Title	Analysis of nonlinear electric circuits having periodic solutions (part II)
Sub Title	
Author	藤田, 廣一(Fujita, Hiroichi)
Publisher	慶應義塾大学藤原記念工学部
Publication year	1955
Jtitle	Proceedings of the Fujihara Memorial Faculty of Engineering Keio University Vol.8, No.28 (1955.) ,p.19(19)- 25(25)
JaLC DOI	
Abstract	This paper is continuation of Part I of the same author's paper in Vol. 7, No. 25, 1954 of this proceedings. Here, some nonlinear circuits are analised as examples for the method described in Part I . For van der Pol's self-exited oscillation, we can obtain the solution which is the same as the first approximate solution obllined by the usual perturbation method. Hartley oscillator, one of self-exited oscillation represented by the third order differential equation, is calculated too. Last example is a saturable reactor with inductive load.
Notes	
Genre	Departmental Bulletin Paper
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=KO50001004-00080028-0019

Analysis of Nonlinear Electric Circuits having Periodic Solutions (Part II)

(Received 21 October, 1955)

By Hiroichi FUJITA*

Abstract

This paper is continuation of Part I of the same auther's paper in Vol. 7, No. 25, 1954 of this proceedings. Here, some nonlinear circuits are analised as examples for the method described in Part I.

For van der Pol's self-exited oscillation, we can obtain the solution which is the same as the first approximate solution obtained by the usual perturbation method. Hartley oscillator, one of self-exited oscillation represented by the third order differential equation, is calculated too. Last example is a saturable reactor with inductive load.

I. Van der Pol's differential equation

Well known van der Pol's equation is

$$\frac{d^2x}{dt^2}+\varepsilon(x^2-1)\frac{dx}{dt}+x=0.$$

This is the equation for a plate tuned vacuum tube oscillator and it is already known that the amplitude of the approximate solution of this equation is 2 by usual perturbation method. My method is equivalent to van der Pol's method but more general than his method.

Now, from the part I we put

$$\left.\begin{aligned}\frac{dx}{dt}&=y\\ \frac{dy}{dt}&=-\ (x^2-1)y-x\end{aligned}\right\}\qquad(1\text{-}1)$$

and

$$x=K\cos(\omega t+\varphi)$$
$$y=-K\sin(\omega t+\varphi)$$

where K and φ are slowly varied with time t and ω is angular frequency of oscillation which is not known now. Then

$$\begin{aligned}&\cos(\omega t+\varphi)\frac{dK}{dt}-\omega K\sin(\omega t+\varphi)\frac{d\varphi}{dt}=0\\ &-\sin(\omega t+\varphi)\frac{dK}{dt}-\omega K\cos(\omega t+\varphi)\frac{d\varphi}{dt}\\ &=\omega K\cos(\omega t+\varphi)+\varepsilon(K^2\cos^2(\omega t+\varphi)-1)K\sin(\omega t+\varphi).\end{aligned}\qquad(1\text{-}2)$$

---

* 藤田廣一: Lecturer at Keio University

Then D. C. component of dK/dt and $d\varphi/dt$ are respectively

$$\begin{aligned}\left[\frac{dK}{dt}\right]_{D.C.} &= \frac{\varepsilon K}{2} - \frac{\varepsilon K^3}{8} \\ \left[\frac{d\varphi}{dt}\right]_{D.C.} &= -\frac{1}{2} + \frac{\omega}{2}\end{aligned} \qquad (1\text{-}3)$$

If (1-3) are not zero, K and φ tend to $\pm\infty$ after a long time.
Then they are equal to zero, and we obtain

$$K=2 \quad \text{and} \quad \omega=1$$

So van der Pol's oscillator has the amplitude 2 and angular frequency 1 approximately.

II. D.C. component of dK/dt and $d\varphi/dt$

Van der Pol's oscillator has a vacuum tube whose characteristics is represented by the first and third power polinomial. If the nonlinearity is of more complex type, say fifth or seventh power polinomial, D. C. component of dK/dt and $d\varphi/dt$ are different from (1-3)

In general case, we must calculate

$$\frac{1}{2\pi}\int_0^{2\pi}\sin(t+\theta)F(t.K.\theta)dt$$

$$\frac{1}{2\pi}\int_0^{2\pi}\cos(t+\theta)F(t.K.\theta)dt$$

When $F(t.K.\theta)$ is the integer function of $\sin t$ and $\cos t$, we can easily expand $F(t.K.\theta)$ to Fourier series of $\sin nt$ and $\cos nt$.

Then the term for $n=0$, is the D.C. component of $F(t.K.\theta)$.

