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On Intermittent Oscillation and Oscillation Hysteresis

(Received April 13, 1955)

Zin-iti NAGUMO*

Abstract

A type of intermittent oscillation which frequently occurs in electrical system is considered. It is pointed out that an oscillation hysteresis of the system plays an important role in the intermittent oscillation. Methods are derived to get intermittent oscillation and to prevent intermittent behaviour of oscillation.

I. Introduction

The definition of "intermittent oscillation" considered here is an oscillation which can be regarded as hamonic and the amplitude of which is modulated periodically by the oscillating system itself. The definition implies, as an extreme case, a harmonic oscillaion which occurs and vanishes periodically.

It seems to be impossible†) to describe such an intermittent oscillation by a single differential equation of pseud-harmonic type:

$$\ddot{x}+x=\mu f(x, \dot{x}, t) \tag{1}$$

where μ is a small parameter. But it is possible if the oscillating system has an element which changes by the oscillation itself and the change of the element affects back the oscillation.

The simplest case of these intermittent oscillations is that oscillating system of the form (1) contains an element expressed by a parameter λ and the change of λ is governed by a first order differential equation:

$$\dot{\lambda}=G(x_0, \lambda) \tag{2}$$

where x_0 is the amplitude of the oscillat on x.

We shall call (1) "harmonic part" and (2) "relaxative part" of the system respectively and consider such an oscillating system that can be regarded, under suitable assumptions and approximations, as being composed of the two parts (Fig. 1).

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†) It is of course possible if the frequency or the phase, as well as the amplitude, is modulated.

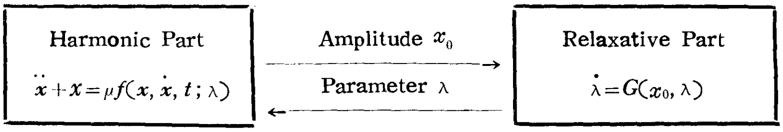

Fig. 1

II. Intermittent Oscillation

We shall consider the harmonic part:

$$\ddot{x}+x=\mu f(x,\dot{x},t;\lambda) \tag{3}$$

containing a parameter λ. The function f is considered to depend only on the parameter λ although it contains $\dot{\lambda}$, $\ddot{\lambda}$ etc. since they affect f so slightly that we may neglect them.

The Stroboscopic System [1] of (3) is written as

$$\begin{cases} \dfrac{d\rho}{d\tau}=f_1(\rho,\phi;\lambda) \\ \dfrac{d\phi}{d\tau}=f_2(\rho,\phi;\lambda) \end{cases} \tag{4}$$

and the stationary oscillation is decided from

$$f_1(\rho,\phi;\lambda)=0\,,\quad f_2(\rho,\phi;\lambda)=0\,. \tag{5}$$

The stationary amplitude x_0 of (3) is given by eliminating ϕ from (5) if the change of λ is sufficiently slow compared with the transient time of (3). We write it in the form

$$F(x_0,\lambda)=0 \tag{6}$$

where $\rho=x_0^2$, and call it "characteristic of the harmonic part".

On the other hand, the behaviour of the relaxative part is governed by the first order differential equation:

$$\dot{\lambda}=G(x_0,\lambda) \tag{7}$$

and the equilibrium point of λ is given by

$$G(x_0,\lambda)=0. \tag{8}$$

We call it "characteristic of the relaxative part". From physical consideration the equilibrium must be stable and λ must change continuously with respect to time.

We shall consider the intermittent behaviour by means of the two characteristics on the (λ,x_0)-plane (Fig. 2).

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[1] N. Minorsky: Accademia delle Scienze dell'Istituto di Bologna (1952); Bulletin Société Française des Mécaniciens (1954)

A representative point on this plane corresponds to a steady oscillation and the ordinate of the point expresses steady amplitude of the oscillation.

When the amplitude of the oscillation is modulated, the value of x_0 varies periodically so that the representative point travels on a part of (6) back and forth. Accordingly, in general, the value of λ varies periodically following the variation of x_0.

Therefore (8) must traverse (6) since λ increasses on one side of (8) and decreases on the other side of (8).

The intersection of the two characteristics is a stable equilibrium with respect to the relaxative part and hence it must be an unstable equilibrium with respect to the harmonic part, since if it is stable with respect to the harmonic part the whole system rests on the intersection and the intermittent behaviour does not occur.

