$\left(\frac{d^2}{dz^2}-\xi^2\right)^2 G_m=0, \tag{13}$$

$$\left(\frac{d^2}{dz^2}-\xi^2\right)H_m=0. \tag{14}$$

The general solutions of (13) and (14) are

$$G_m(\xi,z)=(A_m+B_m z)e^{\xi z}+(C_m+D_m z)e,^{-\xi z} \tag{15}$$

$$H_m(\xi, z)=E_m e^{\xi z}+F_m e,^{-\xi z} \tag{16}$$

where the arbitrary constants $A_m \sim F_m$ are to be determined from the given boundary conditions. Once these constants have been determined, G_m and H_m are known functions of z and the parameter ξ and the expressions for Φ_m and ψ_m may then be obtained by means of the Hankel transforms;

$$\Phi_m(r, z)=\int_0^\infty \xi G_m(\xi, z)J_m(\xi r)d\xi\,, \tag{17}$$

$$\psi_m(r, z) = \int_0^\infty \xi H_m(\xi, z)J_m(\xi r)d\xi. \tag{18}$$

Next, we consider the transformation of the expressions for the displacement and stress components into relations involving G_m, H_m and their derivatives. Substituting one term of (6) into the expression of w in (3), we have

$$w=\left[2(1-\nu)\nabla_m^2\Phi_m-\frac{\partial^2\Phi_m}{\partial z^2}\right]\cos m\,\theta.$$

If we multiply both sides of the above equation by $rJ_m(\xi r)$ and integrate with respect to r over the range 0, ∞, we obtain

$$\int_0^\infty wrJ_m(\xi r)dr=\left[(1-2\nu)\frac{d^2G_m}{dz^2}-2(1-\nu)\xi^2 G_m\right]\cos m\,\theta.$$

Inverting the results by the Hankel transform theorem, we have

$$w=\int_0^\infty\left[(1-2\nu)\frac{d^2G_m}{dz^2}-2(1-\nu)\xi^2 G_m\right]\cos m\theta\xi J_m(\xi r)d\xi\,. \tag{19}$$

By a similar procedure, the expression for σ_z can be obtained. The single expression of the remaining components for displacement and stress, however, does not permit the transformation in terms of G_m, H_m and their deivatives. So, constructing the following pairs of the components and carrying out similar calculations, we have

$$\left.\begin{aligned}
&\left(\frac{u}{\cos m\theta}+\frac{v}{\sin m\theta}\right)=\int_0^\infty\left(\frac{dG_m}{dz}+2H_m\right)\xi^2J_{m+1}(\xi r)d\xi\,,\\
&\left(\frac{u}{\cos m\theta}-\frac{v}{\sin m\theta}\right)=-\int_0^\infty\left(\frac{dG_m}{dz}-2H_m\right)\xi^2J_{m-1}(\xi r)d\xi\,,\\
&\left(\frac{\sigma_r}{\cos m\theta}+\frac{\sigma_e}{\cos m\theta}\right)=2\mu\int_0^\infty\left[2\nu\frac{d^3G_m}{dz^3}+(1-2\nu)\xi^2\frac{dG_m}{dz}\right]\xi J_m(\xi r)d\xi\,,\\
&\left(\frac{\tau_{\theta z}}{\sin m\theta}+\frac{\tau_{zr}}{\cos m\theta}\right)=2\mu\int_0^\infty\left[\nu\frac{d^2G_m}{dz^2}+(1-\nu)\xi^2G_m+\frac{dH_m}{dz}\right]\xi^2J_{m+1}(\xi r)d\xi,\\
&\left(\frac{\tau_{\theta z}}{\sin m\theta}-\frac{\tau_{zr}}{\cos m\theta}\right)=2\mu\int_0^\infty\left[\nu\frac{d^2G_m}{dz^2}+(1-\nu)\xi^2G_m-\frac{dH_m}{dz}\right]\xi^2J_{m-1}(\xi r)d\xi\,,\\
&\left(\frac{\sigma_r}{\cos m\theta}+\frac{2\mu u}{r\cos m\theta}+\frac{2\mu mv}{r\sin m\theta}\right)\\
&\qquad\qquad=2\mu\int_0^\infty\left[\nu\frac{d^3G_m}{dz^3}+(1-\nu)\xi^2G_m\right]\xi J_m(\xi r)d\xi\,,\\
&\left(\frac{\tau_{r\theta}}{\sin m\theta}+\frac{2\mu mu}{r\cos m\theta}+\frac{2\mu v}{r\sin m\theta}\right)\\
&\qquad\qquad=2\mu\int_0^\infty H_m\xi^3J_m(\xi r)d\xi\,.
\end{aligned}\right\}\tag{20}$$

Solving these equations, we can find the expressions for the displacement and stress components in terms of G_m, H_m and their derivatives. Summing them up with respect to m, the general expressions for the displacement and stress components are obtained as follows.