When $F(t.K.\theta)$ is polinomial of $\sin^n t \cos^m t$, and $m,n \leq 3$, we obtain the D.C. terms using triangular formula. But for $m,n>3$, the calculation is very troublesome. So we will try to make the formula for general m and n.

i) For the terms of odd integer n ($n=2p-1$, $p=1, 2, \cdots$)

$$\frac{1}{2\pi}\int_0^{2\pi}\sin^n t\ dt=0$$

$$\frac{1}{2\pi}\int_0^{2\pi}\cos^n t\ dt=0$$

So here is no D.C. component

ii) For the terms of even integer n ($n=2p$, $p=1, 2, \cdots$)

$$\frac{1}{2\pi}\int_0^{2\pi}\sin^n t\ dt=\frac{1}{2\pi}\int_0^{2\pi}\cos^n t\ dt = \left(\frac{1}{2}\right)^n {}_nC_{\frac{n}{2}}$$

The above is proved by using

$$\sin t = \frac{e^{it}-e^{-it}}{2i} \qquad \cos t = \frac{e^{it}-e^{-it}}{2}$$

iii) For $\frac{1}{2\pi}\int_0^{2\pi} \sin^n t \cos^m t \, dt$

$$\sin^n t \cos^m t = \left(\frac{1}{2i}\right)^n \left(\frac{1}{2}\right)^m \sum_{r=0}^{n} \sum_{s=0}^{m} (-)^r {}_nC_r \, {}_mC_s e^{i\{m+n-2(r+s)t\}}$$

If $n+m-2(r+s) \neq 0$, it has no D.C. component. If $n+m-2(r+s)=0$ and n or m is odd integer, D.C. components vanishes. When n and m are both even integer, there are D.C. component. The result is shown in next table. $F(t,K,\theta)$ is generally regular function which is not always integer function. But the integral can be calculated after the transformation

$$u=\tan\frac{t}{2} \quad \text{or} \quad u=\tan t$$

Table

$n,m.$	0	2	4	6	8
0	1				
2	$\frac{1}{2}$	$\frac{1}{8}$			
4	$\frac{3}{8}$	$\frac{1}{16}$	$\frac{3}{128}$		
6	$\frac{5}{16}$	$\frac{5}{128}$	$\frac{3}{256}$	$\frac{5}{1924}$	
8	$\frac{35}{280}$	$\frac{7}{259}$	$\frac{7}{1024}$	$\frac{5}{2048}$	$\frac{15}{32768}$
10	$\frac{93}{256}$	$\frac{21}{1024}$	$\frac{9}{2048}$	$\frac{45}{32768}$	
12	$\frac{231}{1024}$	$\frac{33}{2048}$	$\frac{99}{32768}$		
14	$\frac{429}{2048}$	$\frac{429}{32768}$			
16	$\frac{6435}{32768}$				

III. Colpittz Oscillator

Colpittz oscillator whose vacuum tube has high amplification facter μ, can be analized by this method. When μ is not very high, i_p is the function of V_p and V_g. Then the circuit equation is not adequate. (II—B Note)

The plate current of high μ vacuum tube is the function of grid voltage V_g only. So it may be written

$$i_p = f(V_g)$$

For so amplification, let $f(V_g)$ be as the following

$$i_p=g_1V_g-g_3V_g^3$$

If we neglect the grid current,* we can obtain the circuit equation by referring to Part I, (II−G) and Fig. 1.

$$LCC_gp^3V_g+rCC_gp^2V_g+(C+C_g)pV_g+g_1V_g-g_3V_g^3=0$$

Since the D.C. components of dK/dt and $d\varphi/dt$ are zero, we obtain following conditions

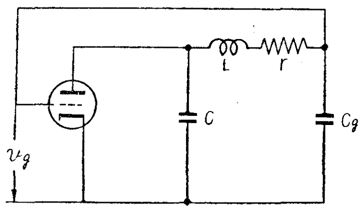

Fig. 1

$$-(r/L)+(g_1/\omega^2CC_g)-(3g_3K^2/\omega^2CC_g)=0$$

$$-(1/2)+(C+C_g)/2LCC_g=0$$

Then the amplitude and angular frequecy are

$$K^2=-4r(C+C_g)/3g_3L_g+4g_1/3g_3$$

$$\omega=(C+C_g)/LCC_g$$

The angular frequency is the same as that calculated by the linear theory. The amplitude is maximum when $r=0$.

IV. Saturable Reactor

Saturable reactor consists of a nonlinear ferromagnetic core and two windings: one of the windings is for A.C. power supply voltage and another is for D.C. signal voltage.

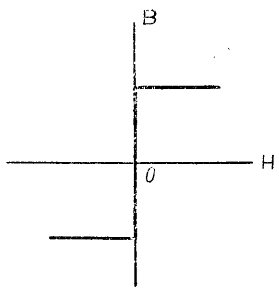

Fig. 2

The nonlinearity of core is often represented by the broken lines shown in Fig. 2. Then, for the only extreme case (namely, zero and infinite impedance of D.C. circuit, corresponding respectively to free magnetization and forced magnetization) may be analized. Here, we will analize the general case.