Moreover, it is easy to see that the intermittent oscillation does not take place if the characteristic of the harmonic part is single-valued, that is, if x_0 in (6) is a one-valued function of λ.

From these considerations we may presume the following result.

The intermittent oscillation occurs when the characteristic of the harmonic part is many-valued and the characteristic of the relaxative part intersects the former at its unstable branch.

An illustrative example which will occur most frequently in practical cases is shown in Fig. 2, where the branch (D B) is the unstable one.

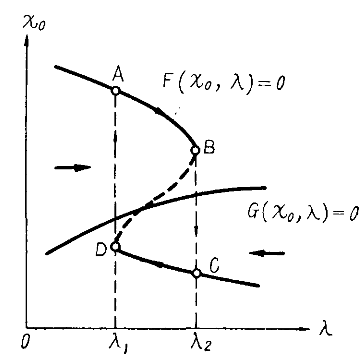

Fig. 2

In this case, the representative point at A travels rightwards on the characteristic of the harmonic part, since $\dot{\lambda}>0$ on the left-side of the characteristic of the relaxative part. As soon as it arrives at B, it jumps to C since λ varies continuously and the transient time may be neglected.

The rep. pt. at C travels leftwards since it is on the right-side of the characteristic of the relaxative part and no soonr it reaches to D than it jumps to A.

Then the rep. pt. repeats the cycling A B C D which expresses an oscillation hysteresis[2)] on the (λ, x_0)-plane.

It is easy to know the period of the intermittent behaviour. Solving (6) with

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2) E. V. Appleton & B. van der Pol : Phil. Mag. **43** (1922) 177
L. Mandelstam & N. Papalexi : Z. f. Phys. **73** (1932) 223
N. Minorsky : J. Franklin Inst. **256** (1953) 147

respect to x_0, we get

$$x_0=\xi_1(\lambda), \qquad x_0=\xi_2(\lambda) \tag{9}$$

where ξ_1 corresponds to the branch $A\ B$ and ξ_2 corresponds to the branch $D\ C$.

Substituting (9) into (7), we have

$$\dot{\lambda}=G\{\xi_1(\lambda),\lambda\}, \qquad \dot{\lambda}=G\{\xi_2(\lambda),\lambda\} \tag{10}$$

corresponding to the branch $A\ B$ and $D\ C$ respectively. Hence the period T of the intermittent behaviour is given by

$$T=\int_{\lambda_1}^{\lambda_2}\frac{d\lambda}{G\{\xi_1(\lambda),\lambda\}}+\int_{\lambda_2}^{\lambda_1}\frac{d\lambda}{G\{\xi_2(\lambda),\lambda\}} \tag{11}$$

where λ_1 is the abscissa of A (or D) and λ_2 is that of B (or C).

A representative point on the (λ, x_0)-plane corresponds to a limit cycle on the $(x, \dot{x})$-plane. At the branch points B and D, a stable limit cycle and an unstable limit cycle coalesce and disappear. In the special case, the lowest stable branch ($D\ C$ in Fig.2) may be a stable singular point ($x=0$, $\dot{x}=0$, i.e., $x_0\equiv 0$).

We shall call a harmonic part which displays an oscillation hysteresis on the (λ, x_0)-plane " hard harmonic part ", analogizing with the naming of the " hard " self-excitation [3] in the case of the autonomous system.

III. Hard Harmonic Part

Examples of the hard harmonic part which appear frequently in electrical systems are as follows.[*]

(A) The ferro-resonance circuit is an example of the hard harmonic part. Under suitable conditions, the amplitude x_0 of the alternative current through the ferro-resonance circuit displays an oscillation hysteresis with respect to the amplitude p ($p\equiv\mu P$) of the alternative source voltage, as shown in Fig.3.