$$\left.\begin{aligned}
u&=\frac{1}{2}\sum_{m=0}^\infty\left[U_{m+1}(r,z)-V_{m-1}(r,z)\right]\cos m\theta\,,\\
v&=\frac{1}{2}\sum_{m=0}^\infty\left[U_{m+1}(r,z)+V_{m-1}(r,z)\right]\sin m\theta\,,\\
w&=\frac{1}{2}\sum_{m=0}^\infty\left[\int_0^\infty\left\{(1-2\nu)\frac{d^2G_m}{dz^2}-2(1-\nu)\xi^2G_m\right\}\xi J_m(\xi r)d\xi\right]\cos m\theta\,.
\end{aligned}\right\}\tag{21}$$

$$\begin{aligned}
\frac{\sigma_r}{2\mu}&=\sum_{m=0}^\infty\Bigg[\int_0^\infty\left\{\nu\frac{d^3G_m}{dz^3}+(1-\nu)\xi^2\frac{dG_m}{dz}\right\}\xi J_m(\xi r)d\xi\\
&\qquad\qquad-\frac{(m+1)}{2r}U_{m+1}-\frac{(m-1)}{2r}V_{m-1}\Bigg]\cos m\theta\,,\\
\frac{\sigma_\theta}{2\mu}&=\sum_{m=0}^\infty\Bigg[\nu\int_0^\infty\left\{\frac{d^3G_m}{dz^3}-\xi^2\frac{dG_m}{dz}\right\}\xi J_m(\xi r)d\xi\\
&\qquad\qquad+\frac{(m+1)}{2r}U_{m+1}+\frac{(m-1)}{2r}V_{m-1}\Bigg]\cos m\theta\,,
\end{aligned}$$

$$\left.\begin{aligned}
\frac{\sigma_z}{2\mu} &= \sum_{m=0}^{\infty}\left[\int_0^{\infty}\left\{(1-\nu)\frac{d^3G_m}{dz^3}-(2-\nu)\xi^2\frac{dG_m}{dz^3}\right\}\xi J_m(\xi r)d\xi\right]\cos m\theta ,\\
\frac{\tau_{\theta z}}{2\mu} &= \frac{2}{1}\sum_{m=0}^{\infty}\left[\int_0^{\infty}\left\{\nu\frac{d^2G_m}{dz^2}+(1-\nu)\xi^2G_m+\frac{dH_m}{dz}\right\}\xi^2J_{m+1}(\xi r)d\xi\right.\\
&\qquad\left.+\int_0^{\infty}\left\{\nu\frac{d^2G_m}{dz^2}+(1-\nu)\xi^2G_m-\frac{dH_m}{dz}\right\}\xi^2J_{m-1}(\xi r)d\xi\right]\sin m\theta ,\\
\frac{\tau_{zr}}{2\mu} &= \frac{1}{2}\sum_{m=0}^{\infty}\left[\int_0^{\infty}\left\{\nu\frac{d^2G_m}{dz^2}+(1-\nu)\xi^2G_m+\frac{dH_m}{dz}\right\}\xi^2J_{m+1}(\xi r)d\xi\right.\\
&\qquad\left.-\int_0^{\infty}\left\{\nu\frac{d^2G_m}{dz^2}+(1-\nu)\xi^2G_m-\frac{dH_m}{dz}\right\}\xi^2J_{m-1}(\xi r)d\xi\right]\cos m\theta ,\\
\frac{\tau_{r\theta}}{2\mu} &= \sum_{m=0}^{\infty}\left[\int_0^{\infty}H_m\xi^3J_m(\xi r)d\xi-\frac{(m+1)}{2r}U_{m+1}+\frac{(m-1)}{2r}V_{m-1}\right]\sin m\theta ,
\end{aligned}\right\} \quad (22)$$

where

$$\left.\begin{aligned}
U_{m+1}(r,z) &= \int_0^{\infty}\left(\frac{dG_m}{dz}+2H_m\right)\xi^2J_{m+1}(\xi r)d\xi ,\\
V_{m-1}(r,z) &= \int_0^{\infty}\left(\frac{dG_m}{dz}-2H_m\right)\xi^2J_{m-1}(\xi r)d\xi .
\end{aligned}\right\} \quad (23)$$

It is easy to see that the equations (21) and (22) are reduced to the results which have been obtained by Harding and Sneddon[3] for the axisymmetric torsion free deformation if we put $m=0$ and $H_m=0$.

General Solution

We shall now consider the stresses produced in a semi-infinite elastic solid under compressive action of a rigid body of a prescribed end shape. It is assumed in the analysis that the contact area between the solid and the rigid body be kept in a circular form. If we assume that the shearing stresses vanish at all points of the boundary, the boundary conditions of this mixed problem for $z=0$ are

$$w = w(r,\theta) = \sum_{m=0}^{\infty} w_m(r)\cos m\theta , \qquad 0\leqslant r\leqslant a \qquad (24)$$

$$\sigma_z = 0 , \qquad r > a \qquad (25)$$

$$\tau_{zr} = \tau_{\theta z} = 0 . \qquad 0\leqslant r < \infty \qquad (26)$$

For the sake of convenience, we consider only a single value of m. From the requirement that the displacement and stress components vanish as z tends to infinity, we assume the solutions of equations (13) and (14) of the forms

$$G_m(\xi,z) = (C_m + D_m z)e^{-\xi z}, \qquad (27)$$

$$H_m(\xi,z) = F_m\, e^{-\xi z}. \qquad (28)$$

Substituting (27) (28) in (20) and setting $z=0$, we find in view of (26) that

$$\frac{1}{2\mu}\left[\frac{\tau_{\theta z}}{\sin m\theta}+\frac{\tau_{zr}}{\cos m\theta}\right]_{z=0}=\int_0^\infty[\xi C_m-2\nu D_m-F_m]\xi^3 J_{m+1}(\xi r)d\xi=0\,,$$

$$\frac{1}{2\mu}\left[\frac{\tau_{\theta z}}{\sin m\theta}-\frac{\tau_{zr}}{\cos m\theta}\right]_{z=0}=\int_0^\infty[\xi C_m-2\nu D_m+F_m]\xi^3 J_{m-1}(\xi r)d\xi=0$$