Fig. 3 is the circuit diagram of a fundamental saturable reactor with inductive load. Induced voltages across A.C. windings are respectively

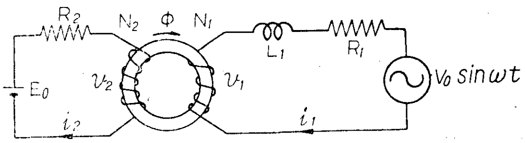

Fig. 3

$$v_1=N_1d\phi/dt$$

$$v_2=N_2d\phi/dt$$

(N_1, N_2 numbers of turns)

---

*) Grid current is a very important factor for the oscillator. But we neglect it here to simplify the problem in order to understand the essential character.

The magnetomotive force is

$$H=(N_1 i_1+N_2 i_2)/l \qquad \text{(ampere turns / meter)}$$

For this case, the four-terminal-network constants $A_{11} B_1 A_{21}$ and B_2 are

$A_{11}=1 \qquad B_1=L_1 p+R_1$

$A_{21}=1 \qquad B_2=R_2$

L_1 is inductance of load in Henry and R_1,R_2 are resistances of A.C. and D.C. circuits in ohms.

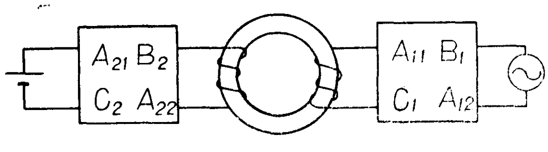

Fig. 4

The nonlinear characteristics of ferromagnetic core is

$$H=g(\phi)$$

Then we obtain the circuit equation (II-F in Part I)

$$\frac{N_1}{l}\frac{V_0 \sin\omega t}{L_1 p+R_1}+\frac{N_1 E_0}{l\ R}=\left(\frac{N_1^2}{l}\frac{1}{Lp+R_1}+\frac{N_2^2}{l}\frac{1}{R_2}\right)\frac{d\phi}{dt}+g(\phi)$$

The differntial equation for this, is

$$Ln_2^2 d^2\phi/dt^2+(n_1^2+n_2^2)d\phi/dt+g(\phi)+Ldg(\phi)/dt=V\sin\omega t+E \qquad (4\text{-}1)$$

where

$$L=\frac{L_1}{R_1} \qquad n_1^2=\frac{N_1^2}{lR_1}$$

$$n_2^2=\frac{N_2^2}{lR_2} \qquad V=\frac{N_1^2}{lR_1}V_0$$

$$E=N_2^2 E_0/lR_2$$

Let the nonlinear characteristics be

$$g(\phi)=\alpha\phi+\beta\phi^3 \qquad (4\text{-}2)$$

If $g(\phi)$ has general power ϕ^n, the calculation will be complicated but possible.

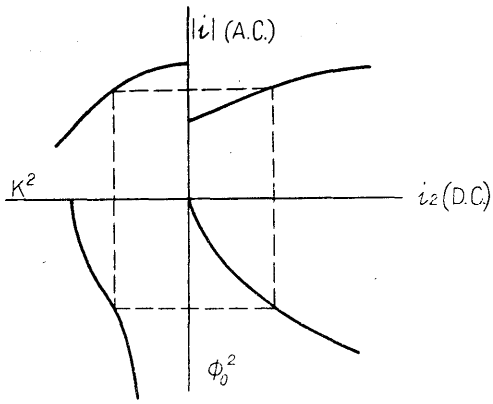

Fig. 5

From (4-1) and (4-2)

$$a\frac{d^2\phi}{dt^2}+(b+c\phi^2)\frac{d\phi}{dt}+\alpha\phi+\beta\phi^3-V\sin\omega t-E=0$$

where

$$a=Ln_1^2 \qquad b=n_1^2+n_2^2+L\alpha \qquad c=3\beta L$$

Then let

$$\phi=\phi_0+K\cos(\tau+\theta) \tag{4-3}$$

where $\tau=\omega t$ (refer III-B)

$$d\phi/d\tau=-\omega K\sin(\tau+\theta)$$

$$d^2\phi/d\tau^2=-\omega^2\sin(\tau+\theta)dK/d\tau-\omega^2K\cos(\tau+\theta)(1+d\theta/d\tau)$$

$$\left[\frac{dK}{dt}\right]_{D.C}=\frac{1}{a\omega^2}\left[-\frac{\omega K}{2}(b+c\phi_0^2)-\frac{1}{8}c\omega^2K^3-\frac{1}{2}V\cos\theta\right]=0 \tag{4-4}$$

$$\left[\frac{d\theta}{dt}\right]_{D.C}=-\frac{1}{2}+\frac{1}{a\omega^2K}\left[\frac{1}{2}K(\alpha+3\beta\phi_0^3)+\frac{3}{8}\beta K^3-\frac{1}{2}V\sin\theta\right]=0$$

