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Oscillation Represented by the Third Order Differential Equations (Part 1)

(Received August 20, 1956)

*Hiroichi FUJITA

Abstract

A self excited oscillation expressed by the third order differential equations is illustrated in the phase space, whose axes are displacement x, velocity $y=dx/dt$ and accelaration $z=d^2x/dt^2$, respectively.

It is found that trajectories of the constant coefficient third order linear differential equation are rather simple. In aperiodic case, corresponding to the real characteristic root, the trajectories are straight lines on a corn $y^2=xz$; in oscillatory case, corresponding to the complex roots, the trajectories are spiral curves on a plane $ax+by+cz=0$.

A nonlinear characteristic curve of a vacuum tube is divided into several sections and a linear approximation is made in each section. At the connecting poit, the aperiodic component is of great importance for the balance concerning to the limit cycle, and this is one of the features of the third order oscillations.

Hartley and Colpitts oscillator are analized.

I. Introduction

The oscillation involving one dependent variable is usually represented by the second order ordinary differential equation. However, if the oscillation is represented by the third order, more complicated phenomena may be represented.

II. Phase space

In the second order ordinary differential equations, we consider the phase plane whose holizontal axis is dependent variable x and transversal axis is its derivative dx/dt. Now, in the third order differential equations, we consider a phase space. It is difficult to select the most suitable axes of phase space. But, here, the axes are dependent variable x, its derivative $y=dx/dt$ and its second derivative $z=d^2x/dt^2$. When we take these axes, the trajectories are not entirely free. Let us divide the phase space into 8 quadrants. For $z>0$, denote I^+, II^+, III^+ and IV^+, the four quadrants of x-y plane, Denote I^-, II^-, III^- and IV^- the four corresponding quardrants of $z<0$. Then, in I^+, since $y=\dot{x}>0$, x increases and since $z=\dot{y}>0$, y increases too. As the result, in I^+ and III^-, the trajectories go away from the origin. In the contrary, in IV^- and II^+, they come to the origin. In III^+, II^+, I^-, and IV^-

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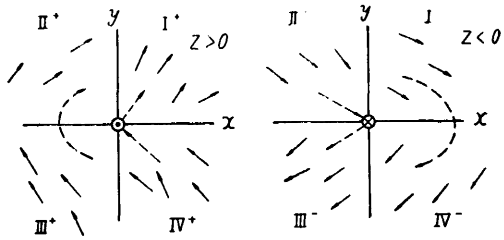

Fig. 1

the trajectories turn around the origin in this order. As described later, the trajectories corresponding to periodic solutions turn on these quadrants in this order. The extremity of x is on the xz plane and the extremity of y is on the xy plane.

III. Singular points

Let x be variation of voltage or current. The differential equation for the oscillator of the third order are generally

$$\frac{dx}{dt}=y \qquad \frac{dy}{dt}=z \qquad \frac{dz}{dt}=f(x,y,z) \qquad f(0,0,0)=0$$

At the origin, the directions of trejectories can not be decided, so the origin is a singular point. In the neighbourhood of origin, the trajectories of general nonlinear differential equation coincide with the trajectories of the third order constant coefficient linear differential equation.

The differential equation has three characteristic roots. If the roots are complex, they are conjugate. Thus their characteristic roots will accord with one or other of the following 8 cases:

(i)	$-\lambda$	$-\alpha\pm j\omega$	(v)	$-\lambda_1$	$-\lambda_2$	$-\lambda_3$
(ii)	$-\lambda$	$\alpha\pm j\omega$	(vi)	λ_1	$-\lambda_2$	$-\lambda_3$
(iii)	λ	$-\alpha\pm j\omega$	(vii)	λ_1	λ_2	$-\lambda_3$
(iv)	λ	$\alpha\pm j\omega$	(viii)	λ_1	λ_2	λ_3

where all letters are real positive numbers. (i) and (v) are stable and the others are all unstable.

IV. The trajectories of the third order constant coefficient linear differential equations.

In order to know the rough characteristics of trajectories of nonlinear differential equations, it is neccessary to investigate the third order constant coefficient linear differential equations.

IV-1 Corn of non-periodic solutions

When a characteristic root is real, say λ_1,

$$\begin{cases} x = Ke^{\lambda_1 t} \\ y = \lambda_1 K e^{\lambda_1 t} = \lambda_1 x \\ z = \lambda_1^2 x = \lambda_1 y \end{cases} \tag{4-1}$$

Where K is an arbitrary constant. (4-1) are equations of lines passing though the origin, that is

(4-2) $\begin{cases} y = \lambda_1 x \\ z = \lambda_1 y \end{cases}$

For any λ_1, the line is on the corn

(4-3) $y^2 = xz$

The axis of this corn is

(4-4) $\begin{cases} x = z \\ y = 0 \end{cases}$

and angle of vertex is π. Near the z axis on the corn, representative point moves rapidly. In I^+ it moves to infinity and in IV^+, to origin.