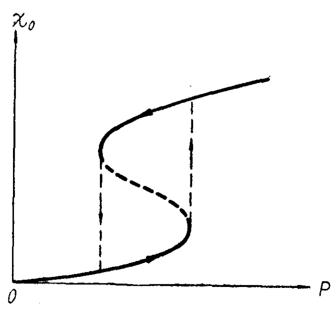

Fig. 3

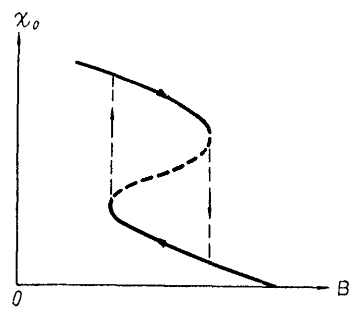

Fig. 4

[3] N. Minorsky: "Nonlinear Mechanics" (1947) p. 87
Andronow & Chaikin: "Theory of Oscillations" (1949) p. 329

[*] Some reasons why these harmonic parts are hard are furnished in Appendix.

(B) Similarly, x_0 displays an oscillation hysteresis with respect to the series resistance β ($\beta \equiv \mu B$) in the ferro-resonance circuit, as shown in Fig.4.

(C) The amplitude x_0 of the current through a ferro-resonance circuit as shown in Fig.5 displays an oscillation hysteresis with respect to the voltage α ($\alpha \equiv \mu A$) (Fig.6).

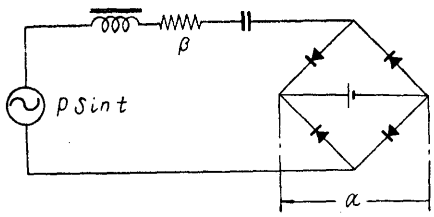

Fig. 5

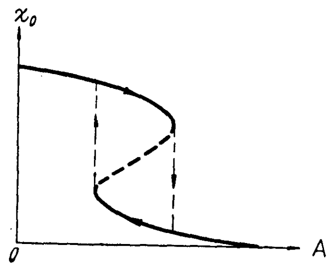

Fig. 6

(D) The self-excitation of a plate-tuning oscillator (Fig.7) (or a grid-tuning oscillator) is hard with respect to the grid bias voltage under suitable condition on the vacuum-tube characteristic. Namely, x_0, which is proportional to the amplitude of the alternative current i, displays an oscillation hysteresis with respect to the grid bias voltage $-V$ in the neighbourhood of its cut-off voltage $-V_0$, as shown in Fig.8.

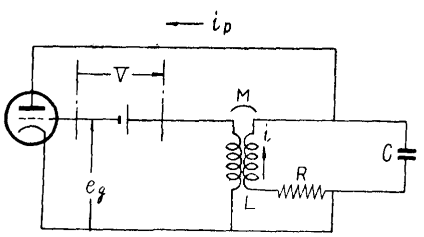

Fig. 7

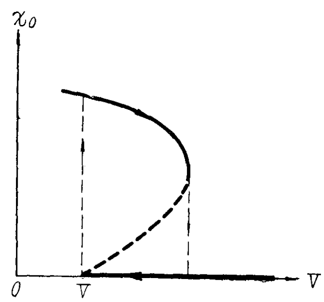

Fig. 8

(E) In the case of forced oscillations, the amplitude x_0 of the oscillation often displays an oscillation hysteresis with respect to the frequency deviation δ ($\delta \equiv \mu D$) of the forcing frequency from the proper frequency of the circuit. (Fig.9).

(F) In the case of parametric excitations, the amplitude x_0 of the oscillation often displays an oscillation hysteresis with respect to the frequency deviation

δ ($\delta \equiv \mu D$) of the parameter frequency from the proper frequency. (Fig.10).

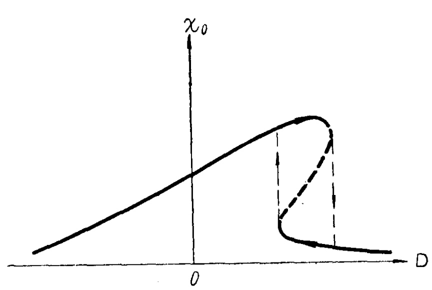

Fig. 9

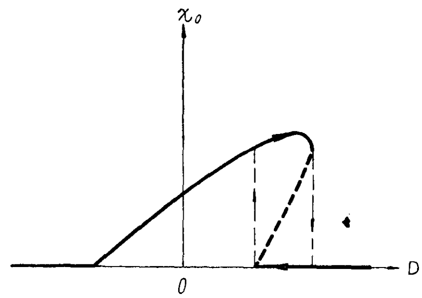

Fig. 10

IV. Relaxative Part

Examples of the relaxative part which appear frequently in electrical systems are as follows.