Then

$$V^2=\omega^2K^2\{b+c(\phi_0^2+K^2/4)\}^2+K^2\{\alpha-a\omega^2+3\beta(\phi_0^2+K^2/4)\}^2$$

Solve (4-5) for ϕ^2 and we obtain

$$\phi_0^2=-\frac{K^2}{4}-\frac{\omega^2bc+3\beta(\alpha-a\omega)^2}{\omega^2c^2+9\beta^2}\pm\sqrt{\left(\frac{\omega^2bc+3\beta(\alpha-a\omega)^2}{\omega^2c^2+9\beta^2}\right)^2-\frac{\omega^2b^2+(\alpha-a\omega^2)-V^2/K^2}{\omega^2c^2+9\beta^2}}$$

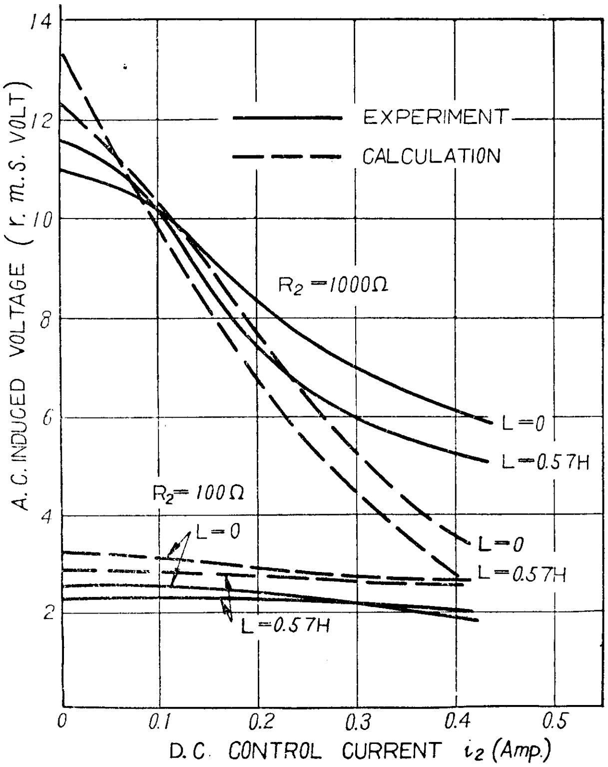

Fig. 6

Relation between A. C. current i_1 and magnetic flux K is

$$|i_1| = \sqrt{\frac{V_0^2 + \omega^2 N_1 K^2 + 2\omega N_1 V_0 K \cos\theta}{R_1^2 + \omega^2 L_1^2}}$$

This is obtained by substitution (4-3) to

$$V_0 \sin\omega t = L_1 di_1/dt + Ri_1 + N_1 d\phi/dt$$

From (5-5) and (5 6)

$$|i_1| = \sqrt{\frac{V_0^2 - 2\omega^2 K^2\{N_2^2/2 + R_1 N_1^2/R_2 + \alpha l L_1 + 3L_1\beta(\phi_0^2 + K^2/4)\}}{R_1^2 + \omega^2 L_1^2}}$$

Thus we have discovered the relations $i_1 - \phi_0$, $K^2 - \phi_0^2$ and $I_1 - K^2$. So from the Fig. 5, the characteristic curve of saturable reactor, $i_2 - i_1$ curve, is obtained. If the load inductance L_1 is zero

$$\phi_0^2 = -K_2/4 - \alpha/3\beta + \sqrt{V^2/K^2 - \omega^2(n_1^2 + n_2^2)^2}\,/3\beta$$

$$|i_1| = \sqrt{V_0^2 - 2\omega^2 K^2 (N_2^2/2 + R_1/R_2) N_1^2}\,/R_1$$

From these equations, the characteritsic curve for pure resistive load is known. Fig. 6, shows the results of the experiment and calculation. Fig. 7 is the characteristic curve of ferromagnetic core.

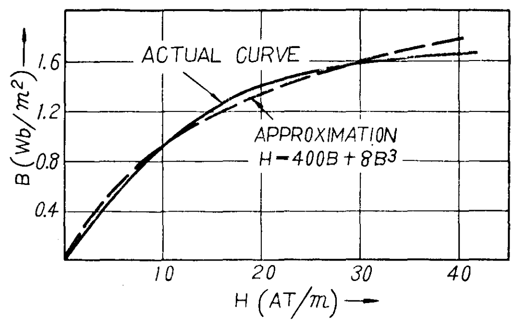

Fig. 7