Hence we have

$$C_m=\frac{2\nu}{\xi}D_m\,,\qquad F_m=0\,, \tag{29}$$

$$G_m=D_m\left[\frac{2\nu}{\xi}+z\right]e^{-\xi z}\,. \tag{30}$$

From the conditions (24) (25), we obtain the relations

$$\left.\begin{aligned}\left[\frac{w}{\cos m\theta}\right]_{z=0}&=-2(1-\nu)\int_0^\infty\xi^2 D_m J_m(\xi r)d\xi=w_m(r)\,, &&0\leqslant r<a\\ \frac{1}{2\mu}\left[\frac{\sigma_z}{\cos m\theta}\right]_{z=0}&=\int_0^\infty\xi^3 D_m J_m(\xi r)d\xi=0\,. &&r<a\end{aligned}\right\} \tag{31}$$

If we now make the substitutions

$$\left.\begin{aligned}&\xi^2 D_m=a^2 f_m(p),\qquad \xi a=p,\qquad r=a\rho\,,\\ &-w_m(r)=2(1-\nu)a g_m(\rho)\,,\end{aligned}\right\} \tag{32}$$

we find that $f_m(p)$ is a solution of the dual integral equations

$$\left.\begin{aligned}&\int_0^\infty f_m(p)J_m(p\rho)dp=g_m(\rho)\,, &&0\leqslant\rho<1\\ &\int_0^\infty p f_m(p)J_m(p\rho)dp=0\,. &&\rho>1\end{aligned}\right\} \tag{33}$$

The dual integral equations of the type (33) have been studied by Busbridge.[10] From her results, we have

$$f_m(p)=\sqrt{\frac{2}{\pi}}\left[\sqrt{p}\;J_{m-\frac{1}{2}}(p)\int_0^1 y^{m+1}(1-y^2)^{-\frac{1}{2}}\,g_m(y)dy\right.$$
$$\left.+\int_0^1 y^{m+1}(1-y^2)^{-\frac{1}{2}}dy\int_0^1 g_m(yu)(pu)^{1+\frac{1}{2}}J_{m+\frac{1}{2}}(pu)du\right]. \tag{34}$$

In the particular case in which

$$g_m(\rho)=\sum_{n=0}^\infty A_m^n\,\rho^{n+m}\,, \tag{35}$$

$f_m(p)$ can be reduced to

$$f_m(p)=\sqrt{2p}\sum_{n=0}^\infty\frac{\Gamma\left(1+\frac{n}{2}+m\right)}{\Gamma\left(\frac{1}{2}+\frac{n}{2}+m\right)}A_m^n\int_0^1 u^{n+m+\frac{1}{2}}J_{m-\frac{1}{2}}(pu)du \tag{36}$$

[10]) Busbridge, London Math. Soc., Proc. Vol. 44, p.115 (1938)

by the aid of the following formulas

$$\int_0^1 y^{n+1}(1-y^2)^{-\frac{1}{2}}dy=\frac{\sqrt{\pi}}{2}\frac{\Gamma\left(1+\frac{n}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{n}{2}\right)},$$

$$p\int_0^1 u^{n+m+\frac{3}{2}}J_{m+\frac{1}{2}}(pn)du=-J_{m-\frac{1}{2}}(p)+(n+2m+1)\int_0^1 u^{n+m+\frac{1}{2}}J_{m-\frac{1}{2}}(pu)du\,.$$

If we adopt the notation

$$I_{m+q}{}^{n}(\rho,\zeta)=\int_0^\infty p^n f_m(p)e^{-p\zeta}J_{m+q}(p\rho)dp\,, \tag{37}$$

where $\zeta = z/a$, we find the displacement and stress components in the solid from equations (21) (22) (30) (32) (36) as follows

$$\left.\begin{aligned}
u&=\frac{1}{2}a\sum_{m=0}^{\infty}\Big[(1-2\nu)(I^0_{m+1}-I^0_{m-1})-\zeta(I^1_{m+1}-I^1_{m-1})\Big]\cos m\theta,\\
v&=\frac{1}{2}a\sum_{m=0}^{\infty}\Big[(1-2\nu)(I^0_{m+1}+I^0_{m-1})-\zeta(I^1_{m+1}+I^1_{m-1})\Big]\sin m\theta,\\
w&=-a\sum_{m=0}^{\infty}\Big[2(1-\nu)I^0_m+\zeta I^1_m\Big]\cos m\theta\,.
\end{aligned}\right\} \tag{38}$$

$$\left.\begin{aligned}
\frac{\sigma_r}{2\mu}&=\sum_{m=0}^{\infty}\Big[I^1_m-\zeta I^2_m-\frac{(1-2\nu)}{2\rho}\Big\{(m+1)I^0_{m+1}+(m-1)I^0_{m-1}\Big\}\\
&\qquad+\frac{\zeta}{2\rho}\Big\{(m+1)I^1_{m+1}+(m-1)I^1_{m-1}\Big\}\Big]\cos m\theta\,,\\
\frac{\sigma_r}{2\mu}+\frac{\sigma_\theta}{2\mu}&=\sum_{m=0}^{\infty}\Big[(1+2\nu)I^1_m-\zeta I^2_m\Big]\cos m\theta\,,\\
\frac{\sigma_z}{2\mu}&=\sum_{m=0}^{\infty}\Big[I^1_m+\zeta I^2_m\Big]\cos m\theta,\\
\frac{\tau_{\theta z}}{2\mu}&=\frac{1}{2}\sum_{m=0}^{\infty}\zeta\Big[I^2_{m+1}+I^2_{m-1}\Big]\sin m\theta\,,\\
\frac{\tau_{zr}}{2\mu}&=\frac{1}{2}\sum_{m=0}^{\infty}\zeta\Big[I^2_{m+1}-I^2_{m-1}\Big]\cos m\theta,\\
\frac{\tau_{r\theta}}{2\mu}&=\sum_{m=0}^{\infty}\Big[-\frac{(1-2\nu)}{2\rho}\Big\{(m+1)I^0_{m+1}-(m-1)I^0_{m-1}\Big\}\\
&\qquad+\frac{\zeta}{2\rho}\Big\{(m+1)I^1_{m+1}-(m-1)I^1_{m-1}\Big\}\Big]\sin m\theta.
\end{aligned}\right\} \tag{39}$$