When a differential equation has three real characteritic roots, the general solution is represented by linear conbination of three non-perodic solutions.

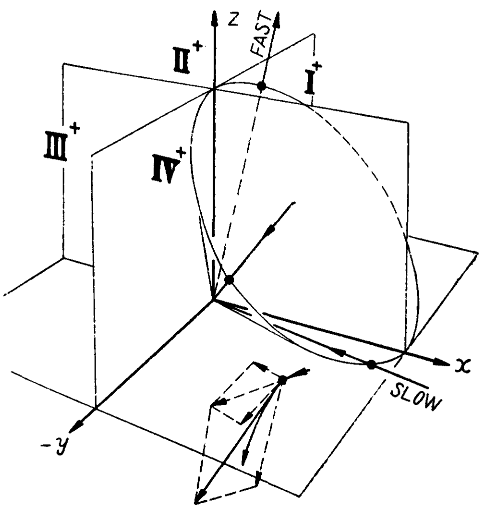

Fig. 2

Define a vector whose direction is tangential to the trajectory and whose amplitude is the velocity of representative point. Then the vector of nonperiodic general solution is decomposed to three vectors which are paralell with the trajectories on the corn corresponding to $K_i e^{\lambda_i t}$ (i=1, 2, 3,)
(Let us called these trajectories nonperiodic solution axes.)

The decomposition and composition of these vectors are done by the parallel quadrilateral method. No trajectories cross a plane decided by the two nonperiodic solution axes and trajectories on this plane never go out from this plane.

IV-2 Plane of periodic solution

When the characteristic roots are complex conjugate, the solution is periodic.

Then

(4-5) $\begin{cases} x = Ae^{-\alpha t} \sin(\omega t+\varphi) \\ y = Ae^{-\alpha t} \{\omega \cos(\omega t+\varphi) - \alpha \sin(\omega t+\varphi)\} \\ z = Ac^{-\alpha t} \{(\alpha^2-\omega^2) \sin(\omega t+\varphi) - 2\alpha\omega \cos(\omega t+\varphi)\} \end{cases}$

Where A and φ are arbitrary constants, From (4-5), eliminate $\sin(\omega t+\varphi)$ $\cos(\omega t+\varphi)$ and $e^{-\alpha t}$, and obtain the followidg

(4-6) $(\alpha^2+\omega^2)x+2\alpha y+z = 0$

(4-6) is the equation of plane.

This plane intersects yz plane ($x=0$) along the $z=-2\alpha y$ and xz and xy planes along the lines $z=-(\alpha^2+\omega^2)x$ and $y=-(\alpha^2+\omega^2)x/2\alpha$. In fig. 3, D is a plane for damped oscillating solution and I is a plane for increased one.

For high frequency oscillation, the velocity and acceleration are large, and an angle between the periodic solution plane and yz plane is small. General solution which contains periodic component, has a vector turning clockwise on the plane.

So, trajectories are on an exponential horn or a hyperbolic horn.