(a) If the capacity C in Fig.11 is so large that the period of the current source can be neglected compared with the time constant CR, the voltage V is given by

$$C\dot{V}+\frac{V}{R}=ki_0 \qquad (k\text{: const.}) \qquad (12)$$

approximately.

Similarly, in Fig. 12,

$$C\dot{V}+\frac{V}{R}=Kv_0 \qquad (K\text{: const.}) \qquad (13)$$

provided that the internal resistance of the rectifier is fairly high.

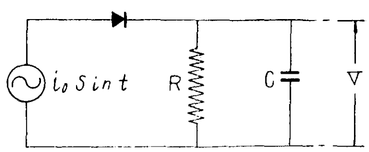

Fig. 11

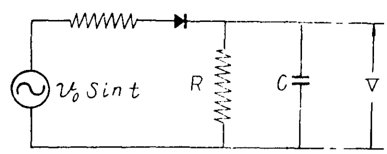

Fig. 12

(b) When an incandescent lamp is ignited by alternative current i_0 sin t, the resistance R of the lamp is approximately given by

$$\frac{\dot{R}}{R}+H(R)=g(i_0) \qquad (14)$$

where $H(R)$, $g(i_0)$ are such functions that they increase monotonically with respective variables.*)

V. Examples of Intermittent Oscillation

It has seen in II that the oscillation hysteresis of the harmonic part plays important role in the intermittent oscillation considered here.

Moreover, it is possible to get intermittent oscillation by combining suitable relaxative part to a hard harmonic part.

The following examples will explain the situation.

Ex.1[4] [Hard harmonic part (C)+Relaxative part (*a*)]

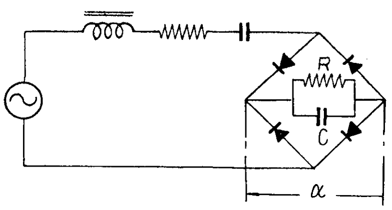

Fig. 13

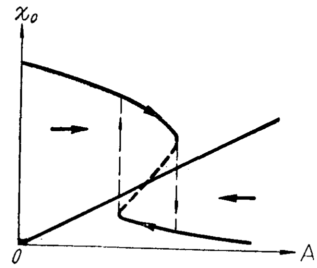

Fig. 14

The circuit of Fig. 5 may be regarded as the harmonic part of the oscillation system of Fig.13, provided capacity C is so large that its impedance to the forcin frequency is very low. Hence Fig.6 gives the characteristic of the harmonic part.

On the other hand, the circuit of Fig.11 is the relaxative part of the system. Hence voltage α is given by

$$C\dot{\alpha}+\frac{\alpha}{R}=kx_0 \tag{15}$$

and the characteristic of the relaxative part becomes

$$\alpha=\mu A=kRx_0$$

or

$$x_0=\frac{\mu}{kR}A\ . \tag{16}$$

An intermittent oscillation takes place when the line (16) intersects the characteristic of the harmonic part as shown in Fig.14.

Ex. 2[5] [Hard harmonic part (B)+Relaxative part (*b*)]

An incandesent lamp connected to alternative voltage source in series with

*) See Ref. (5) Nagumo

4) R. S. Mackay: App. Phys. **24** (1953) 1163

5) R. S. Mackay: App. Phys. **24** (1953) 311

Z. Nagumo: Proc. Faculty of Engineering, Keio University (Japan) Vol. **6**, No. 20(1953) 1 an iron-cored choke and a condenser flashes intermittently (Fig. 15). The period of "on and off" is sufficiently large compared with that of the voltage source. Therefore Fig.4 may be regarded as the characteristic of the harmonic part of the system.