Indentation by a Slightly Inclined Flat Ended Cylinder

As an example, we shall consider the case where the flat ended rigid circular cylinder which, at first instance, was indented normally to the plane surface of the elastic solid is inclined by a angle $\Delta\theta$ as shown in Fig. 1 by the moment acting on the cylinder. As far as $\Delta\theta$ is very small, this problem is identical with the one shown in Fig. 2, where the flat end of the cylinder makes a small angle $\Delta\theta$ with a plane normal to the axis of the cylinder. If we take the solution to the problem shown in Fig. 2 as S, S may be considered as the sum of the solutions S_1, S_2 for the two problems indicated in Fig. 3, (a), (b).

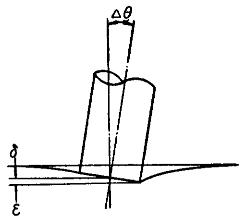

Fig. 1

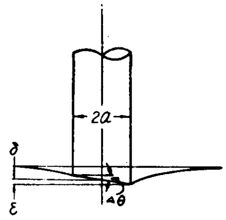

Fig. 2

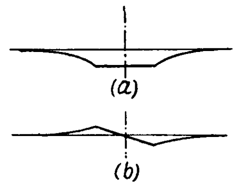

Fig. 3

The z component of the surface displacement for $z=0$ is

$$w=\delta+\varepsilon\frac{x}{a}=\delta+\varepsilon\rho\cos\theta\,. \tag{40}$$

Using the notations (35) and considering (32), we see that

$$\left.\begin{aligned} g_0(\rho) &= A_0^0=-\frac{1}{2(1-\nu)}\frac{\delta}{a}, \\ g_1(\rho) &= A_1^0\,\rho=-\frac{1}{2(1-\nu)}\frac{\varepsilon}{a}\rho\,. \end{aligned}\right\} \tag{41}$$

From (36), we have

$$f_0(p)=\sqrt{\frac{2}{\pi p}}\,A_0^0\,J_{\frac{1}{2}}(p), \qquad f_1(p)=2\sqrt{\frac{2}{\pi p}}\,A_1^0\,J_{\frac{3}{2}}(p). \tag{42}$$

Since the displacement and stress fields corresponding to S_1 have been studied by Sneddon[11], we shall consider only the ones corresponding to S_2.

It is easily seen from (38) (39) that the necessary integrals for the determination of the displacement and stress components on the surface of the solid are $I_2^0(\rho.o)$, $I_0^0(\rho.0)$, $I_1^0(\rho.0)$, $I_1^1(\rho.0)$ which are reduced to the particular cases of the following Sonnine Schafheitlin discontinuous integral[12]

$$\int_0^\infty \frac{J_{\alpha-\beta}(ap)J_{\gamma-1}(bp)}{p^{\gamma-\alpha-\beta}}\,dp=\frac{b^{\gamma-1}\Gamma(\alpha)}{2^{\gamma-\alpha-\beta}a^{\alpha+\beta}\Gamma(\gamma)\Gamma(1-\beta)}F\Big(\alpha,\beta;\gamma;\frac{b^2}{a^2}\Big)\,. \tag{43}$$

The hypergeometric series obtained by inserting corresponding values in places of

[11] Sneddon, Cambrige Phil, Soc., Proc. Vol. 42, p. 29 (1946)

[12] Watson, "Theory of Bessel Functions" p.401 (1922)

$\alpha, \beta, \gamma, a, b,$ can be reduced to the elementary functions by the aid of the recurrence formulas of Gauss.

$$\left.\begin{aligned}
I_2^0(\rho.0) &= \frac{A_1^0}{\pi}\rho^2 F(2, \tfrac{1}{2}; 3; \rho^2) \\
&= \frac{2A_1^0}{\pi}\left[\frac{1}{(1+\sqrt{1-\rho^2})^2} + \frac{1}{\rho^2}\left\{1 - \frac{2}{3\rho^2}\left[1-(1-\rho^2)^{\frac{3}{2}}\right]\right\}\right], && 0 \leqslant \rho \leqslant 1 \\
I_2^0(\rho.0) &= \frac{8}{3\pi}A_1^0\frac{1}{\rho^2}, && 1 \leqslant \rho \\
I_0^0(\rho.0) &= \frac{4A_1}{\pi}(1-\rho^2)^{\frac{1}{2}}, && 0 \leqslant \rho \leqslant 1 \\
I_0^0(\rho.o) &= 0 && 1 \leqslant \rho \\
I_1^0(\rho,0) &= A_1^0\rho, && 0 \leqslant \rho \leqslant 1 \\
I_1^0(\rho.0) &= \frac{4}{3\pi}A_1^0 F\left(\frac{3}{2}, \frac{1}{2}; \frac{5}{2}; \frac{1}{\rho^2}\right) \\
&= \frac{2}{\pi}A_1^0\left[\rho \arcsin\frac{1}{\rho} - \sqrt{1-\frac{1}{\rho^2}}\right], && 1 \leqslant \rho \\
I_1^1(\rho.0) &= \frac{4}{\pi}A_1^0\frac{\rho}{\sqrt{1-\rho^2}}, && 0 \leqslant \rho < 1 \\
I_1^1(\rho.0) &= 0. && 1 < \rho
\end{aligned}\right\} \quad (44)$$