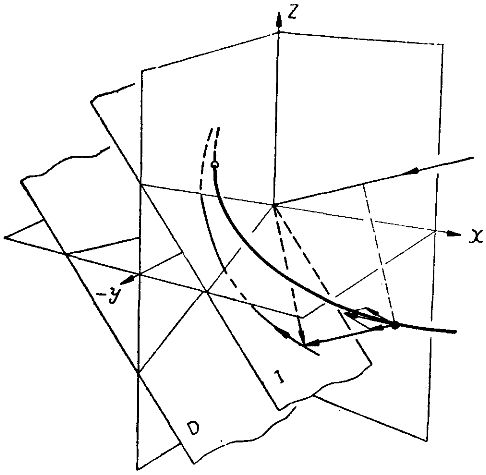

Fig. 3

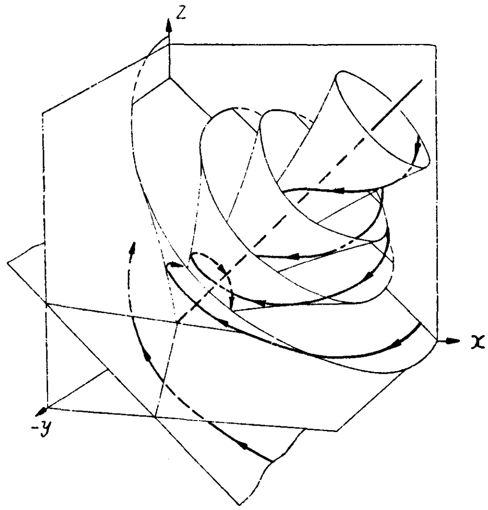

Fig. 4

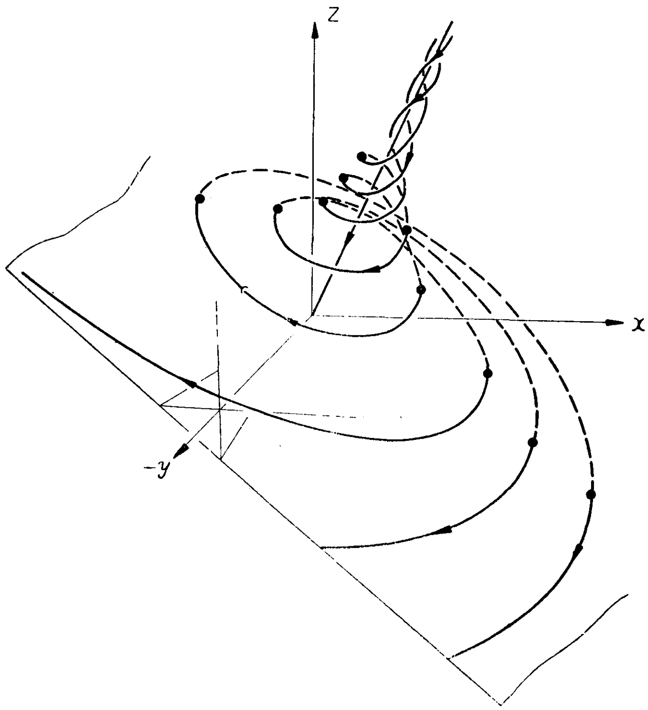

Fig. 5

V. Oscillators represented by the third order nonlinear differential equations.

We will apply the above representation to the anlysis of oscillators. At first, Hartley and Colpittz oscillator are analized.

V-1 Hartley oscillator

Using notation shown in fig. 6, we obtain the circuit equation for grid voltage

$$V_g \equiv x$$

$$L_pL_g\ C\frac{d^3f(x)}{dt^3}+(L_pR_g+L_gR_p)\ C\frac{d^2f(x)}{dt^2}+(L_p+L_g)\ C\frac{d^2x}{dt^2}$$

$$+(R_p+R_g)\ C\frac{dx}{dt}+R_pR_g\ C\frac{df(x)}{dt}+x=0\ *$$

where $-f(x)=i_p$ neglecting grid current.

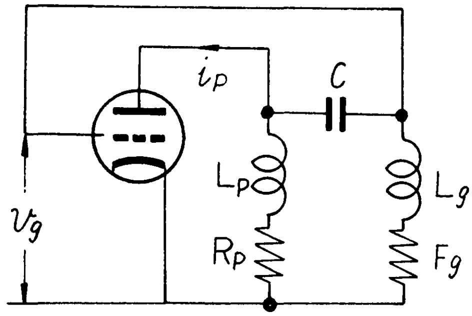

Fig. 6

Grid current plays an important role in the oscillator but in order to simplify the problem, we neglect it here and later (part II) we will consider it in grid tuned oscillator.

For the convinience of numerical calculation, let the time normalize

$$\omega t=\tau$$

where ω is an angular frequency of oscillator decided by linear network theory.

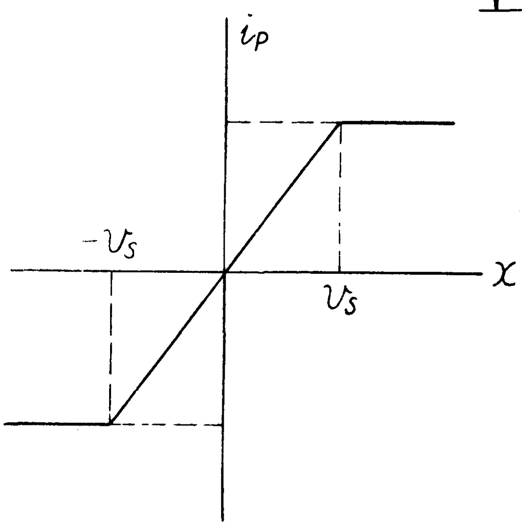

Fig. 7

The nonlinear characterisitic of vacuum tube is divided into three sections and approximated by linear characteristics in each sections, shown in fig. 7 Denote V_s as critical value between linear zone and satulating zone.