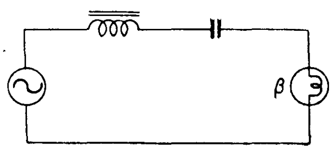

Fig. 15

On the other hand, the relaxative part is given by

$$\frac{\dot{\beta}}{\beta}+H(\beta)=g(x_0) \quad (17)$$

Hence the characteristic of the relaxative part becomes

$$H(\beta)=H(\mu B)=g(x_0). \quad (18)$$

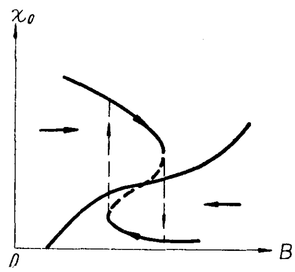

Fig. 16

The lamp flashes intermittently when the monotonic curve (18) intersects the characteristic of the harmonic part as shown in Fig. 16.

Ex. 3 [6]. [Hard harmonic part (D)+Relaxative part (a)]

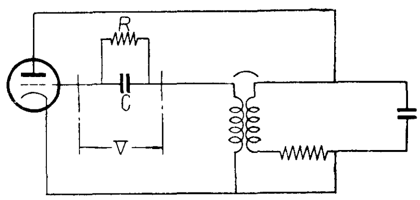

Fig. 17

Let us consider the plate-tuning oscillator with grid-leak and grid-condenser shown in Fig.17. If the coupling of the transformer is rough and the internal grid resistance is fairly high, the grid current is small and its reaction to tank circuit may be neglected. Therefore, if the capacity C is so large that its impedance to the proper frequency of the tank circuit can be neglebted, then the circuit of Fig.7 may be regarded as the harmonic part of the oscillator. Hence Fig.8 is the characteristic of the harmonic part.

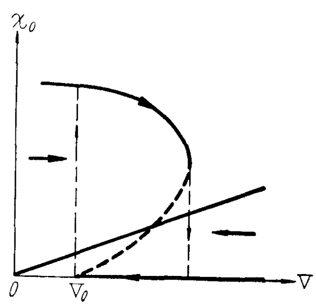

Fig. 18

On the other hand, the circuit of Fig. 12 may be thought as the relaxative part of the oscillator. Therefore the voltage V is given by

$$C\dot{V}+\frac{V}{R}=Kx_0 \quad (19)$$

6) W. A. Edson: "Vacuum-tube Oscillators" (1953) p. 227

and the characteristic of the relaxative part becomes

$$V = KRx_0$$

or

$$x_0 = \frac{1}{KR} V. \tag{20}$$

An intermittent self-excitation occurs when the line (20) intersects the characteristic of the harmonic part as shown in Fig.18.

If the intermittent behaviour of the oscillation is undesired, we can prevent it by shifting the characteristic of the relaxative part and making the intersection in the stable branch of the characteristic of the harmonic part.

For example, in the last case, we can prevent the intermittent behaviour by reducing R so that the characteristic of the relaxative part intersects one of the harmonic part in its upper stable branch.

VI. Summary

It has been pointed out that the oscillation hysteresis of the harmonic part plays an important role in the intermittent oscillation cansidered here.

The intermittent oscillation takes place when the characteristic of the relaxative part intersects the characteristic of the harmontc part in the unstable branch of the latter. It is possible to get intermittent oscillation by combining suitable relaxative part to a hard harmonic part, that is, a harmonic part the characteristic of which displays an oscillation hysteresis.

On the contrary, if the intermittent behaviour of the oscillation is undesired, we can prevent it by shifting the characteristic of the relaxative part and making the intersection in the stable branch of the characteristic of the harmonic part.

Acknowledgment

The writer is indebted to Prof. M. Mashima of Keio University for his leadership and encouragement.

Appendix

As being well known,[7] the ferro-resonance phenomena can be explained by the following differential equation.

$$\ddot{x} + \beta\dot{x} + x - \delta x + \varepsilon x^3 = p \sin t. \tag{1}$$

[7] J. J. Stoker: "Nonlinear Vibrations" (1950) p. 91

If β, δ, ε and p are small quantities of the same order with μ, one gets by Stroboscopic Method[1)]

$$9E^2x_0^6-24DEx_0^4+16(B^2+D^2)x_0^2-16P^2=0 \tag{2}$$

where x_0 is the stationary amplitude of x and $\beta/B=\delta/D=\varepsilon/E=p/P=\mu$.

Under suitable conditions, Eq. (2) takes the form shown in Fig.3, Fig.4 and Fig.9 by choosing P, B and D as respective parameter.