It is easily seen from (38) (39) (44) that the boundary conditions (24) $\sim$ (26) are satisfied. For the evaluation of the stresses in the interior of the solid, the integrals $I_1^1, I_1^2, I_2^0, I_2^1, I_2^2, I_0^2$ should be computed as seen from (39). On the z axis ($\rho=0$), all the integrals vanish except I_0^2 which takes the from

$$I_0^2(o,\zeta) = 2\sqrt{\frac{2}{\pi}}A_1^0\int_0^\infty J_{\frac{3}{2}}(p)e^{-p\zeta}p_{\frac{3}{2}}\,dp = \frac{8}{\pi}A_1^0\frac{1}{(1+\zeta^2)^2}. \quad (45)$$

For arbitrary values of ρ anə ζ, these integrals can be evaluated in a same procedure as adopted by Terazawa[13] and by Sneddon.[11] For I_1^1, we have

$$\begin{aligned}
I_1^1(\rho,\zeta) &= 2\sqrt{\frac{2}{\pi}}A_1^0\int_0^\infty \sqrt{p}\, e^{-p\zeta}J_{\frac{3}{2}}(p)J_1(p\rho)dp \\
&= \frac{4}{\pi}A_1^0\int_0^\infty\left(-\cos p + \frac{\sin p}{p}\right)J_1(p\rho)\,e^{-p\zeta}dp \\
&= \frac{4}{\pi}A_1^0\Big[-Re\left(\int_0^\infty e^{-p(\zeta-i)}J_1(p\rho)\right) \\
&\qquad + I_m\left(\int_0^\infty \frac{e^{-p(\zeta-i)}}{p}J_1(p\rho)\,dp\right)\Big] \\
&= \frac{4}{\pi}A_1^0\frac{R^{-\frac{1}{2}}}{\rho}\left[\zeta\cos\frac{\phi}{2} + (1-R)\sin\frac{\phi}{2}\right],
\end{aligned} \quad (46)$$

where R_e and I_m represent the real and the imaginary part of the function in the

[13] Terazawa, K. Jour. Coll. Sci. Imp. Univ. Tokyo, Vol 37, Art.7, p.56

parentheses, respectively, and

$$\left.\begin{aligned} &R^2=(\rho^2+\zeta^2-1)^2+4\zeta^2, \\ &\tan\phi=\frac{2\zeta}{\rho^2+\zeta^2-1}. \end{aligned}\right\} \tag{47}$$

The results of the evaluation for the other integrals are;

$$\left.\begin{aligned} I_1^2(\rho,\zeta) &= \frac{4}{\pi}A_1^0\Big[-\rho R^{-\frac{3}{2}}\cos\frac{3}{2}\phi+\frac{R^{-\frac{1}{2}}}{\rho}\Big(\cos\frac{1}{2}\phi-\zeta\sin\frac{1}{2}\phi\Big)\Big], \\ I_0^2(\rho,\zeta) &= \frac{2}{\rho}I_1^1(\rho,\zeta)-I_2^2(\rho,\zeta), \\ I_2^1(\rho,\zeta) &= \frac{4}{\pi}A_1^0\Big[\Big\{R^{-\frac{1}{2}}-\frac{R^{\frac{1}{2}}}{\rho^2}\Big\}\cos\frac{1}{2}\phi+\frac{\zeta}{\rho^2}R^{\frac{1}{2}}\sin\frac{1}{2}\phi\Big], \\ I_2^2(\rho,\zeta) &= \frac{4}{\pi}A_1^0\Big[\frac{2R^{-\frac{1}{2}}}{\rho^2}\Big(\zeta\cos\frac{1}{2}\phi+\sin\frac{1}{2}\phi\Big) \\ &\qquad -\Big(R^{-\frac{1}{2}}+\frac{2R^{\frac{1}{2}}}{\rho^2}\Big)\sin\frac{1}{2}\phi+R^{-\frac{3}{2}}\Big(\zeta\cos\frac{3}{2}\phi+\sin\frac{3}{2}\phi\Big)\Big], \\ I_0^2(\rho,\zeta) &= \frac{4}{\pi}A_1^0\Big[\frac{R^{\frac{1}{2}}}{\rho^2}\Big(\zeta\cos\frac{1}{2}\phi-\sin\frac{1}{2}\phi\Big)-\frac{R^{\frac{3}{2}}}{3\rho^2}\sin\frac{3}{2}\phi+\frac{2}{3\rho^2}\Big]. \end{aligned}\right\} \tag{48}$$

For the sake of later convenience, we record here the expressions of σ_z for the solution S_1 from Sueddon's paper.[14)]

$$\left.\begin{aligned} &\frac{\sigma_z(\rho,0)}{2\mu}=\frac{2A_0}{\pi}\frac{1}{\sqrt{1-\rho^2}}, && 0\leqslant\rho<1 \\ &\frac{\sigma_z(\rho,0)}{2\mu}=0, && 1<\rho \\ &\frac{\sigma_z(0,\zeta)}{2\mu}=\frac{2A^0}{\pi}\frac{1+3\zeta^2}{(1+\zeta^2)^2}, && 0\leqslant\zeta \\ &\frac{\sigma_z(0,\zeta)}{2\mu}=\frac{2A_0}{\pi}\Big[R^{-\frac{1}{2}}\sin\frac{1}{2}\phi+\zeta R^{-\frac{3}{2}}\Big(\zeta\sin\frac{3}{2}\phi-\cos\frac{3}{2}\phi\Big)\Big]. \end{aligned}\right\} \tag{49}$$

In view of (39) (43) and (49), it is easily seen that σ_z on the surface of the solid is, for $r\leq a$,

$$\Big[\frac{\sigma_z}{2\mu}\Big]_{z=0}=-\frac{\delta/a}{(1-\nu)\pi}\frac{2}{\sqrt{1-\rho^2}}(1+2\,\varepsilon/\delta\,\rho\cos\theta). \tag{50}$$

Eq. (50) enables us to evaluate the force P to penerate the cylinder to a depth δ below the level of the undisturbed boundary and the moment M to incline the cylinder by $\varDelta\theta=\varepsilon/a$ to its original position.