For $|x|<V_s$ (linear zone)

$$g_1\dddot{x}+(1+g_2)\ \ddot{x}+\sqrt{\frac{c}{L_p+L_g}}(g_3+R_p+R_g)\ \dot{x}+x=0$$

$|x|>V_s$ (satulaing zone)

$$\ddot{x}+\sqrt{\frac{c}{L_p+L_g}}(R_p+R_g)\ \dot{x}+x=0$$

*H. Fujita, This Proceedings, 7. 29 (1954)

where
$$g_1 \equiv \omega^3 L_p L_g C g_m$$
$$g_2 \equiv \omega^3 C(L_p R_g + L_g R_p) g_m$$
$$g_3 \equiv R_p R_g g_m$$

For ordinary circuit elements of oscillator, the origin in the phase space is singular point (ii). In the satulating zone, the differential equation has the complex characteristic roots whose real parts are negative and the trejectories lie only on the plane of periodic solution.

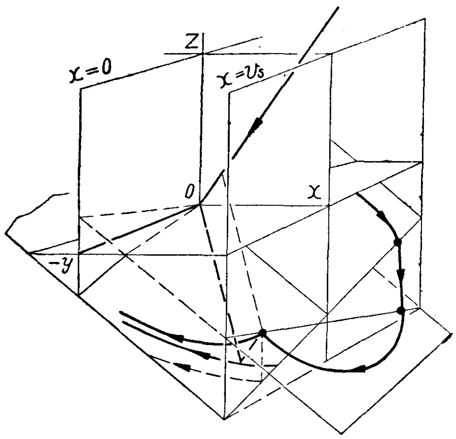

Fig. 8

If a small initial deviation from the origin is given, the representative point reaches at $|x| = V_s$ on the plane of increasing periodic solution of linear zone, for nonperiodic component will become negligibly small.

At $|x| = V_s$ representative point jumps onto the plane of decreasing periodic solution of satulating zone. Here, the solution and its derivative are continuous and second derivative is discontinuous. If the characteristics of vacuum tube has continuous derivative, the representative point does not jump but comes very rapidly to the plane of satulating zone.

From here, the representative point goes through the maximum value of x and comes to $|x| = V_s$ again and enters the linear zone. But here the representative point is not on the plane of periodic solution.

So, the solution must have nonperiodic component. Therefore the periodic component is suppressed. This suppression at the boundary plane between linear and satulating zones does not occur in oscillations represented by the second order differential equations.

On the limit cycle.

The increase in the linear zone and the decrease in the satulating zone and at boundary planes $|x| = V_s$, are balanced. Then the solution is stately periodic.

V-2. Colpittz oscillator

The differential equations for the Colpittz oscillator are obtained in the same way as for the Hartley oscillator.

In the linear zone
$$\dddot{x} + R\ddot{x} + \dot{x} + gx = 0$$
and in the satulating zone
$$\dddot{x} + R\ddot{x} + \dot{x} + i_{po} = 0$$

where

$$R \equiv r/\omega L = \frac{1}{Q}$$

$$g = g_m/\omega(C+C_g)$$

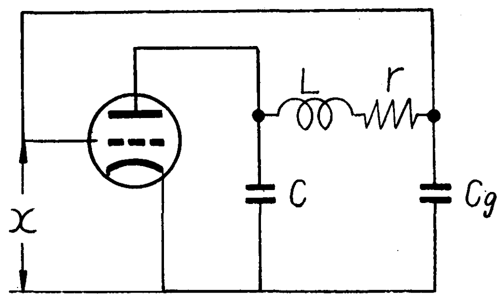

Fig. 9

In the linear zone, trajectories are same type as the Hartley oscillator. But in the satulating zone, the solution is

$$x = Ae^{-Rt/2}\,sin(\omega t+\varphi)-i_{po}t+K$$

where A, φ and K are arbitrary constants.

Therefore, in satulating zone, the nonperiodic axis is on x-y plane and parallel to x axis. So representative point is on exponential horns shown in fig. 10.

Physical meaning of $-i_{po}t$ is constant current discharge from C_g in satulating zone. Stately periodic trajectory occurs in the same way as with the Hartley oscillator.

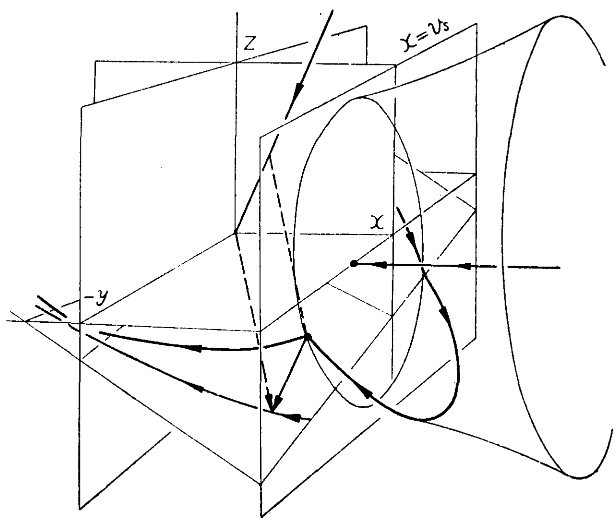

Fig. 10