If α is a small quantity of the same order with μ, it would not be a great mistake to explain a ferro-resonance phenomenon of the circuit shown in Fig. 5 by the differential equation of the form:

$$\ddot{x}+\beta\dot{x}+x-\delta x+\varepsilon\dot{x}^2x+\alpha\,\mathrm{sgn}\,\dot{x}=p\sin t. \tag{3}$$

Then one has

$$\frac{E^2}{16}x_0^6-\frac{DE}{2}x_0^4+(B^2+D^2)x_0^2+\frac{8AB}{\pi}x_0+\left(\frac{16A^2}{\pi^2}-P^2\right)=0 \tag{4}$$

or

$$\frac{4}{\pi}A=-Bx_0+\sqrt{P^2-x_0^2\left(\frac{E}{4}x_0^2-D\right)^2} \tag{5}$$

where $\alpha=\mu A$.

Under suitable conditions, Eq. (5) takes the form as shown in Fig. 6.

In considering the oscillation hysteresis of the plate-tuning oscillator (Fig. 7) with respect to gridbias voltage, one may, for simplicity, idealize the vacuum-tube characteristic as follows:

$$i_p=\frac{I_s}{2}\{1+\mathrm{sgn}(e_g+V_0)\} \tag{6}$$

where I_s is the saturated plate current (Fig. A).

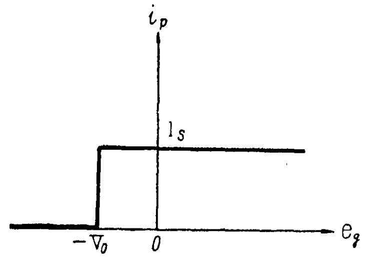

Fig. A

The differential equations of the circuit are

$$L\frac{di}{dt}+Ri+\frac{1}{C}\int(i-i_p)dt=0$$

$$e_g=M\frac{di}{dt}-V. \tag{7}$$

Putting $x=\frac{M}{\sqrt{LC}}i$, $\beta=\sqrt{\frac{C}{L}}R$,

$p=\frac{1}{2}\frac{M}{\sqrt{LC}}I_s$, $V_1=V-V_0$ and $t'=\frac{1}{\sqrt{LC}}t\left(\dot{x}=\frac{dx}{dt'}\right)$, one has

$$\ddot{x}+\beta\dot{x}+x=p\{1+\mathrm{sgn}(\dot{x}-V_1)\}. \tag{8}$$

The Stroboscopic System of (8) becomes

$$\frac{d\rho}{d\tau}=-B\rho+Pf_\rho\ , \quad \frac{d\phi}{d\tau}=0 \tag{9}$$

where

$$f_\rho=\frac{4}{\pi}\sqrt{\rho-V_1^2}\ (\rho>V_1^2),\ =0\ (\rho<V_1^2).$$

Hence $\rho\to0$ if $\rho<V_1^2$, $\rho\to\rho_0$ if $\rho>V_1^2$
where

$$\rho_0=x_0^2=\frac{8P^2}{B^2\pi^2}\pm\sqrt{\frac{64P^4}{B^4\pi^4}-\frac{16P^2V_1^2}{B^2\pi^2}} \tag{10}$$

From (10) and consideration of stability one has Fig. 8.

As an example of parametric excitations, we shall consider the differential equation of the form

$$\ddot{x}+(\beta+\gamma x^2)\dot{x}+(1+\alpha\cos2t)x-\delta x+\varepsilon x^3=0 \tag{11}$$

where γ is a small quantity of the same order with μ ($\gamma=\mu C$).

The Stroboscopic System becomes

$$\left\{\begin{aligned}\frac{d\rho}{d\tau}&=-B\rho-\frac{C}{4}\rho^2-\frac{A}{2}\rho\sin2\phi\\ \frac{d\phi}{d\tau}&=\frac{D}{2}-\frac{3E}{8}\rho-\frac{A}{4}\cos2\phi\ .\end{aligned}\right. \tag{12}$$

Hence stationary amplitude x_0 ($x_0\neq0$) is given by

$$D=\frac{3E}{4}\pm\frac{1}{2}\sqrt{(A^2-4B^2)-2BCx_0^2-\frac{C^2}{4}x_0^4}\ . \tag{13}$$

From (13) and consideration of stability, one has Fig. 10.