[14)] See (11) or (6), p.458. The last expression in (49) differs a little from the one in the Sneddon's paper, but, they are the same.

$$\left.\begin{aligned} P &= \int_0^a\int_0^{2\pi}[\sigma_z]_{z=0}\, r d\theta\, dr = -\frac{2\pi a^2\mu}{(1-\nu)}\cdot\frac{\delta}{a}, \\ M &= \int_0^a\int_0^{2\pi}[\sigma_z]_{z=0}\, x\, r d\theta\, dr = -\frac{8a^3\mu}{3(1-\nu)}\cdot\frac{\varepsilon}{a}. \end{aligned}\right\} \tag{51}$$

In order to arrive at a reasonable result, the tensile stress should not exist on the contact surface beteween the cylinder and the solid. In view of (50), the validity of the present solution is, therefore, restricted to the range

$$\varepsilon \leqslant \frac{\delta}{2}.$$

In the following calculations, 0.25 is used for the value of Poisson's ratio. The variations of σ_z with ρ and ζ are shown for the cases $\varepsilon/\delta=0, 0.25, 0.5$ in Figs. 4, 5, 6, respectively. These figures show that the distribution of stress is strongly affected by the inclination of the cylinder. The variations of the stresses due to the solution S_2 (nonsymmetric part of the solution S) with ρ and ζ are shown in Figs. 7~12. In Fig. 12, $\tau_{r\theta}$ at $\rho=1$ are plotted against ζ, since the point, where $\tau_{r\theta}$ on a plane parallel to the surface, attains its maximum value are at or very near to $\rho=1$.

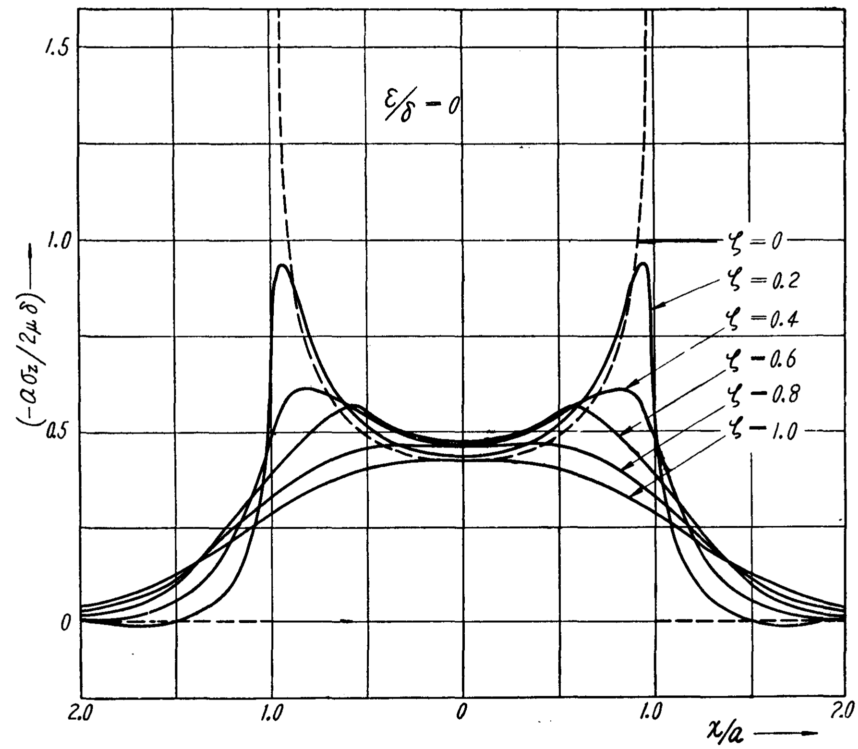

Fig. 4

The infinite principal stresses on the circumference of the rigid cylinder indicate that a certain amount of plastic flow will occur. For low loading, however, as suggested by Sneddon for the axisymmetric case, the elastic stress is predominant except in the vicinity of the cicumference of the rigid body and the results derived above will approximate closely to the true state of the stess distribution.

Acknowledgement

The author wishes to express his gratitude to Professor T. Suhara for his valuable guidance throughout the course of the work. His thanks are also due to Professor F. Kito and Assistant Professor R. Araki for kindly reading the manuscript of the paper and to Assistant Professor H. Miyamoto of Chiba Institute of Technology for his helpful suggestions and discussions.

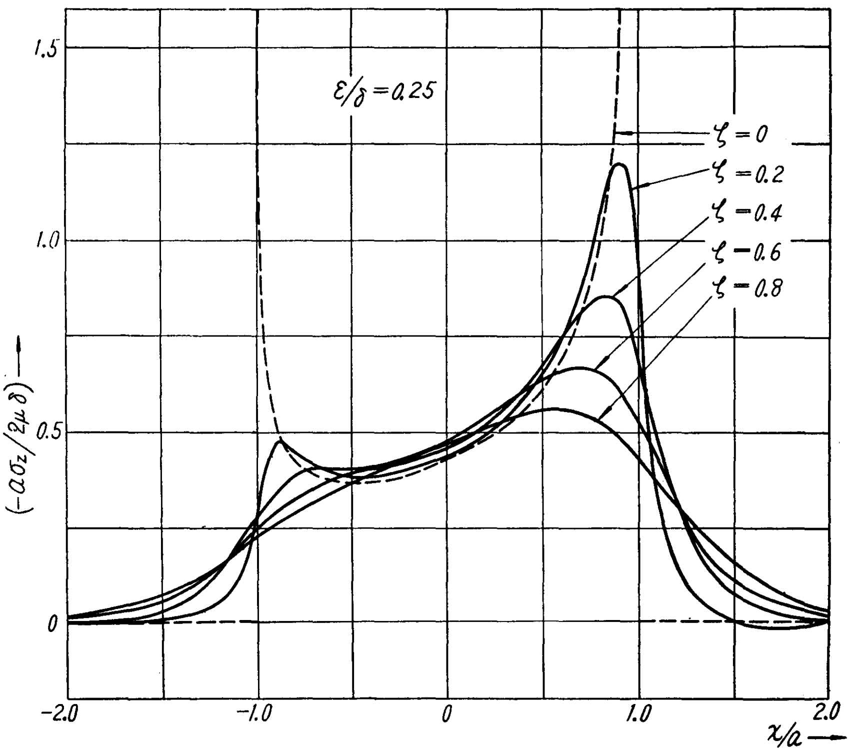

Fig. 5

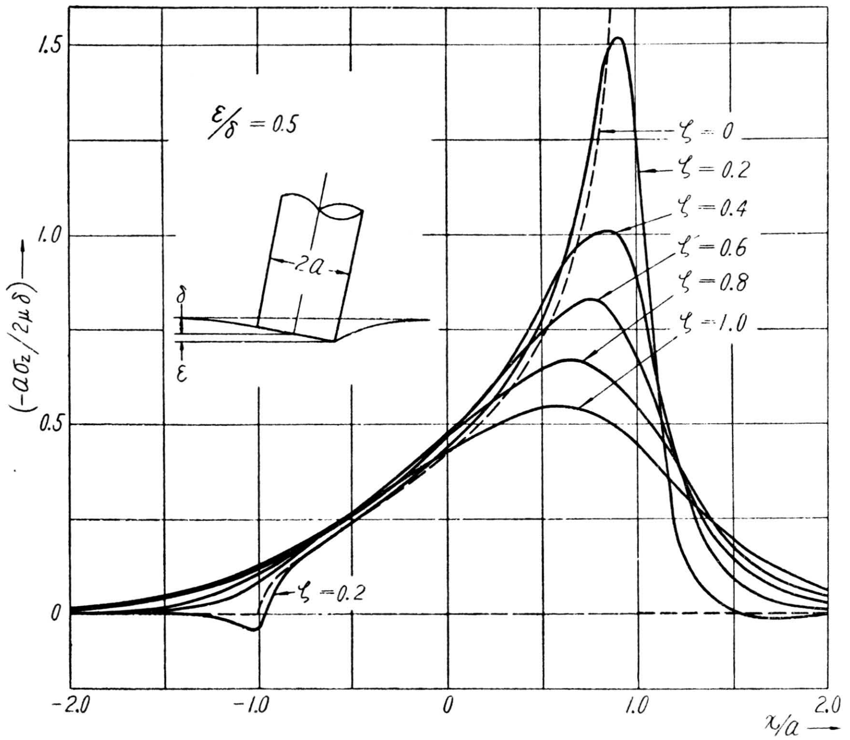

Fig. 6

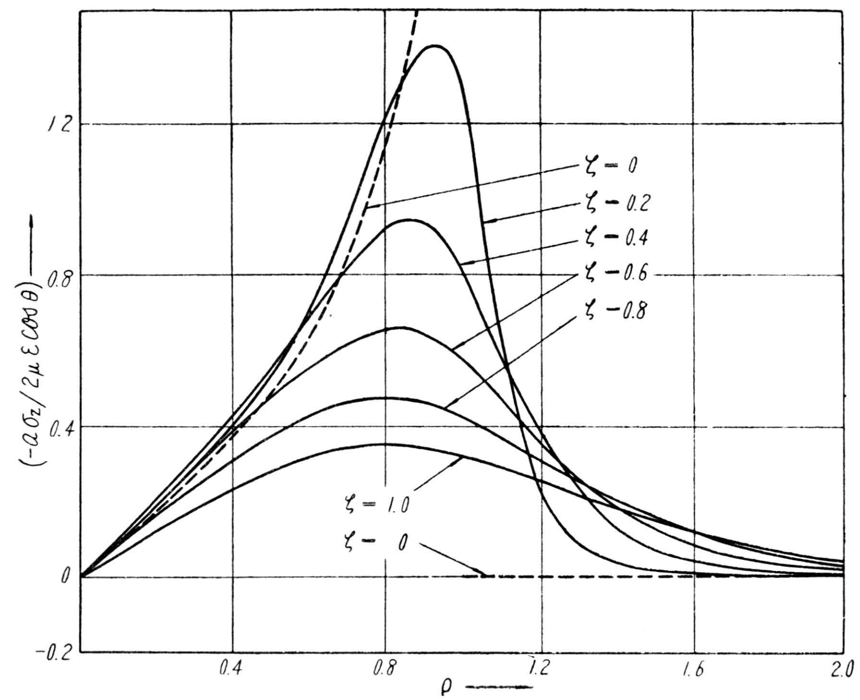

Fig. 7

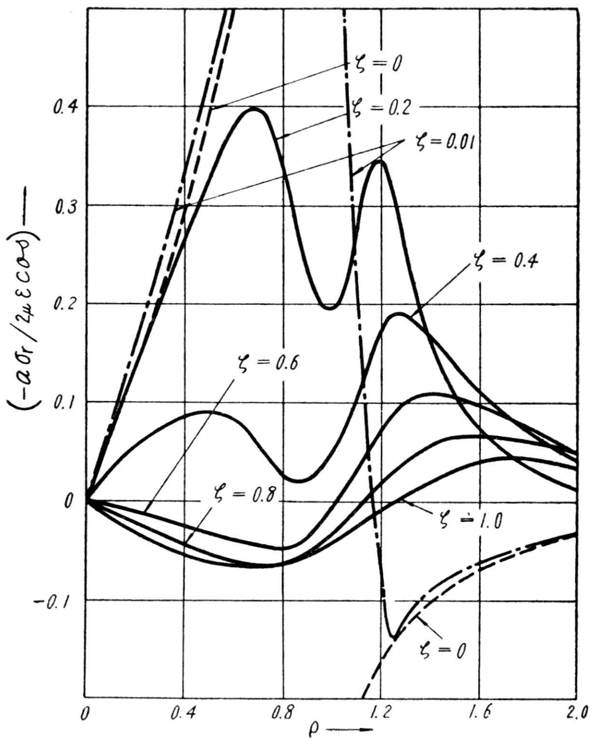

Fig. 8

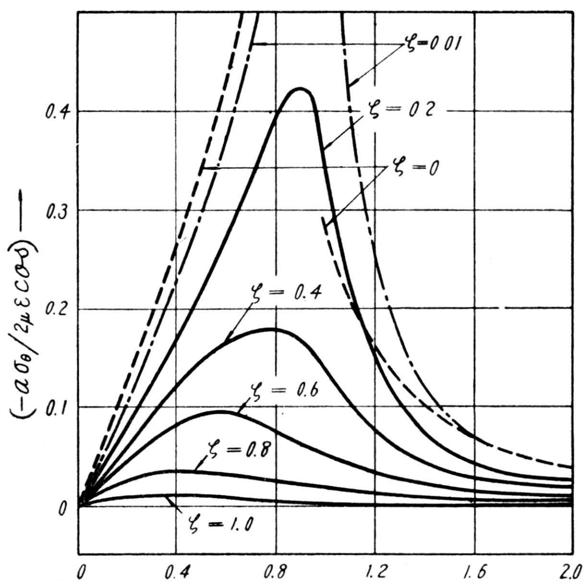

Fig. 9

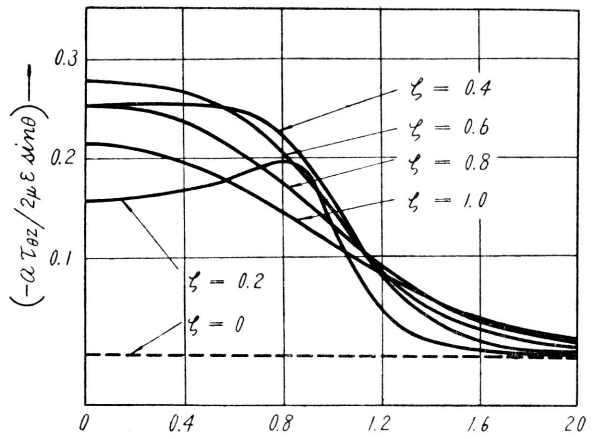

Fig. 10

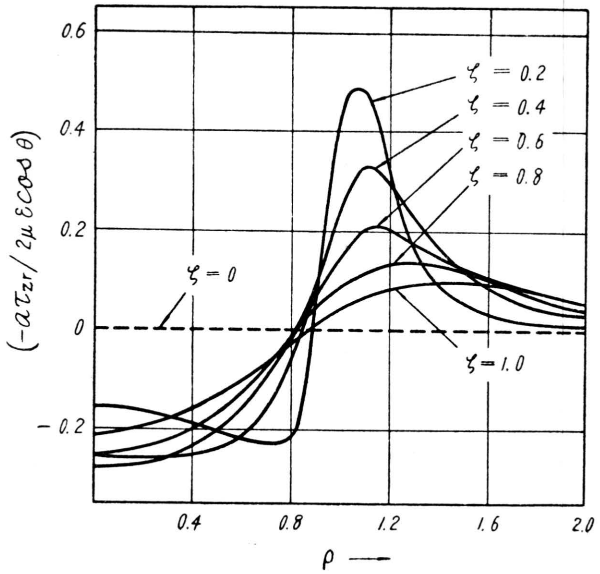

Fig. 11

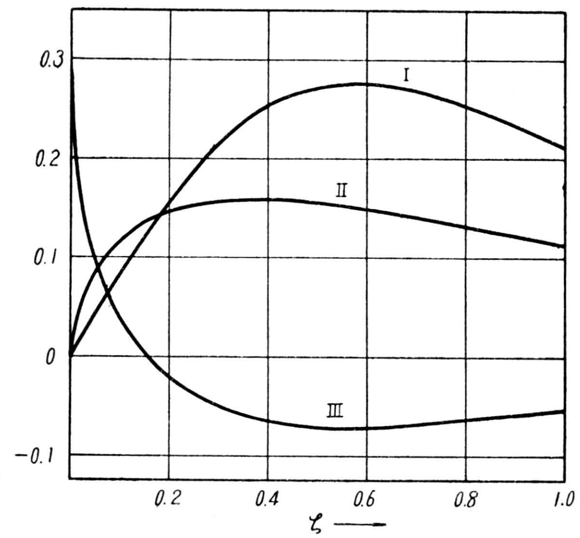

Fig. 12